FHSST Authors

The Free High School Science Texts: Textbooks for High School Students Studying the Sciences Mathematics Grades 10 - 12

Version 0
September 17, 2008

STOP!!!!

Did you notice the FREEDOMS we've granted you?

Our copyright license is **different!** It grants freedoms rather than just imposing restrictions like all those other textbooks you probably own or use.

- We know people copy textbooks illegally but we would LOVE it if you copied our's - go ahead copy to your hearts content, **legally!**
- Publishers revenue is generated by controlling the market, we don't want any money, go ahead, distribute our books far and wide - we DARE you!
- Ever wanted to change your textbook? Of course you have! Go ahead change ours, make your own version, get your friends together, rip it apart and put it back together the way you like it. That's what we really want!
- Copy, modify, adapt, enhance, share, critique, adore, and contextualise. Do it all, do it with your colleagues, your friends or alone but get involved! Together we can overcome the challenges our complex and diverse country presents.
- So what is the catch? The only thing you can't do is take this book, make a few changes and then tell others that they can't do the same with your changes. It's share and share-alike and we know you'll agree that is only fair.
- These books were written by volunteers who want to help support education, who want the facts to be freely available for teachers to copy, adapt and re-use. Thousands of hours went into making them and they are a gift to everyone in the education community.

FHSST Core Team

Mark Horner ; Samuel Halliday ; Sarah Blyth ; Rory Adams ; Spencer Wheaton

FHSST Editors

Jaynie Padayachee ; Joanne Boulle ; Diana Mulcahy ; Annette Nell ; René Toerien ; Donovan Whitfield

FHSST Contributors

Rory Adams ; Prashant Arora ; Richard Baxter ; Dr. Sarah Blyth ; Sebastian Bodenstein ; Graeme Broster ; Richard Case ; Brett Cocks ; Tim Crombie ; Dr. Anne Dabrowski ; Laura Daniels ; Sean Dobbs ; Fernando Durrell ; Dr. Dan Dwyer ; Frans van Eeden ; Giovanni Franzoni ; Ingrid von Glehn ; Tamara von Glehn ; Lindsay Glesener ; Dr. Vanessa Godfrey ; Dr. Johan Gonzalez ; Hemant Gopal ; Umeshree Govender ; Heather Gray ; Lynn Greeff ; Dr. Tom Gutierrez ; Brooke Haag ; Kate Hadley ; Dr. Sam Halliday ; Asheena Hanuman ; Neil Hart ; Nicholas Hatcher ; Dr. Mark Horner ; Mfandaidza Hove ; Robert Hovden ; Jennifer Hsieh ; Clare Johnson ; Luke Jordan ; Tana Joseph ; Dr. Jennifer Klay ; Lara Kruger ; Sihle Kubheka ; Andrew Kubik ; Dr. Marco van Leeuwen ; Dr. Anton Machacek ; Dr. Komal Maheshwari ; Kosma von Maltitz ; Nicole Masureik ; John Mathew ; JoEllen McBride ; Nikolai Meures ; Riana Meyer ; Jenny Miller ; Abdul Mirza ; Asogan Moodaly ; Jothi Moodley ; Nolene Naidu ; Tyrone Negus ; Thomas O'Donnell ; Dr. Markus Oldenburg ; Dr. Jaynie Padayachee ; Nicolette Pekeur ; Sirika Pillay ; Jacques Plaut ; Andrea Prinsloo ; Joseph Raimondo ; Sanya Rajani ; Prof. Sergey Rakityansky ; Alastair Ramlakan ; Razvan Remsing ; Max Richter ; Sean Riddle ; Evan Robinson ; Dr. Andrew Rose ; Bianca Ruddy ; Katie Russell ; Duncan Scott ; Helen Seals ; Ian Sherratt ; Roger Sieloff ; Bradley Smith ; Greg Solomon ; Mike Stringer ; Shen Tian ; Robert Torregrosa ; Jimmy Tseng ; Helen Waugh ; Dr. Dawn Webber ; Michelle Wen ; Dr. Alexander Wetzler ; Dr. Spencer Wheaton ; Vivian White ; Dr. Gerald Wigger ; Harry Wiggins ; Wendy Williams ; Julie Wilson ; Andrew Wood ; Emma Wormauld ; Sahal Yacoob ; Jean Youssef

Contributors and editors have made a sincere effort to produce an accurate and useful resource. Should you have suggestions, find mistakes or be prepared to donate material for inclusion, please don't hesitate to contact us. We intend to work with all who are willing to help make this a continuously evolving resource!

www.fhsst.org

Contents

Part I

Basics

Chapter 1

Introduction to Book

1.1 The Language of Mathematics

The purpose of any language, like English or Zulu, is to make it possible for people to communicate. All languages have an alphabet, which is a group of letters that are used to make up words. There are also rules of grammar which explain how words are supposed to be used to build up sentences. This is needed because when a sentence is written, the person reading the sentence understands exactly what the writer is trying to explain. Punctuation marks (like a full stop or a comma) are used to further clarify what is written.

Mathematics is a language, specifically it is the language of Science. Like any language, mathematics has letters (known as numbers) that are used to make up words (known as expressions), and sentences (known as equations). The punctuation marks of mathematics are the different signs and symbols that are used, for example, the plus sign ($+$), the minus sign (-), the multiplication sign ($\times$), the equals sign ($=$) and so on. There are also rules that explain how the numbers should be used together with the signs to make up equations that express some meaning.

Part II

Grade 10

Chapter 2

Review of Past Work

2.1 Introduction

This chapter describes some basic concepts which you have seen in earlier grades, and lays the foundation for the remainder of this book. You should feel confident with the content in this chapter, before moving on with the rest of the book.

So try out your skills on the exercises throughout this chapter and ask your teacher for more questions just like them. You can also try making up your own questions, solve them and try them out on your classmates to see if you get the same answers.

Practice is the only way to get good at maths!

2.2 What is a number?

A number is a way to represent quantity. Numbers are not something that you can touch or hold, because they are not physical. But you can touch three apples, three pencils, three books. You can never just touch three, you can only touch three of something. However, you do not need to see three apples in front of you to know that if you take one apple away, that there will be two apples left. You can just think about it. That is your brain representing the apples in numbers and then performing arithmetic on them.

A number represents quantity because we can look at the world around us and quantify it using numbers. How many minutes? How many kilometers? How many apples? How much money? How much medicine? These are all questions which can only be answered using numbers to tell us "how much" of something we want to measure.

A number can be written many different ways and it is always best to choose the most appropriate way of writing the number. For example, "a half" may be spoken aloud or written in words, but that makes mathematics very difficult and also means that only people who speak the same language as you can understand what you mean. A better way of writing "a half" is as a fraction $\frac{1}{2}$ or as a decimal number $0{,}5$. It is still the same number, no matter which way you write it.

In high school, all the numbers which you will see are called *real numbers* and mathematicians use the symbol $\mathbb{R}$ to stand for the *set of all real numbers*, which simply means all of the real numbers. Some of these real numbers can be written in a particular way and some cannot. Different types of numbers are described in detail in Section 1.12.

2.3 Sets

A *set* is a group of objects with a well-defined criterion for membership. For example, the criterion for belonging to a set of apples, is that it must be an apple. The set of apples can then be divided into red apples and green apples, but they are all still apples. All the red apples form another set which is a *sub-set* of the set of apples. A sub-set is part of a set. All the green apples form another sub-set.

Now we come to the idea of a **union**, which is used to combine things. The symbol for **union** is $\cup$. Here we use it to combine two or more intervals. For example, if x is a real number such that $1 < x \leq 3$ or $6 \leq x < 10$, then the set of all the possible x values is

$$(1,3] \cup [6,10) \tag{2.1}$$

where the $\cup$ sign means the union (or combination) of the two intervals. We use the set and interval notation and the symbols described because it is easier than having to write everything out in words.

2.4 Letters and Arithmetic

The simplest things that can be done with numbers is to add, subtract, multiply or divide them. When two numbers are added, subtracted, multiplied or divided, you are performing *arithmetic*[1]. These four basic operations can be performed on any two real numbers.

Mathematics as a language uses special notation to write things down. So instead of:

one plus one is equal to two

mathematicians write

$$1 + 1 = 2$$

In earlier grades, place holders were used to indicate missing numbers in an equation.

$$\begin{aligned} 1 + \square &= 2 \\ 4 - \square &= 2 \\ \square + 3 - 2\square &= 2 \end{aligned}$$

However, place holders only work well for simple equations. For more advanced mathematical workings, letters are usually used to represent numbers.

$$\begin{aligned} 1 + x &= 2 \\ 4 - y &= 2 \\ z + 3 - 2z &= 2 \end{aligned}$$

These letters are referred to as **variables**, since they can take on any value depending on what is required. For example, $x = 1$ in Equation 2.2, but $x = 26$ in $2 + x = 28$.
A **constant** has a fixed value. The number 1 is a constant. The *speed of light* in a vacuum is also a constant which has been defined to be exactly 299 792 458 m·s^{-1}(read metres per second). The speed of light is a big number and it takes up space to always write down the entire number. Therefore, letters are also used to represent some constants. In the case of the speed of light, it is accepted that the letter c represents the speed of light. Such constants represented by letters occur most often in physics and chemistry.

Additionally, letters can be used to describe a situation, mathematically. For example, the following equation

$$x + y = z \tag{2.2}$$

can be used to describe the situation of finding how much change can be expected for buying an item. In this equation, y represents the price of the item you are buying, x represents the amount of change you should get back and z is the amount of money given to the cashier. So, if the price is R10 and you gave the cashier R15, then write R15 instead of z and R10 instead of y and the change is then x.

$$x + 10 = 15 \tag{2.3}$$

We will learn how to "solve" this equation towards the end of this chapter.

[1]Arithmetic is derived from the Greek word *arithmos* meaning *number*.

2.5 Addition and Subtraction

Addition (+) and subtraction (-) are the most basic operations between numbers but they are very closely related to each other. You can think of subtracting as being the opposite of adding since adding a number and then subtracting the same number will not change what you started with. For example, if we start with a and add b, then subtract b, we will just get back to a again

$$\begin{aligned} a+b-b&=a \\ 5+2-2&=5 \end{aligned} \tag{2.4}$$

If we look at a number line, then addition means that we move to the right and subtraction means that we move to the left.

The order in which numbers are added does not matter, but the order in which numbers are subtracted does matter. This means that:

$$\begin{aligned} a+b &= b+a \\ a-b &\neq b-a \quad \text{if } a\neq b \end{aligned} \tag{2.5}$$

The sign $\neq$ means "is not equal to". For example, $2+3=5$ and $3+2=5$, but $5-3=2$ and $3-5=-2$. -2 is a negative number, which is explained in detail in Section 2.8.

Extension: Commutativity for Addition

The fact that $a+b=b+a$, is known as the *commutative* property for addition.

2.6 Multiplication and Division

Just like addition and subtraction, multiplication ($\times$, $\cdot$) and division ($\div$, $/$) are opposites of each other. Multiplying by a number and then dividing by the same number gets us back to the start again:

$$\begin{aligned} a\times b\div b&=a \\ 5\times 4\div 4&=5 \end{aligned} \tag{2.6}$$

Sometimes you will see a multiplication of letters as a dot or without any symbol. Don't worry, its exactly the same thing. Mathematicians are lazy and like to write things in the shortest, neatest way possible.

$$\begin{aligned} abc &= a\times b\times c \\ a\cdot b\cdot c &= a\times b\times c \end{aligned} \tag{2.7}$$

It is usually neater to write known numbers to the left, and letters to the right. So although $4x$ and $x4$ are the same thing, it looks better to write $4x$. In this case, the "4" is a constant that is referred to as the *coefficient* of x.

Extension: Commutativity for Multiplication

The fact that $ab=ba$ is known as the *commutative* property of multiplication. Therefore, both addition and multiplication are described as commutative operations.

2.7 Brackets

Brackets[2] in mathematics are used to show the order in which you must do things. This is important as you can get different answers depending on the order in which you do things. For

[2] Sometimes people say "parenthesis" instead of "brackets".

example

$$(5 \times 5) + 20 = 45 \quad (2.8)$$

whereas

$$5 \times (5 + 20) = 125 \quad (2.9)$$

If there are no brackets, you should always do multiplications and divisions first and then additions and subtractions[3]. You can always put your own brackets into equations using this rule to make things easier for yourself, for example:

$$\begin{aligned} a \times b + c \div d &= (a \times b) + (c \div d) \quad (2.10) \\ 5 \times 5 + 20 \div 4 &= (5 \times 5) + (20 \div 4) \end{aligned}$$

If you see a multiplication outside a bracket like this

$$\begin{aligned} &a(b + c) \quad (2.11) \\ &3(4 - 3) \end{aligned}$$

then it means you have to multiply each part inside the bracket by the number outside

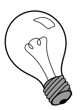
$$\begin{aligned} a(b + c) &= ab + ac \quad (2.12) \\ 3(4 - 3) &= 3 \times 4 - 3 \times 3 = 12 - 9 = 3 \end{aligned}$$

unless you can simplify everything inside the bracket into a single term. In fact, in the above example, it would have been smarter to have done this

$$3(4 - 3) = 3 \times (1) = 3 \quad (2.13)$$

It can happen with letters too

$$3(4a - 3a) = 3 \times (a) = 3a \quad (2.14)$$



Extension: Distributivity
The fact that $a(b + c) = ab + ac$ is known as the *distributive* property.

If there are two brackets multiplied by each other, then you can do it one step at a time

$$\begin{aligned} (a + b)(c + d) &= a(c + d) + b(c + d) \quad (2.15) \\ &= ac + ad + bc + bd \\ (a + 3)(4 + d) &= a(4 + d) + 3(4 + d) \\ &= 4a + ad + 12 + 3d \end{aligned}$$

2.8 Negative Numbers

2.8.1 What is a negative number?

Negative numbers can be very confusing to begin with, but there is nothing to be afraid of. The numbers that are used most often are greater than zero. These numbers are known as *positive numbers*.

A negative number is simply a number that is less than zero. So, if we were to take a positive number a and subtract it from zero, the answer would be the negative of a.

$$0 - a = -a$$

[3] Multiplying and dividing can be performed in any order as it doesn't matter. Likewise it doesn't matter which order you do addition and subtraction. Just as long as you do any $\times \div$ before any $+-$.

On a number line, a negative number appears to the left of zero and a positive number appears to the right of zero.

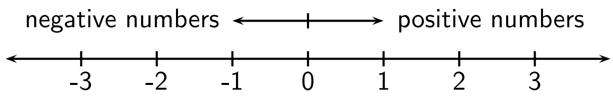


Figure 2.1: On the number line, numbers increase towards the right and decrease towards the left. Positive numbers appear to the right of zero and negative numbers appear to the left of zero.

2.8.2 Working with Negative Numbers

When you are adding a negative number, it is the same as subtracting that number if it were positive. Likewise, if you subtract a negative number, it is the same as adding the number if it were positive. Numbers are either positive or negative, and we call this their *s*ign. A positive number has positive sign (+), and a negative number has a negative sign (-).

Subtraction is actually the same as adding a *negative number*.

In this example, a and b are positive numbers, but $-b$ is a negative number

$$\begin{aligned} a - b &= a + (-b) \\ 5 - 3 &= 5 + (-3) \end{aligned} \tag{2.16}$$

So, this means that subtraction is simply a short-cut for adding a negative number, and instead of writing $a + (-b)$, we write $a - b$. This also means that $-b + a$ is the same as $a - b$. Now, which do you find easier to work out?

Most people find that the first way is a bit more difficult to work out than the second way. For example, most people find $12 - 3$ a lot easier to work out than $-3 + 12$, even though they are the same thing. So, $a - b$, which looks neater and requires less writing, is the accepted way of writing subtractions.

Table 2.1 shows how to calculate the sign of the answer when you multiply two numbers together. The first column shows the sign of the first number, the second column gives the sign of the second number, and the third column shows what sign the answer will be. So multiplying or

a	b	$a \times b$ or $a \div b$
+	+	+
+	-	-
-	+	-
-	-	+

Table 2.1: Table of signs for multiplying or dividing two numbers.

dividing a negative number by a positive number always gives you a negative number, whereas multiplying or dividing numbers which have the same sign always gives a positive number. For example, $2 \times 3 = 6$ and $-2 \times -3 = 6$, but $-2 \times 3 = -6$ and $2 \times -3 = -6$.

Adding numbers works slightly differently, have a look at Table 2.2. The first column shows the sign of the first number, the second column gives the sign of the second number, and the third column shows what sign the answer will be.

a	b	$a + b$
+	+	+
+	-	?
-	+	?
-	-	-

Table 2.2: Table of signs for adding two numbers.

If you add two positive numbers you will always get a positive number, but if you add two negative numbers you will always get a negative number. If the numbers have different sign, then the sign of the answer depends on which one is bigger.

2.8.3 Living Without the Number Line

The number line in Figure 2.1 is a good way to visualise what negative numbers are, but it can get very inefficient to use it every time you want to add or subtract negative numbers. To keep things simple, we will write down three tips that you can use to make working with negative numbers a little bit easier. These tips will let you work out what the answer is when you add or subtract numbers which may be negative and will also help you keep your work tidy and easier to understand.

Negative Numbers Tip 1

If you are given an equation like $-a+b$, then it is easier to move the numbers around so that the equation looks easier. For this case, we have seen that adding a negative number to a positive number is the same as subtracting the number from the positive number. So,

$$\begin{aligned} -a+b &= b-a \\ -5+10 &= 10-5=5 \end{aligned} \quad (2.17)$$

This makes equations easier to understand. For example, a question like "What is $-7+11$?" looks a lot more complicated than "What is $11-7$?", even though they are exactly the same question.

Negative Numbers Tip 2

When you have two negative numbers like $-3-7$, you can calculate the answer by simply adding together the numbers as if they were positive and then putting a negative sign in front.

$$\begin{aligned} -c-d &= -(c+d) \\ -7-2 &= -(7+2)=-9 \end{aligned} \quad (2.18)$$

Negative Numbers Tip 3

In Table 2.2 we saw that the sign of two numbers added together depends on which one is bigger. This tip tells us that all we need to do is take the smaller number away from the larger one, and remember to put a negative sign before the answer if the bigger number was subtracted to begin with. In this equation, F is bigger than e.

$$\begin{aligned} e-F &= -(F-e) \\ 2-11 &= -(11-2)=-9 \end{aligned} \quad (2.19)$$

You can even combine these tips together, so for example you can use Tip 1 on $-10+3$ to get $3-10$, and then use Tip 3 to get $-(10-3)=-7$.

Exercise: Negative Numbers

1. Calculate:
   (a) $(-5)-(-3)$ (b) $(-4)+2$ (c) $(-10)\div(-2)$
   (d) $11-(-9)$ (e) $-16-(6)$ (f) $-9\div 3\times 2$
   (g) $(-1)\times 24\div 8\times(-3)$ (h) $(-2)+(-7)$ (i) $1-12$
   (j) $3-64+1$ (k) $-5-5-5$ (l) $-6+25$
   (m) $-9+8-7+6-5+4-3+2-1$

2. Say whether the sign of the answer is + or -

   (a) $-5+6$ (b) $-5+1$ (c) $-5 \div -5$
   (d) $-5 \div 5$ (e) $5 \div -5$ (f) $5 \div 5$
   (g) -5×-5 (h) -5×5 (i) 5×-5
   (j) 5×5

2.9 Rearranging Equations

Now that we have described the basic rules of negative and positive numbers and what to do when you add, subtract, multiply and divide them, we are ready to tackle some real mathematics problems!

Earlier in this chapter, we wrote a general equation for calculating how much change (x) we can expect if we know how much an item costs (y) and how much we have given the cashier (z). The equation is:

$$x + y = z \tag{2.20}$$

So, if the price is R10 and you gave the cashier R15, then write R15 instead of z and R10 instead of y.

$$x + 10 = 15 \tag{2.21}$$

Now, that we have written this equation down, how exactly do we go about finding what the change is? In mathematical terms, this is known as solving an equation for an unknown (x in this case). We want to re-arrange the terms in the equation, so that only x is on the left hand side of the $=$ sign and everything else is on the right.

The most important thing to remember is that an equation is like a set of weighing scales. In order to keep the scales balanced, whatever, is done to one side, must be done to the other.

Method: Rearranging Equations

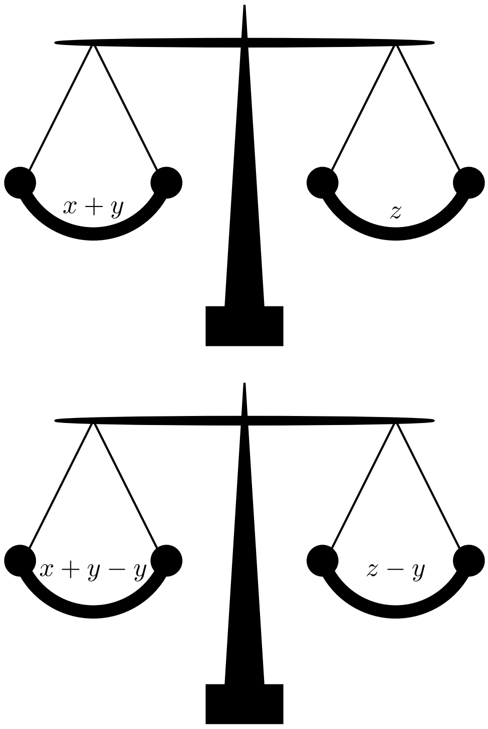
You can add, subtract, multiply or divide both sides of an equation by any number you want, as long as you always do it to both sides.

So for our example we could subtract y from both sides

$$\begin{aligned} x + y &= z \qquad (2.22)\\ x + y - y &= z - y \\ x &= z - y \\ x &= 15 - 10 \\ &= 5 \end{aligned}$$

so now we can find the change is the price subtracted from the amount handed over to the cashier. In the example, the change should be R5. In real life we can do this in our head, the human brain is very smart and can do arithmetic without even knowing it.

When you subtract a number from both sides of an equation, it looks just like you moved a positive number from one side and it became a negative on the other, which is exactly what happened. Likewise if you move a multiplied number from one side to the other, it looks like it changed to a divide. This is because you really just divided both sides by that number, and a



divide the other side too.

Figure 2.2: An equation is like a set of weighing scales. In order to keep the scales balanced, you must do the same thing to both sides. So, if you add, subtract, multiply or divide the one side, you must add, subtract, multiply or
divide the other side too.

number divided by itself is just 1

$$\begin{aligned} a(5+c) &= 3a \\ a(5+c) \div a &= 3a \div a \\ \frac{a}{a} \times (5+c) &= 3 \times \frac{a}{a} \\ 1 \times (5+c) &= 3 \times 1 \\ 5+c &= 3 \\ c &= 3-5=-2 \end{aligned} \tag{2.23}$$

However you must be careful when doing this, as it is easy to make mistakes.

The following is the wrong thing to do

$$\begin{aligned} 5a+c &= 3a \\ 5+c &\neq^{4} 3a \div a \end{aligned} \tag{2.24}$$

Can you see why it is wrong? It is wrong because we did not divide the c term by a as well. The correct thing to do is

$$\begin{aligned} 5a+c &= 3a \\ 5+c \div a &= 3 \\ c \div a &= 3-5=-2 \end{aligned} \tag{2.25}$$

?

Exercise: Rearranging Equations

1. If $3(2r - 5) = 27$, then $2r - 5 =$ .....
2. Find the value for x if $0{,}5(x - 8) = 0{,}2x + 11$
3. Solve $9 - 2n = 3(n + 2)$
4. Change the formula $P = A + Akt$ to $A =$
5. Solve for x: $\frac{1}{ax} + \frac{1}{bx} = 1$

2.10 Fractions and Decimal Numbers

A fraction is one number divided by another number. There are several ways to write a number divided by another one, such as $a \div b$, a/b and $\frac{a}{b}$. The first way of writing a fraction is very hard to work with, so we will use only the other two. We call the number on the top, the *numerator* and the number on the bottom the *denominator*. For example,

$$\frac{1}{5} \qquad \frac{\text{numerator} = 1}{\text{denominator} = 5} \tag{2.26}$$

Extension: Definition - Fraction
The word *fraction* means *part of a whole*.

The *reciprocal* of a fraction is the fraction turned upside down, in other words the numerator becomes the denominator and the denominator becomes the numerator. So, the reciprocal of $\frac{2}{3}$ is $\frac{3}{2}$.
A fraction multiplied by its reciprocal is always equal to 1 and can be written

$$\frac{a}{b} \times \frac{b}{a} = 1 \tag{2.27}$$

This is because dividing by a number is the same as multiplying by its reciprocal.

Extension: Definition - Multiplicative Inverse
The reciprocal of a number is also known as the multiplicative inverse.

A decimal number is a number which has an integer part and a fractional part. The integer and the fractional parts are separated by a *decimal point*, which is written as a comma in South Africa. For example the number $3\frac{14}{100}$ can be written much more cleanly as $3{,}14$.

All real numbers can be written as a decimal number. However, some numbers would take a huge amount of paper (and ink) to write out in full! Some decimal numbers will have a number which will repeat itself, such as $0{,}33333\ldots$ where there are an infinite number of 3's. We can write this decimal value by using a dot above the repeating number, so $0{,}\dot{3} = 0{,}33333\ldots$. If there are two repeating numbers such as $0{,}121212\ldots$ then you can place dots[5] on each of the repeated numbers $0{,}\dot{1}\dot{2} = 0{,}121212\ldots$. These kinds of repeating decimals are called *recurring decimals*.

Table 2.3 lists some common fractions and their decimal forms.

[5] or a bar, like $0{,}\overline{12}$

Fraction	Decimal Form
$\frac{1}{20}$	0,05
$\frac{1}{16}$	0,0625
$\frac{1}{10}$	0,1
$\frac{1}{8}$	0,125
$\frac{1}{6}$	$0{,}16\dot{6}$
$\frac{1}{5}$	0,2
$\frac{1}{2}$	0,5
$\frac{3}{4}$	0,75

Table 2.3: Some common fractions and their equivalent decimal forms.

2.11 Scientific Notation

In science one often needs to work with very large or very small numbers. These can be written more easily in scientific notation, which has the general form

$$a \times 10^m \tag{2.28}$$

where a is a decimal number between 0 and 10 that is rounded off to a few decimal places. The m is an integer and if it is positive it represents how many zeros should appear to the right of a. If m is negative then it represents how many times the decimal place in a should be moved to the left. For example $3{,}2 \times 10^3$ represents 32000 and $3{,}2 \times 10^{-3}$ represents $0{,}0032$.

If a number must be converted into scientific notation, we need to work out how many times the number must be multiplied or divided by 10 to make it into a number between 1 and 10 (i.e. we need to work out the value of the exponent m) and what this number is (the value of a). We do this by counting the number of decimal places the decimal point must move.

For example, write the speed of light which is 299 792 458 ms^{-1} in scientific notation, to two decimal places. First, determine where the decimal point must go for two decimal places (to find a) and then count how many places there are after the decimal point to determine m.

In this example, the decimal point must go after the first 2, but since the number after the 9 is a 7, $a = 3{,}00$.

So the number is $3{,}00 \times 10^m$, where $m = 8$, because there are 8 digits left after the decimal point. So the speed of light in scientific notation, to two decimal places is $3{,}00 \times 10^8 ms^{-1}$.

As another example, the size of the HI virus is around 120×10^{-9} m. This is equal to $120 \times 0{,}000000001$ m which is 0,00000012 m.

2.12 Real Numbers

Now that we have learnt about the basics of mathematics, we can look at what real numbers are in a little more detail. The following are examples of real numbers and it is seen that each number is written in a different way.

$$\sqrt{3}, \quad 1{,}2557878, \quad \frac{56}{34}, \quad 10, \quad 2{,}1, \quad -5, \quad -6{,}35, \quad -\frac{1}{90} \tag{2.29}$$

Depending on how the real number is written, it can be further labelled as either rational, irrational, integer or natural. A set diagram of the different number types is shown in Figure 2.3.

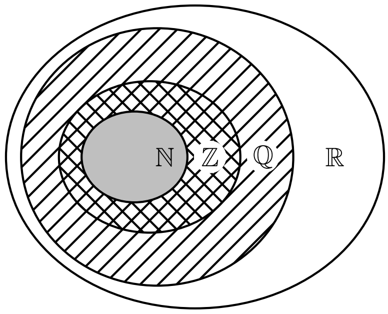
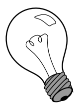


Figure 2.3: Set diagram of all the real numbers $\mathbb{R}$, the rational numbers $\mathbb{Q}$, the integers $\mathbb{Z}$ and the natural numbers $\mathbb{N}$. The irrational numbers are the numbers not inside the set of rational numbers. All of the integers are also rational numbers, but not all rational numbers are integers.



Extension: Non-Real Numbers
All numbers that are not real numbers have *imaginary* components. We will not see imaginary numbers in this book but you will see that they come from $\sqrt{-1}$. Since we won't be looking at numbers which are not real, if you see a number you can be sure it is a real one.

2.12.1 Natural Numbers

The first type of numbers that are learnt about are the numbers that were used for counting. These numbers are called *natural numbers* and are the simplest numbers in mathematics.

$$0, 1, 2, 3, 4 \ldots \tag{2.30}$$

Mathematicians use the symbol $\mathbb{N}$ to mean the *set of all natural numbers*. The natural numbers are a *subset* of the real numbers since every natural number is also a real number.

2.12.2 Integers

The integers are all of the natural numbers and their negatives

$$\ldots - 4, -3, -2, -1, 0, 1, 2, 3, 4 \ldots \tag{2.31}$$

Mathematicians use the symbol $\mathbb{Z}$ to mean *the set of all integers*. The integers are a subset of the real numbers, since every integer is a real number.

2.12.3 Rational Numbers

The natural numbers and the integers are only able to describe quantities that are whole or complete. For example you can have 4 apples, but what happens when you divide one apple into 4 equal pieces and share it among your friends? Then it is not a whole apple anymore and a different type of number is needed to describe the apples. This type of number is known as a rational number.

A rational number is any number which can be written as:

$$\frac{a}{b} \tag{2.32}$$

where a and b are integers and $b \neq 0$.

The following are examples of rational numbers:

$$\frac{20}{9}, \quad \frac{-1}{2}, \quad \frac{20}{10}, \quad \frac{3}{15} \tag{2.33}$$

Extension: Notation Tip
Rational numbers are any number that can be expressed in the form $\frac{a}{b}; a, b \in \mathbb{Z}; b \neq 0$ which means "the set of numbers $\frac{a}{b}$ when a and b are integers".

Mathematicians use the symbol $\mathbb{Q}$ to mean *the set of all rational numbers*. The set of rational numbers contains all numbers which can be written as terminating or repeating decimals.

Extension: Rational Numbers
All integers are rational numbers with denominator 1.

You can add and multiply rational numbers and still get a rational number at the end, which is very useful. If we have 4 integers, a, b, c and d, then the rules for adding and multiplying rational numbers are

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd} \quad (2.34)$$

$$\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd} \quad (2.35)$$

Extension: Notation Tip
The statement "4 integers a, b, c and d" can be written formally as $\{a, b, c, d\} \in \mathbb{Z}$ because the $\in$ symbol means *in* and we say that a, b, c and d are *in* the set of integers.

Two rational numbers ($\frac{a}{b}$ and $\frac{c}{d}$) represent the same number if $ad = bc$. It is always best to simplify any rational number so that the denominator is as small as possible. This can be achieved by dividing both the numerator and the denominator by the same integer. For example, the rational number $1000/10000$ can be divided by 1000 on the top and the bottom, which gives $1/10$. $\frac{2}{3}$ of a pizza is the same as $\frac{8}{12}$ (Figure 2.4).

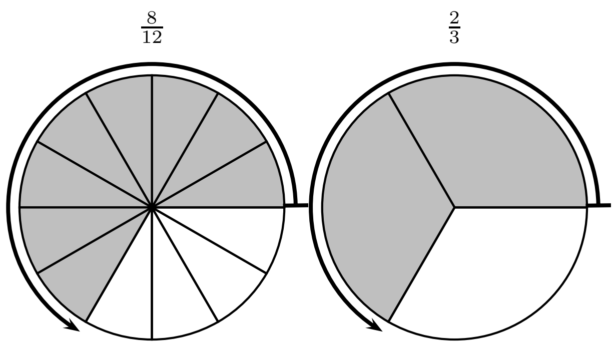


Figure 2.4: $\frac{8}{12}$ of the pizza is the same as $\frac{2}{3}$ of the pizza.

You can also add rational numbers together by finding a *lowest common denominator* and then adding the numerators. Finding a lowest common denominator means finding the lowest number that both denominators are a *factor*[6] of. A factor of a number is an integer which evenly divides that number without leaving a remainder. The following numbers all have a factor of 3

$$3, 6, 9, 12, 15, 18, 21, 24 \ldots$$

and the following all have factors of 4

$$4, 8, 12, 16, 20, 24, 28 \ldots$$

[6] Some people say *divisor* instead of factor.

The common denominators between 3 and 4 are all the numbers that appear in both of these lists, like 12 and 24. The lowest common denominator of 3 and 4 is the number that has both 3 and 4 as factors, which is 12.

For example, if we wish to add $\frac{3}{4}+\frac{2}{3}$, we first need to write both fractions so that their denominators are the same by finding the lowest common denominator, which we know is 12. We can do this by multiplying $\frac{3}{4}$ by $\frac{3}{3}$ and $\frac{2}{3}$ by $\frac{4}{4}$. $\frac{3}{3}$ and $\frac{4}{4}$ are really just complicated ways of writing 1. Multiplying a number by 1 doesn't change the number.

$$\begin{aligned}\frac{3}{4}+\frac{2}{3} &= \frac{3}{4}\times\frac{3}{3}+\frac{2}{3}\times\frac{4}{4} \\ &= \frac{3\times 3}{4\times 3}+\frac{2\times 4}{3\times 4} \\ &= \frac{9}{12}+\frac{8}{12} \\ &= \frac{9+8}{12} \\ &= \frac{17}{12}\end{aligned} \tag{2.36}$$

Dividing by a rational number is the same as multiplying by its reciprocal, as long as neither the numerator nor the denominator is zero:

$$\frac{a}{b}\div\frac{c}{d}=\frac{a}{b}.\frac{d}{c}=\frac{ad}{bc} \tag{2.37}$$

A rational number may be a *proper* or *improper* fraction.

Proper fractions have a numerator that is smaller than the denominator. For example,

$$\frac{-1}{2},\frac{3}{15},\frac{-5}{-20}$$

are proper fractions.

Improper fractions have a numerator that is larger than the denominator. For example,

$$\frac{-10}{2},\frac{13}{15},\frac{-53}{-20}$$

are improper fractions. Improper fractions can always be written as the sum of an integer and a proper fraction.

Converting Rationals into Decimal Numbers

Converting rationals into decimal numbers is very easy.

If you use a calculator, you can simply divide the numerator by the denominator.

If you do not have a calculator, then you unfortunately have to use long division.

Since long division, was first taught in primary school, it will not be discussed here. If you have trouble with long division, then please ask your friends or your teacher to explain it to you.

2.12.4 Irrational Numbers

An *irrational number* is any real number that is not a rational number. When expressed as decimals these numbers can never be fully written out as they have an infinite number of decimal places which never fall into a repeating pattern, for example $\sqrt{2}=1{,}41421356\ldots$, $\pi=3{,}14159265\ldots$. π is a Greek letter and is pronounced "pie".

?

Exercise: Real Numbers

1. Identify the number type (rational, irrational, real, integer) of each of the following numbers:
   (a) $\frac{c}{d}$ if c is an integer and if d is irrational.
   (b) $\frac{3}{2}$
   (c) -25
   (d) 1,525
   (e) $\sqrt{10}$
2. Is the following pair of numbers real and rational or real and irrational? Explain. $\sqrt{4}$;$\frac{1}{8}$

2.13 Mathematical Symbols

The following is a table of the meanings of some mathematical signs and symbols that you should have come across in earlier grades.

Sign or Symbol	Meaning
$>$	greater than
$<$	less than
$\geq$	greater than or equal to
$\leq$	less than or equal to

So if we write $x > 5$, we say that x is greater than 5 and if we write $x \geq y$, we mean that x can be greater than or equal to y. Similarly, $<$ means 'is less than' and $\leq$ means 'is less than or equal to'. Instead of saying that x is between 6 and 10, we often write $6 < 10$. This directly means 'six is less than x which in turn is less than ten'.

Exercise: Mathematical Symbols

1. Write the following in symbols:
   (a) x is greater than 1
   (b) y is less than or equal to z
   (c) a is greater than or equal to 21
   (d) p is greater than or equal to 21 and p is less than or equal to 25

2.14 Infinity

Infinity (symbol ∞) is usually thought of as something like "the largest possible number" or "the furthest possible distance". In mathematics, infinity is often treated as if it were a number, but it is clearly a very different type of "number" than the integers or reals.

When talking about recurring decimals and irrational numbers, the term *infinite* was used to describe *never-ending* digits.

2.15 End of Chapter Exercises

1. Calculate

   (a) $18 - 6 \times 2$

   (b) $10 + 3(2 + 6)$

   (c) $50 - 10(4 - 2) + 6$

   (d) $2 \times 9 - 3(6 - 1) + 1$

   (e) $8 + 24 \div 4 \times 2$

   (f) $30 - 3 \times 4 + 2$

   (g) $36 \div 4(5 - 2) + 6$

   (h) $20 - 4 \times 2 + 3$

   (i) $4 + 6(8 + 2) - 3$

   (j) $100 - 10(2 + 3) + 4$

2. If $p = q + 4r$, then $r =$ .....

3. Solve $\frac{x-2}{3} = x - 3$

Chapter 3

Rational Numbers - Grade 10

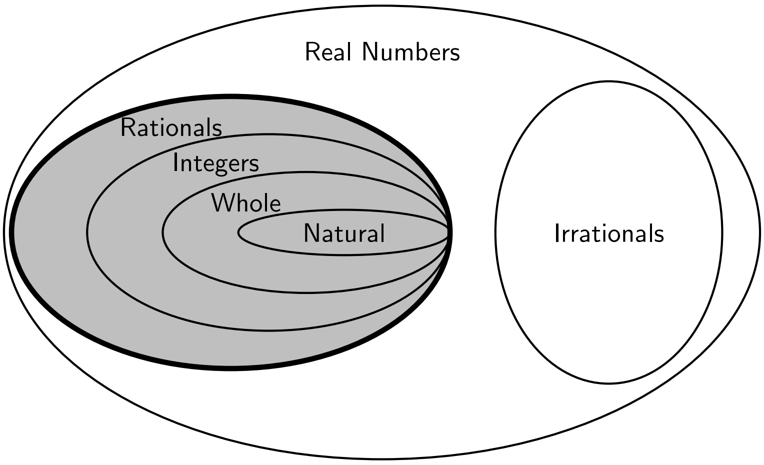
3.1 Introduction

As described in Chapter 2, a number is a way of representing quantity. The numbers that will be used in high school are all real numbers, but there are many different ways of writing any single real number.

This chapter describes *rational numbers*.

3.2 The Big Picture of Numbers

All numbers inside the grey oval are rational numbers.



3.3 Definition

The following numbers are all rational numbers.

$$\frac{10}{1}, \quad \frac{21}{7}, \quad \frac{-1}{-3}, \quad \frac{10}{20}, \quad \frac{-3}{6} \tag{3.1}$$

You can see that all the denominators and all the numerators are integers[1].

Definition: Rational Number
A rational number is any number which can be written as:

$$\frac{a}{b} \tag{3.2}$$

where a and b are integers and $b \neq 0$.

[1]Integers are the counting numbers (1, 2, 3, ...), their opposites (-1, -2, -3, ...), and 0.

Important: Only fractions which have a numerator and a denominator that are integers are rational numbers.

This means that all integers are rational numbers, because they can be written with a denominator of 1.

Therefore, while

$$\frac{\sqrt{2}}{7}, \quad \frac{-1{,}33}{-3}, \quad \frac{\pi}{20}, \quad \frac{-3}{6{,}39} \tag{3.3}$$

are **not examples** of rational numbers, because in each case, either the numerator or the denominator is not an integer.

Exercise: Rational Numbers

1. If a is an integer, b is an integer and c is not an integer, which of the following are rational numbers:

   (a) $\frac{5}{6}$ (b) $\frac{a}{3}$ (c) $\frac{b}{2}$ (d) $\frac{1}{c}$

2. If $\frac{a}{1}$ is a rational number, which of the following are valid values for a?

   (a) 1 (b) -10 (c) $\sqrt{2}$ (d) $2{,}1$

3.4 Forms of Rational Numbers

All integers and fractions with integer numerators and denominators are rational numbers. There are two more forms of rational numbers.

Activity :: Investigation : Decimal Numbers

You can write the rational number $\frac{1}{2}$ as the decimal number 0,5. Write the following numbers as decimals:

1. $\frac{1}{4}$
2. $\frac{1}{10}$
3. $\frac{2}{5}$
4. $\frac{1}{100}$
5. $\frac{2}{3}$

Do the numbers after the decimal comma end or do they continue? If they continue, is there a repeating pattern to the numbers?

You can write a rational number as a decimal number. Therefore, you should be able to write a decimal number as a rational number. Two types of decimal numbers can be written as rational numbers:

1. decimal numbers that end or *terminate*, for example the fraction $\frac{4}{10}$ can be written as 0,4.

2. decimal numbers that have a repeating pattern of numbers, for example the fraction $\frac{1}{3}$ can be written as $0{,}33333\overline{3}$.

For example, the rational number $\frac{5}{6}$ can be written in decimal notation as $0{,}8333\overline{3}$, and similarly, the decimal number 0,25 can be written as a rational number as $\frac{1}{4}$.

Important: Notation for Repeating Decimals

You can use a bar over the repeated numbers to indicate that the decimal is a repeating decimal.

3.5 Converting Terminating Decimals into Rational Numbers

A decimal number has an integer part and a fractional part. For example, $10{,}589$ has an integer part of 10 and a fractional part of $0{,}589$ because $10 + 0{,}589 = 10{,}589$. The fractional part can be written as a rational number, i.e. with a numerator and a denominator that are integers.

Each digit after the decimal point is a fraction with denominator in increasing powers of ten. For example:

- $\frac{1}{10}$ is $0{,}1$
- $\frac{1}{100}$ is $0{,}01$

This means that:

$$\begin{aligned} 2{,}103 &= 2 + \frac{1}{10} + \frac{0}{100} + \frac{3}{1000} \\ &= 2\frac{103}{1000} \\ &= \frac{2103}{1000} \end{aligned}$$

Exercise: Fractions

1. Write the following as fractions:

(a) $0{,}1$ (b) $0{,}12$ (c) $0{,}58$ (d) $0{,}2589$

3.6 Converting Repeating Decimals into Rational Numbers

When the decimal is a repeating decimal, a bit more work is needed to write the fractional part of the decimal number as a fraction. We will explain by means of an example.

If we wish to write $0{,}\overline{3}$ in the form $\frac{a}{b}$ (where a and b are integers) then we would proceed as follows

$$\begin{aligned} x &= 0{,}33333\ldots && (3.4) \\ 10x &= 3{,}33333\ldots \quad \text{multiply by 10 on both sides} && (3.5) \\ 9x &= 3 \quad \text{subtracting (3.4) from (3.5)} \\ x &= \frac{3}{9} = \frac{1}{3} \end{aligned}$$

And another example would be to write $5{,}\overline{432}$ as a rational fraction

$$\begin{aligned} x &= 5{,}432432432\ldots && (3.6)\\ 1000x &= 5432{,}432432432\ldots \quad \text{multiply by 1000 on both sides} && (3.7)\\ \\ 999x &= 5427 \quad \text{subtracting (3.6) from (3.7)}\\ x &= \frac{5427}{999} = \frac{201}{37} \end{aligned}$$

For the first example, the decimal number was multiplied by 10 and for the second example, the decimal number was multiplied by 1000. This is because for the first example there was only one number (i.e. 3) that recurred, while for the second example there were three numbers (i.e. 432) that recurred.

In general, if you have one number recurring, then multiply by 10, if you have two numbers recurring, then multiply by 100, if you have three numbers recurring, then multiply by 1000. Can you spot the pattern yet?

The number of zeros after the 1 is the same as the number of recurring numbers.

But not all decimal numbers can be written as rational numbers, because some decimal numbers like $\sqrt{2} = 1{,}4142135...$ is an irrational number and cannot be written with an integer numerator and an integer denominator. However, when possible, you should always use rational numbers or fractions instead of decimals.

Exercise: Repeated Decimal Notation

1. Write the following using the repeated decimal notation:
   (a) $0{,}11111111\ldots$
   (b) $0{,}1212121212\ldots$
   (c) $0{,}123123123123\ldots$
   (d) $0{,}11414541454145\ldots$
2. Write the following in decimal form, using the repeated decimal notation:
   (a) $\frac{2}{3}$
   (b) $1\frac{3}{11}$
   (c) $4\frac{5}{6}$
   (d) $2\frac{1}{9}\ldots$
3. Write the following decimals in fractional form:
   (a) $0{,}6333\ldots$
   (b) $5{,}3131\overline{31}$
   (c) $11{,}570571\ldots$
   (d) $0{,}999999\ldots$

3.7 Summary

The following are rational numbers:

- Fractions with both denominator and numerator as integers.
- Integers.
- Decimal numbers that end.
- Decimal numbers that repeat.

3.8 End of Chapter Exercises

1. If a is an integer, b is an integer and c is not an integer, which of the following are rational numbers:
   (a) $\frac{5}{6}$
   (b) $\frac{a}{3}$
   (c) $\frac{b}{2}$
   (d) $\frac{1}{c}$
2. Write each decimal as a simple fraction:
   (a) $0{,}5$
   (b) $0{,}12$
   (c) $0{,}6$
   (d) $1{,}59$
   (e) $12{,}27\overline{7}$
3. Show that the decimal $3{,}2\dot{1}\dot{8}$ is a rational number.
4. Showing all working, express $0{,}7\dot{8}$ as a fraction $\frac{a}{b}$ where $a, b \in Z$.

Chapter 4

Exponentials - Grade 10

4.1 Introduction

In this chapter, you will learn about the short-cuts to writing $2 \times 2 \times 2 \times 2$. This is known as writing a number in *exponential notation*.

4.2 Definition

Exponential notation is a short way of writing the same number multiplied by itself many times. For example, instead of $5 \times 5 \times 5$, we write 5^3 to show that the number 5 is multiplied by itself 3 times and we say "5 to the power of 3". Likewise 5^2 is 5×5 and 3^5 is $3 \times 3 \times 3 \times 3 \times 3$. We will now have a closer look at writing numbers using exponential notation.

Definition: Exponential Notation
Exponential notation means a number written like

$$a^n$$

when n is an integer and a can be any real number. a is called the *base* and n is called the *exponent*.

The nth power of a is defined as:

$$a^n = 1 \times a \times a \times \ldots \times a \qquad (n \text{ times}) \tag{4.1}$$

with a multiplied by itself n times.

We can also define what it means if we have a negative index, $-n$. Then,

$$a^{-n} = 1 \div a \div a \div \ldots \div a \qquad (n \text{ times}) \tag{4.2}$$

Important: Exponentials

If n is an even integer, then a^n will always be positive for any non-zero real number a. For example, although -2 is negative, $(-2)^2 = 1 \times -2 \times -2 = 4$ is positive and so is $(-2)^{-2} = 1 \div -2 \div -2 = \frac{1}{4}$.

4.3 Laws of Exponents

There are several laws we can use to make working with exponential numbers easier. Some of these laws might have been seen in earlier grades, but we will list all the laws here for easy reference, but we will explain each law in detail, so that you can understand them, and not only remember them.

$$a^0 = 1 \quad (4.3)$$
$$a^m \times a^n = a^{m+n} \quad (4.4)$$
$$a^{-n} = \frac{1}{a^n} \quad (4.5)$$
$$a^m \div a^n = a^{m-n} \quad (4.6)$$
$$(ab)^n = a^n b^n \quad (4.7)$$
$$(a^m)^n = a^{mn} \quad (4.8)$$

4.3.1 Exponential Law 1: $a^0 = 1$

Our definition of exponential notation shows that

$$a^0 = 1, (a \neq 0) \quad (4.9)$$

For example, $x^0 = 1$ and $(1\ 000\ 000)^0 = 1$.

Exercise: Application using Exponential Law 1: $a^0 = 1, (a \neq 0)$

1. $16^0 = 1$
2. $16a^0 = 16$
3. $(16 + a)^0 = 1$
4. $(-16)^0 = 1$
5. $-16^0 = -1$

4.3.2 Exponential Law 2: $a^m \times a^n = a^{m+n}$

Our definition of exponential notation shows that

$$\begin{aligned} a^m \times a^n &= 1 \times a \times \ldots \times a \quad (m \text{ times}) \\ &\quad \times 1 \times a \times \ldots \times a \quad (n \text{ times}) \\ &= 1 \times a \times \ldots \times a \quad (m+n \text{ times}) \\ &= a^{m+n} \end{aligned} \quad (4.10)$$

For example,

$$\begin{aligned} 2^7 \times 2^3 &= (2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2) \times (2 \times 2 \times 2) \\ &= 2^{10} \\ &= 2^{7+3} \end{aligned}$$

This simple law is the reason why exponentials were originally invented. In the days before calculators, all multiplication had to be done by hand with a pencil and a pad of paper. Multiplication takes a very long time to do and is very tedious. Adding numbers however, is very easy and quick to do. If you look at what this law is saying you will realise that it means that adding the exponents of two exponential numbers (of the same base) is the same as multiplying the two numbers together. This meant that for certain numbers, there was no need to actually multiply the numbers together in order to find out what their multiple was. This saved mathematicians a lot of time, which they could use to do something more productive.

Exercise: Application using Exponential Law 2: $a^m \times a^n = a^{m+n}$

1. $x^2.x^5 = x^7$
2. $2x^3y \times 5x^2y^7 = 10x^5y^8$
3. $2^3.2^4 = 2^7$ *[Take note that the base (2) stays the same.]*
4. $3 \times 3^{2a} \times 3^2 = 3^{2a+3}$

4.3.3 Exponential Law 3: $a^{-n} = \frac{1}{a^n}, a \neq 0$

Our definition of exponential notation for a negative exponent shows that

$$\begin{aligned} a^{-n} &= 1 \div a \div \ldots \div a \qquad (n \text{ times}) \\ &= \frac{1}{1 \times a \times \cdots \times a} \qquad (n \text{ times}) \\ &= \frac{1}{a^n} \end{aligned} \tag{4.11}$$

This means that a minus sign in the exponent is just another way of writing that the whole exponential number is to be divided instead of multiplied.

For example,

$$\begin{aligned} 2^{-7} &= \frac{1}{2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2} \\ &= \frac{1}{2^7} \end{aligned}$$

Exercise: Application using Exponential Law 3: $a^{-n} = \frac{1}{a^n}, a \neq 0$

1. $2^{-2} = \frac{1}{2^2} = \frac{1}{4}$
2. $\frac{2^{-2}}{3^2} = \frac{1}{2^2.3^2} = \frac{1}{36}$
3. $(\frac{2}{3})^{-3} = (\frac{3}{2})^3 = \frac{27}{8}$

4. $\frac{m}{n^{-4}} = mn^4$
5. $\frac{a^{-3}.x^4}{a^5.x^{-2}} = \frac{x^4.x^2}{a^3.a^5} = \frac{x^6}{a^8}$

4.3.4 Exponential Law 4: $a^m \div a^n = a^{m-n}$

We already realised with law 3 that a minus sign is another way of saying that the exponential number is to be divided instead of multiplied. Law 4 is just a more general way of saying the same thing. We can get this law by just multiplying law 3 by a^m on both sides and using law 2.

$$\begin{aligned} \frac{a^m}{a^n} &= a^m a^{-n} \\ &= a^{m-n} \end{aligned} \tag{4.12}$$

For example,

$$\begin{aligned} 2^7 \div 2^3 &= \frac{2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2}{2 \times 2 \times 2} \\ &= 2 \times 2 \times 2 \times 2 \\ &= 2^4 \\ &= 2^{7-3} \end{aligned}$$

?

Exercise: Exponential Law 4: $a^m \div a^n = a^{m-n}$

1. $\frac{a^6}{a^2} = a^{6-2} = a^4$
2. $\frac{3^2}{3^6} = 3^{2-6} = 3^{-4} = \frac{1}{3^4}$ *[Always give final answer with positive index]*
3. $\frac{32a^2}{4a^8} = 8a^{-6} = \frac{8}{a^6}$
4. $\frac{a^{3x}}{a^4} = a^{3x-4}$

4.3.5 Exponential Law 5: $(ab)^n = a^n b^n$

The order in which two real numbers are multiplied together does not matter. Therefore,

$$\begin{aligned} (ab)^n &= a \times b \times a \times b \times \ldots \times a \times b \quad (n \text{ times}) \\ &= a \times a \times \ldots \times a \quad (n \text{ times}) \\ &\quad \times b \times b \times \ldots \times b \quad (n \text{ times}) \\ &= a^n b^n \end{aligned} \tag{4.13}$$

For example,

$$\begin{aligned} (2 \cdot 3)^4 &= (2 \cdot 3) \times (2 \cdot 3) \times (2 \cdot 3) \times (2 \cdot 3) \\ &= (2 \times 2 \times 2 \times 2) \times (3 \times 3 \times 3 \times 3) \\ &= (2^4) \times (3^4) \\ &= 2^4 3^4 \end{aligned}$$

?

Exercise: Exponential Law 5: $(ab)^n = a^n b^n$

1. $(2x^2y)^3 = 2^3x^{2\times3}y^5 = 8x^6y^5$
2. $(\frac{7a}{b^3})^2 = \frac{49a^2}{b^6}$
3. $(5a^{n-4})^3 = 125a^{3n-12}$

4.3.6 Exponential Law 6: $(a^m)^n = a^{mn}$

We can find the exponential of an exponential just as well as we can for a number. After all, an exponential number is a real number.

$$\begin{aligned}(a^m)^n &= a^m \times a^m \times \ldots \times a^m \qquad (n \text{ times}) \\ &= a \times a \times \ldots \times a \qquad (m \times n \text{ times}) \\ &= a^{mn}\end{aligned} \tag{4.14}$$

For example,

$$\begin{aligned}(2^2)^3 &= (2^2) \times (2^2) \times (2^2) \\ &= (2 \times 2) \times (2 \times 2) \times (2 \times 2) \\ &= (2^6) \\ &= 2^{(2\times3)}\end{aligned}$$

?

Exercise: Exponential Law 6: $(a^m)^n = a^{mn}$

1. $(x^3)^4 = x^{12}$
2. $[(a^4)^3]^2 = a^{24}$
3. $(3^{n+3})^2 = 3^{2n+6}$

Worked Example 1: Simlifying indices

Question: Simplify: $\frac{5^{2x-1}.9^{x-2}}{15^{2x-3}}$
Answer
Step 1 : Factorise all bases into prime factors:

$$\begin{aligned} &= \frac{5^{2x-1}.(3^2)^{x-2}}{(5.3)^{2x-3}} \\ &= \frac{5^{2x-1}.3^{2x-4}}{5^{2x-3}.3^{2x-3}}\end{aligned}$$

Step 2 : Add and subtract the indices of the sam bases as per laws 2 and 4:

$$= 5^{2x-1-2x-3}.3^{2x-4-2x+3}$$
$$= 5^2.3^{-1}$$

Step 3 : Write simplified answer with positive indices:

$$= \frac{25}{3}$$

Activity :: Investigation : Exponential Numbers

Match the answers to the questions, by filling in the correct answer into the **Answer** column. Possible answers are: $\frac{3}{2}$, 1, -1, -3, 8.

Question	Answer
2^3	
7^{3-3}	
$(\frac{2}{3})^{-1}$	
8^{7-6}	
$(-3)^{-1}$	
$(-1)^{23}$	

4.4 End of Chapter Exercises

1. Simplify as far as possible:

   (a) 302^0 (b) 1^0 (c) $(xyz)^0$ (d) $[(3x^4y^7z^{12})^5(-5x^9y^3z^4)^2]^0$
   (e) $(2x)^3$ (f) $(-2x)^3$ (g) $(2x)^4$ (h) $(-2x)^4$

2. Simplify without using a calculator. Leave your answers with positive exponents.

   (a) $\frac{3x^{-3}}{(3x)^2}$

   (b) $5x^0 + 8^{-2} - (\frac{1}{2})^{-2}.1^x$

   (c) $\frac{5^{b-3}}{5^{b+1}}$

3. Simplify, showing all steps:

   (a) $\frac{2^{a-2}.3^{a+3}}{6^a}$

   (b) $\frac{a^{2m+n+p}}{a^{m+n+p}.a^m}$

   (c) $\frac{3^n.9^{n-3}}{27^{n-1}}$

   (d) $(\frac{2x^{2a}}{y^{-b}})^3$

(e) $\frac{2^{3x-1}.8^{x+1}}{4^{2x-2}}$

(f) $\frac{6^{2x}.11^{2x}}{22^{2x-1}.3^{2x}}$

4. Simplify, without using a calculator:

(a) $\frac{(-3)^{-3}.(-3)^{2}}{(-3)^{-4}}$.

(b) $(3^{-1}+2^{-1})^{-1}$

(c) $\frac{9^{n-1}.27^{3-2n}}{81^{2-n}}$

(d) $\frac{2^{3n+2}.8^{n-3}}{4^{3n-2}}$

Chapter 5

Estimating Surds - Grade 10

5.1 Introduction

You should known by now what the nth root of a number means. If the nth root of a number cannot be simplified to a rational number, we call it a *surd*. For example, $\sqrt{2}$ and $\sqrt[3]{6}$ are surds, but $\sqrt{4}$ is not a surd because it can be simplified to the rational number 2.

In this chapter we will only look at surds that look like $\sqrt[n]{a}$, where a is any positive number, for example $\sqrt{7}$ or $\sqrt[3]{5}$. It is very common for n to be 2, so we usually do not write $\sqrt[2]{a}$. Instead we write the surd as just $\sqrt{a}$, which is much easier to read.

It is sometimes useful to know the approximate value of a surd without having to use a calculator. For example, we want to be able to guess where a surd like $\sqrt{3}$ is on the number line. So how do we know where surds lie on the number line? From a calculator we know that $\sqrt{3}$ is equal to $1{,}73205...$. It is easy to see that $\sqrt{3}$ is above 1 and below 2. But to see this for other surds like $\sqrt{18}$ without using a calculator, you must first understand the following fact:

If a and b are positive whole numbers, and $a < b$, then $\sqrt[n]{a} < \sqrt[n]{b}$. (Challenge: Can you explain why?)

If you don't believe this fact, check it for a few numbers to convince yourself it is true.

How do we use this fact to help us guess what $\sqrt{18}$ is? Well, you can easily see that $18 < 25$? Using our rule, we also know that $\sqrt{18} < \sqrt{25}$. But we know that $5^2 = 25$ so that $\sqrt{25} = 5$. Now it is easy to simplify to get $\sqrt{18} < 5$. Now we have a better idea of what $\sqrt{18}$ is.

Now we know that $\sqrt{18}$ is less than 5, but this is only half the story. We can use the same trick again, but this time with 18 on the right-hand side. You will agree that $16 < 18$. Using our rule again, we also know that $\sqrt{16} < \sqrt{18}$. But we know that 16 is a perfect square, so we can simplify $\sqrt{16}$ to 4, and so we get $4 < \sqrt{18}$!

Can you see now that we now have shown that $\sqrt{18}$ is between 4 and 5? If we check on our calculator, we can see that $\sqrt{18} = 4{,}1231...$, and we see that our idea was right! You will notice that our idea used perfect squares that were close to the number 18. We found the closest perfect square underneath 18, which was $4^2 = 16$, and the closest perfect square above 18, which was $5^2 = 25$. Here is a quick summary of what a perfect square or cube is:

A perfect square is the number obtained when an integer is squared. For example, 9 is a perfect square since $3^2 = 9$. Similarly, a perfect cube is a number which is the cube of an integer. For example, 27 is a perfect cube, because $3^3 = 27$.

To make it easier to use our idea, we will create a list of some of the perfect squares and perfect cubes. The list is shown in Table 5.1.

Table 5.1: Some perfect squares and perfect cubes

Integer	Perfect Square	Perfect Cube
0	0	0
1	1	1
2	4	8
3	9	27
4	16	64
5	25	125
6	36	216
7	49	343
8	64	512
9	81	729
10	100	1000

Similarly, when given the surd $\sqrt[3]{52}$ you should be able to tell that it lies somewhere between 3 and 4, because $\sqrt[3]{27} = 3$ and $\sqrt[3]{64} = 4$ and 52 is between 27 and 64. In fact $\sqrt[3]{52} = 3{,}73\ldots$ which is indeed between 3 and 4.

5.2 Drawing Surds on the Number Line (Optional)

How can we accurately draw a surd like $\sqrt{5}$ on the number line? Well, we *could* use a calculator to find $\sqrt{5} = 2{,}2360...$ and measure the distance along the number line using a ruler. But for some surds, there is a much easier way.

Let us call the surd we are working with $\sqrt{a}$. Sometimes, we can write a as the sum of two perfect squares, so $a = b^2 + c^2$. We know from Pythagoras' theorem that $\sqrt{a} = \sqrt{b^2 + c^2}$ is the length of the hypotenuse of a triangle that has sides that have lengths of b and c. So if we draw a triangle on the number line with sides of length b and c, we can use a compass to draw a circle from the top of the hypotenuse down to the number line. The intersection will be the point $\sqrt{b}$ on the number line!

Interesting Fact

Not all numbers can be written as the sum of two squares. See if you can find a pattern of the numbers that can.

Worked Example 2: Estimating Surds

Question: Find the two consecutive integers such that $\sqrt{26}$ lies between them. (Remember that consecutive numbers that are two numbers one after the other, like 5 and 6 or 8 and 9.)

Answer

Step 1 : From the table find the largest perfect square below 26

This is $5^2 = 25$. Therefore $5 < \sqrt{26}$.

Step 2 : From the table find smallest perfect square above 26
This is $6^2 = 36$. Therefore $\sqrt{26} < 6$.
Step 3 : Put the inequalities together
Our answer is $5 < \sqrt{26} < 6$.

Worked Example 3: Estimating Surds

Question: $\sqrt[3]{49}$ lies between: (a) 1 and 2 (b) 2 and 3 (c) 3 and 4 (d) 4 and 5
Answer
Step 1 : Consider (a) as the solution
If $1 < \sqrt[3]{49} < 2$ then cubing all terms gives $1 < 49 < 2^3$. Simplifying gives $1 < 49 < 8$ which is false. So $\sqrt[3]{49}$ does not lie between 1 and 2.
Step 2 : Consider (b) as the solution
If $2 < \sqrt[3]{49} < 3$ then cubing all terms gives $2^3 < 49 < 3^3$. Simplifying gives $8 < 49 < 27$ which is false. So $\sqrt[3]{49}$ does not lie between 2 and 3.
Step 3 : Consider (c) as the solution
If $3 < \sqrt[3]{49} < 4$ then cubing all terms gives $3^3 < 49 < 4^3$. Simplifying gives $27 < 49 < 64$ which is true. So $\sqrt[3]{49}$ lies between 3 and 4.

5.3 End of Chapter Excercises

1.	$\sqrt{5}$ lies between	(a) 1 and 2	(b) 2 and 3	(c) 3 and 4	(d) 4 and 5
2.	$\sqrt{10}$ lies between	(a) 1 and 2	(b) 2 and 3	(c) 3 and 4	(d) 4 and 5
3.	$\sqrt{20}$ lies between	(a) 2 and 3	(b) 3 and 4	(c) 4 and 5	(d) 5 and 6
4.	$\sqrt{30}$ lies between	(a) 3 and 4	(b) 4 and 5	(c) 5 and 6	(d) 6 and 7
5.	$\sqrt[3]{5}$ lies between	(a) 1 and 2	(b) 2 and 3	(c) 3 and 4	(d) 4 and 5
6.	$\sqrt[3]{10}$ lies between	(a) 1 and 2	(b) 2 and 3	(c) 3 and 4	(d) 4 and 5
7.	$\sqrt[3]{20}$ lies between	(a) 2 and 3	(b) 3 and 4	(c) 4 and 5	(d) 5 and 6
8.	$\sqrt[3]{30}$ lies between	(a) 3 and 4	(b) 4 and 5	(c) 5 and 6	(d) 6 and 7

Chapter 6

Irrational Numbers and Rounding Off - Grade 10

6.1 Introduction

You have seen that repeating decimals may take a lot of paper and ink to write out. Not only is that impossible, but writing numbers out to many decimal places or *a high accuracy* is very inconvenient and rarely gives better answers. For this reason we often estimate the number to a certain number of decimal places or to a given number of *significant figures*, which is even better.

6.2 Irrational Numbers

Activity :: Investigation : Irrational Numbers
Which of the following cannot be written as a rational number?
Remember: A rational number is a fraction with numerator and denominator as integers. Terminating decimal numbers or repeating decimal numbers are rational.

1. $\pi = 3{,}14159265358979323846264338327950288419716939937510\ldots$
2. 1,4
3. $1{,}618\,033\,989\ldots$
4. 100

Irrational numbers are numbers that cannot be written as a rational number. You should know that a rational number can be written as a fraction with the numerator and denominator as integers. This means that any number that is *not* a terminating decimal number or a repeating decimal number are irrational. Examples of irrational numbers are:

$$\sqrt{2}, \quad \sqrt{3}, \quad \sqrt[3]{4}, \quad \pi,$$
$$\frac{1+\sqrt{5}}{2} \approx 1{,}618\,033\,989$$

Important: When irrational numbers are written in decimal form, they go on forever and there is no repeated pattern of digits.

Important: Irrational Numbers

If you are asked to identify whether a number is rational or irrational, first write the number in decimal form. If the number is terminated then it is rational. If it goes on forever, then look for a repeated pattern of digits. If there is no repeated pattern, then the number is irrational.

When you write irrational numbers in decimal form, you may (if you have a lot of time and paper!) continue writing them for many, many decimal places. However, this is not convenient and it is often necessary to round off.

6.3 Rounding Off

Rounding off or approximating a decimal number to a given number of decimal places is the quickest way to approximate a number. For example, if you wanted to round-off $2{,}6525272$ to three decimal places then you would first count three places after the decimal.

$$2{,}652|5272$$

All numbers to the right of | are ignored after you determine whether the number in the third decimal place must be rounded up or rounded down. You *round up* the final digit if the first digit after the | was greater or equal to 5 and *round down* (leave the digit alone) otherwise.

So, since the first digit after the | is a 5, we must round up the digit in the third decimal place to a 3 and the final answer of $2{,}6525272$ rounded to three decimal places is

$$2{,}653$$

Worked Example 4: Rounding-Off

Question: Round-off the following numbers to the indicated number of decimal places:

1. $\frac{120}{99} = 1{,}21212121\dot{1}\dot{2}$ to 3 decimal places
2. $\pi = 3{,}141592654\ldots$ to 4 decimal places
3. $\sqrt{3} = 1{,}7320508\ldots$ to 4 decimal places

Answer

Step 1 : Determine the last digit that is kept and mark the cut-off point with |.

1. $\frac{120}{99} = 1{,}212|1212121\dot{1}\dot{2}$
2. $\pi = 3{,}1415|92654\ldots$
3. $\sqrt{3} = 1{,}7320|508\ldots$

Step 2 : Determine whether the last digit is rounded up or down.

1. The last digit of $\frac{120}{99} = 1{,}212|1212121\dot{1}\dot{2}$ must be rounded-down.
2. The last digit of $\pi = 3{,}1415|92654\ldots$ must be rounded-up.
3. The last digit of $\sqrt{3} = 1{,}7320|508\ldots$ must be rounded-up.

Step 3 : Write the final answer.

1. $\frac{120}{99} = 1{,}212$ rounded to 3 decimal places
2. $\pi = 3{,}1416$ rounded to 4 decimal places
3. $\sqrt{3} = 1{,}7321$ rounded to 4 decimal places

6.4 End of Chapter Exercises

1. Write the following rational numbers to 2 decimal places:
   (a) $\frac{1}{2}$
   (b) 1
   (c) $0{,}11111\overline{1}$
   (d) $0{,}99999\overline{1}$
2. Write the following irrational numbers to 2 decimal places:
   (a) $3{,}141592654\ldots$
   (b) $1{,}618\,033\,989\ldots$
   (c) $1{,}41421356\ldots$
   (d) $2{,}71828182845904523536\ldots$
3. Use your calculator and write the following irrational numbers to 3 decimal places:
   (a) $\sqrt{2}$
   (b) $\sqrt{3}$
   (c) $\sqrt{5}$
   (d) $\sqrt{6}$
4. Use your calculator (where necessary) and write the following irrational numbers to 5 decimal places:
   (a) $\sqrt{8}$
   (b) $\sqrt{768}$
   (c) $\sqrt{100}$
   (d) $\sqrt{0{,}49}$
   (e) $\sqrt{0{,}0016}$
   (f) $\sqrt{0{,}25}$
   (g) $\sqrt{36}$
   (h) $\sqrt{1960}$
   (i) $\sqrt{0{,}0036}$
   (j) $-8\sqrt{0{,}04}$
   (k) $5\sqrt{80}$
5. Write the following irrational numbers to 3 decimal places and then write them as a rational number to get an approximation to the irrational number. For example, $\sqrt{3} = 1{,}73205\ldots$. To 3 decimal places, $\sqrt{3} = 1{,}732$. $1{,}732 = 1\frac{732}{1000} = 1\frac{183}{250}$. Therefore, $\sqrt{3}$ is approximately $1\frac{183}{250}$.
   (a) $3{,}141592654\ldots$
   (b) $1{,}618\,033\,989\ldots$
   (c) $1{,}41421356\ldots$
   (d) $2{,}71828182845904523536\ldots$

Chapter 7

Number Patterns - Grade 10

In earlier grades you saw patterns in the form of pictures and numbers. In this chapter we learn more about the mathematics of patterns.Patterns are recognisable regularities in situations such as in nature, shapes, events, sets of numbers. For example, spirals on a pineapple, snowflakes, geometric designs on quilts or tiles, the number sequence 0, 4, 8, 12, 16,....

Activity :: Investigation : Patterns
Can you spot any patterns in the following lists of numbers?

1. 2; 4; 6; 8; 10; ...
2. 1; 2; 4; 7; 11; ...
3. 1; 4; 9; 16; 25; ...
4. 5; 10; 20; 40; 80; ...

7.1 Common Number Patterns

Numbers can have interesting patterns. Here we list the most common patterns and how they are made.

Examples:

1. $1, 4, 7, 10, 13, 16, 19, 22, 25, ...$

   This sequence has a difference of 3 between each number. The pattern is continued by adding 3 to the last number each time.

2. $3, 8, 13, 18, 23, 28, 33, 38, ...$

   This sequence has a difference of 5 between each number. The pattern is continued by adding 5 to the last number each time.

3. $2, 4, 8, 16, 32, 64, 128, 256, ...$

   This sequence has a factor of 2 between each number. The pattern is continued by multiplying the last number by 2 each time.

4. $3, 9, 27, 81, 243, 729, 2187, ...$

   This sequence has a factor of 3 between each number. The pattern is continued by multiplying the last number by 3 each time.

7.1.1 Special Sequences

Triangular Numbers

$1, 3, 6, 10, 15, 21, 28, 36, 45, ...$

This sequence is generated from a pattern of dots which form a triangle. By adding another row of dots and counting all the dots we can find the next number of the sequence.

Square Numbers

$1, 4, 9, 16, 25, 36, 49, 64, 81, ...$

The next number is made by squaring where it is in the pattern. The second number is 2 squared (2^2 *or* 2×2) The seventh number is 7 squared (7^2 *or* 7×7) etc

Cube Numbers

$1, 8, 27, 64, 125, 216, 343, 512, 729, ...$

The next number is made by cubing where it is in the pattern. The second number is 2 cubed (2^3 *or* $2 \times 2 \times 2$) The seventh number is 7 cubed (7^3 *or* $7 \times 7 \times 7$) etc

Fibonacci Numbers

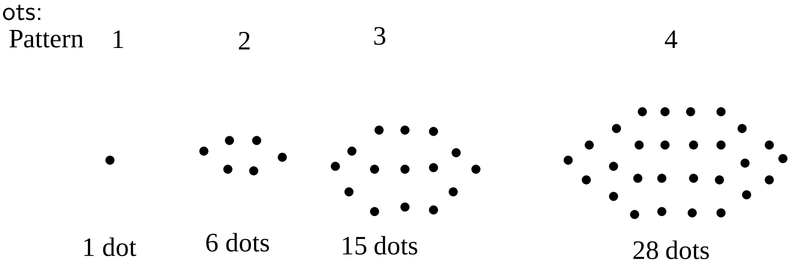
$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, ...$

The next number is found by adding the two numbers before it together. The 2 is found by adding the two numbers in front of it $(1 + 1)$ The 21 is found by adding the two numbers in front of it $(8 + 13)$ The next number in the sequence above would be 55 $(21 + 34)$

Can you figure out the next few numbers?

7.2 Make your own Number Patterns

You can make your own number patterns using coins or matchsticks. Here is an example using dots:



How many dots would you need for pattern 5 ? Can you make a formula that will tell you how many coins are needed for any size pattern? For example if the pattern 20? The formula may look something like

$$dots = pattern \times pattern + ...$$

Worked Example 5: Study Table

Question: Say you and 3 friends decide to study for Maths, and you are seated at a square table. A few minutes later, 2 other friends join you and would like to sit at your table and help you study. Naturally, you move another table and add it to the existing one. Now six of you sit at the table. Another two of your friends join your table, and you take a third table and add it to the existing tables. Now 8 of you can sit comfortably.

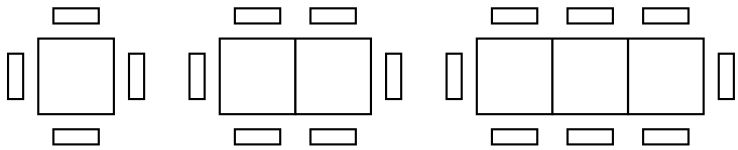


Figure 7.1: Two more people can be seated for each table added.

Examine how the number of people sitting is related to the number of tables.

Answer

Step 1 : Tabulate a few terms to see if there is a pattern

Number of Tables, n	**Number of people seated**
1	$4 = 4$
2	$4 + 2 = 6$
3	$4 + 2 + 2 = 8$
4	$4 + 2 + 2 + 2 = 10$
$\vdots$	$\vdots$
n	$4 + 2 + 2 + 2 + \ldots + 2$

Step 2 : Describe the pattern

We can see for 3 tables we can seat 8 people, for 4 tables we can seat 10 people and so on. We started out with 4 people and added two the whole time. Thus, for each table added, the number of persons increases by two.

7.3 Notation

A sequence does not have to follow a pattern but when it does we can often write down a formula to calculate the n^{th}-term, a_n. In the sequence

$$1; 4; 9; 16; 25; \ldots$$

where the sequence consists of the squares of integers, the formula for the n^{th}-term is

$$a_n = n^2 \tag{7.1}$$

You can check this by looking at:

$$\begin{aligned} a_1 &= 1^2 = 1 \\ a_2 &= 2^2 = 4 \\ a_3 &= 3^2 = 9 \\ a_4 &= 4^2 = 16 \\ a_5 &= 5^2 = 25 \\ &\ldots \end{aligned}$$

Therefore, using (7.1), we can generate a pattern, namely squares of integers.

Worked Example 6: Study Table continued ....

Question: As before, you and 3 friends are studying for Maths, and you are seated at a square table. A few minutes later, 2 other friends join you move another table and add it to the existing one. Now six of you sit at the table. Another two of your friends join your table, and you take a third table and add it to the existing tables. Now 8 of you sit comfortably as illustrated:

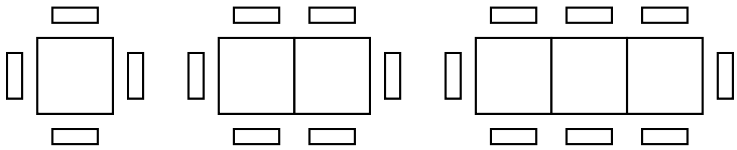


Figure 7.2: Two more people can be seated for each table added.

Find the expression for the number of people seated at n tables. Then, use the general formula to determine how many people can sit around 12 tables and how many tables are needed for 20 people.

Answer

Step 1 : Tabulate a few terms to see if there is a pattern

Number of Tables, n	Number of people seated	Formula
1	$4 = 4$	$= 4 + 2 \cdot (0)$
2	$4 + 2 = 6$	$= 4 + 2 \cdot (1)$
3	$4 + 2 + 2 = 8$	$= 4 + 2 \cdot (2)$
4	$4 + 2 + 2 + 2 = 10$	$= 4 + 2 \cdot (3)$
$\vdots$	$\vdots$	$\vdots$
n	$4 + 2 + 2 + 2 + \ldots + 2$	$= 4 + 2 \cdot (n - 1)$

Step 2 : Describe the pattern

The number of people seated at n tables is:

$$a_n = 4 + 2 \cdot (n - 1)$$

Step 3 : Calculate the 12^{th} term

Using the general formula (36.1) and considering the example from the previous section, how many people can sit around, say, 12 tables? We are looking for a_{12}, that is, where $n = 12$:

$$\begin{aligned} a_n &= a_1 + d \cdot (n - 1) \\ a_{12} &= 4 + 2 \cdot (12 - 1) \\ &= 4 + 2(11) \\ &= 4 + 22 \\ &= 26 \end{aligned}$$

Step 4 : Calculate the number of terms if $a_n = 20$

$$\begin{aligned} a_n &= a_1 + d \cdot (n-1) \\ 20 &= 4 + 2 \cdot (n-1) \\ 20 - 4 &= 2 \cdot (n-1) \\ 16 \div 2 &= n - 1 \\ 8 + 1 &= n \\ n &= 9 \end{aligned}$$

Step 5 : Final Answer

26 people can be seated at 12 tables and 9 tables are needed to seat 20 people.

It is also important to note the difference between n and a_n. n can be compared to a place holder, while a_n is the value at the place "held" by n. Like our "Study Table"-example above, the first table (Table 1) holds 4 people. Thus, at place $n = 1$, the value of $a_1 = 4$, and so on:

n	1	2	3	4	...
a_n	4	6	8	10	...

Activity :: Investigation : General Formula

1. Find the general formula for the following sequences and then find a_{10}, a_{50} and a_{100}:
   (a) $2, 5, 8, 11, 14, \ldots$
   (b) $0, 4, 8, 12, 16, \ldots$
   (c) $2, -1, -4, -7, -10, \ldots$
2. The general term has been given for each sequence below. Work out the missing terms.
   (a) $0; 3; ...; 15; 24$ $\quad n^2 - 1$
   (b) $3; 2; 1; 0; ...; 2$ $\quad -n + 4$
   (c) $11; ...; 7; ...; 3$ $\quad -13 + 2n$

7.3.1 Patterns and Conjecture

In mathematics, a conjecture is a mathematical statement which appears to be true, but has not been formally proven to be true under the rules of mathematics. Other words that have a similar in meaning to conjecture are: hypothesis, theory, assumption and premise.
For example: Make a **conjecture** about the next number based on the pattern $2; 6; 11; 17 : ...$
The numbers increase by 4, 5, and 6.
Conjecture: The next number will increase by 7. So, it will be $17 + 7$ or 24.

Worked Example 7: Number patterns

Question: Consider the following pattern.

$$\begin{aligned} 1^2 + 1 &= 2^2 - 2 \\ 2^2 + 2 &= 3^2 - 3 \\ 3^2 + 3 &= 4^2 - 4 \\ 4^2 + 4 &= 5^2 - 5 \end{aligned}$$

1. Add another two rows to the end of the pattern.
2. Make a conjecture about this pattern. Write your conjecture in words.
3. Generalise your conjecture for this pattern (in other words, write your conjecture algebraically).
4. Prove that your conjecture is true.

Answer
Step 1 : The next two rows

$$5^2 + 5 = 6^2 - 6$$
$$6^2 + 6 = 7^2 - 7$$

Step 2 : Conjecture
Squaring a number and adding the same number gives the same result as squaring the next number and subtracting that number.
Step 3 : Generalise
We have chosen to use x here. You could choose any letter to generalise the pattern.

$$x^2 + x = (x+1)^2 - (x+1)$$

Step 4 : Proof

$$Left\ side: \ x^2 + x$$

$$Right\ side: \ (x+1)^2 - (x+1)$$

$$\begin{aligned} Right\ side &= x^2 + 2x + 1 - x - 1 \\ &= x^2 + x \\ &= left\ side \\ Therefore\ x^2 + x &= (x+1)^2 - (x+1) \end{aligned}$$

7.4 Exercises

1. Find the n^{th} term for: $3, 7, 11, 15, \ldots$
2. Find the general term of the following sequences:

(a) $-2{,}1{,}4{,}7,\ldots$

(b) $11, 15, 19, 23, \ldots$

(c) $x - 1{,}2x + 5{,}5x + 1, \ldots$

(d) sequence with $a_3 = 7$ and $a_8 = 15$

(e) sequence with $a_4 = -8$ and $a_{10} = 10$

3. The seating in a section of a sports stadium can be arranged so the first row has 15 seats, the second row has 19 seats, the third row has 23 seats and so on. Calculate how many seats are in the row 25.

4. Consider the following pattern:

$$\begin{aligned} 2^2 + 2 &= 3^2 - 3 \\ 3^2 + 3 &= 4^2 - 4 \\ 4^2 + 4 &= 5^2 - 5 \end{aligned}$$

(a) Add at least two more rows to the pattern and check whether or not the pattern continues to work.

(b) Describe in words any patterns that you have noticed.

(c) Try to generalise a rule using algebra i.e. find the general term for the pattern.

(d) Prove or disprove that this rule works for all values.

5. The profits of a small company for the last four years has been: R10 000, R15 000, R19 000 and R23 000. If the pattern continues, what is the expected profit in the 10 years (i.e. in the 14^{th} year of the company being in business)?

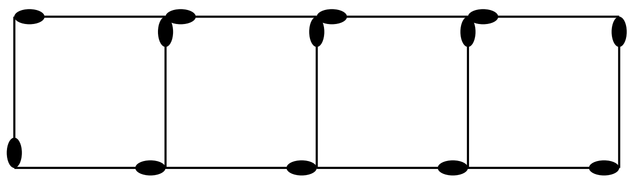
6. A single square is made from 4 matchsticks. Two squares in a row needs 7 matchsticks and 3 squares in a row needs 10 matchsticks. Determine:

(a) the first term

(b) the common difference

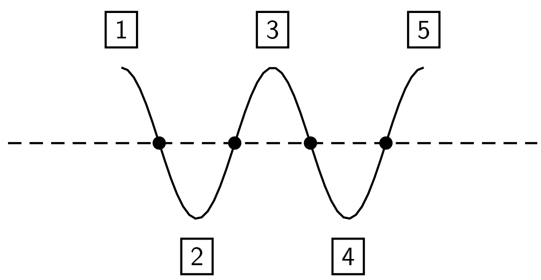
(c) the formula for the general term

(d) how many matchsticks are in a row of 25 squares



7. You would like to start saving some money, but because you have never tried to save money before, you have decided to start slowly. At the end of the first week you deposit R5 into your bank account. Then at the end of the second week you deposit R10 into your bank account. At the end of the third week you deposit R15. After how many weeks, do you deposit R50 into your bank account?

8. A horizontal line intersects a piece of string at four points and divides it into five parts, as shown below.



If the piece of string is intersected in this way by 19 parallel lines, each of which intersects it at four points, find the number of parts into which the string will be divided.

Chapter 8

Finance - Grade 10

8.1 Introduction

Should you ever find yourself stuck with a mathematics question on a television quiz show, you will probably wish you had remembered the how many even prime numbers there are between 1 and 100 for the sake of R1 000 000. And who does not want to be a millionaire, right?

Welcome to the Grade 10 Finance Chapter, where we apply maths skills to everyday financial situations that you are likely to face both now and along your journey to purchasing your first private jet.

If you master the techniques in this chapter, you will grasp the concept of *compound interest*, and how it can ruin your fortunes if you have credit card debt, or make you millions if you successfully invest your hard-earned money. You will also understand the effects of fluctuating exchange rates, and its impact on your spending power during your overseas holidays!

8.2 Foreign Exchange Rates

Is \$500 ("500 US dollars") per person per night a good deal on a hotel in New York City? The first question you will ask is "How much is that worth in Rands?". A quick call to the local bank or a search on the Internet (for example on `http://www.x-rates.com/`) for the Dollar/Rand exchange rate will give you a basis for assessing the price.

A foreign exchange rate is nothing more than the price of one currency in terms of another. For example, the exchange rate of 6,18 Rands/US Dollars means that \$1 costs R6,18. In other words, if you have \$1 you could sell it for R6,18 - or if you wanted \$1 you would have to pay R6,18 for it.

But what drives exchange rates, and what causes exchange rates to change? And how does this affect you anyway? This section looks at answering these questions.

8.2.1 How much is R1 really worth?

We can quote the price of a currency in terms of any other currency, but the US Dollar, British Pounds Sterling or even the Euro are often used as a market standard. You will notice that the financial news will report the South African Rand exchange rate in terms of these three major currencies.

So the South African Rand could be quoted on a certain date as 6,7040 ZAR per USD (i.e. \$1,00 costs R6,7040), or 12,2374 ZAR per GBP. So if I wanted to spend \$1 000 on a holiday in the United States of America, this would cost me R6 704,00; and if I wanted £1 000 for a weekend in London it would cost me R12 237,40.

This seems obvious, but let us see how we calculated that: The rate is given as ZAR per USD, or ZAR/USD such that \$1,00 buys R6,7040. Therefore, we need to multiply by 1 000 to get the

Table 8.1: Abbreviations and symbols for some common currencies.

Currency	Abbreviation	Symbol
South African Rand	ZAR	R
United States Dollar	USD	$
British Pounds Sterling	GBP	£
Euro	EUR	€

number of Rands per $1 000.

Mathematically,

$$\begin{aligned} \$1{,}00 &= \text{R}6{,}0740 \\ \therefore \quad 1\ 000 \times \$1{,}00 &= 1\ 000 \times \text{R}6{,}0740 \\ &= \text{R}6\ 074{,}00 \end{aligned}$$

as expected.

What if you have saved R10 000 for spending money for the same trip and you wanted to use this to buy USD? How much USD could you get for this? Our rate is in ZAR/USD but we want to know how many USD we can get for our ZAR. This is easy. We know how much $1,00 costs in terms of Rands.

$$\begin{aligned} \$1{,}00 &= \text{R}6{,}0740 \\ \therefore \quad \frac{\$1{,}00}{6{,}0740} &= \frac{\text{R}6{,}0740}{6{,}0740} \\ \$\frac{1{,}00}{6{,}0740} &= \text{R}1{,}00 \\ \text{R}1{,}00 &= \$\frac{1{,}00}{6{,}0740} \\ &= \$0{,}164636 \end{aligned}$$

As we can see, the final answer is simply the reciprocal of the ZAR/USD rate. Therefore, R10 000 will get:

$$\begin{aligned} \text{R}1{,}00 &= \$\frac{1{,}00}{6{,}0740} \\ \therefore \quad 10\ 000 \times \text{R}1{,}00 &= 10\ 000 \times \$\frac{1{,}00}{6{,}0740} \\ &= \$1\ 646{,}36 \end{aligned}$$

We can check the answer as follows:

$$\begin{aligned} \$1{,}00 &= \text{R}6{,}0740 \\ \therefore \quad 1\ 646{,}36 \times \$1{,}00 &= 1\ 646{,}36 \times \text{R}6{,}0740 \\ &= \text{R}10\ 000{,}00 \end{aligned}$$

Six of one and half a dozen of the other

So we have two different ways of expressing the same exchange rate: Rands per Dollar (ZAR/USD) and Dollar per Rands (USD/ZAR). Both exchange rates mean the same thing and express the value of one currency in terms of another. You can easily work out one from the other - they are just the reciprocals of the other.

If the South African Rand is our Domestic (or home) Currency, we call the ZAR/USD rate a "direct" rate, and we call a USD/ZAR rate an "indirect" rate.

In general, a direct rate is an exchange rate that is expressed as units of Home Currency per units of Foreign Currency, i.e., Domestic Currency / Foreign Currency.

The Rand exchange rates that we see on the news are usually expressed as Direct Rates, for example you might see:

Table 8.2: Examples of exchange rates

Currency Abbreviation	Exchange Rates
1 USD	R6,9556
1 GBP	R13,6628
1 EUR	R9,1954

The exchange rate is just the price of each of the Foreign Currencies (USD, GBP and EUR) in terms of our Domestic Currency, Rands.

An indirect rate is an exchange rate expressed as units of Foreign Currency per units of Home Currency, i.e. Foreign Currency / Domestic Currency

Defining exchange rates as direct or indirect depends on which currency is defined as the Domestic Currency. The Domestic Currency for an American investor would be USD which is the South African investor's Foreign Currency. So direct rates from the perspective of the American investor (USD/ZAR) would be the same as the indirect rate from the perspective of the South Africa investor.

Terminology

Since exchange rates are simple prices of currencies, movements in exchange rates means that the price or value of the currency changed. The price of petrol changes all the time, so does the price of gold, and currency prices also move up and down all the time.

If the Rand exchange rate moved from say R6,71 per USD to R6,50 per USD, what does this mean? Well, it means that $1 would now cost only R6,50 instead of R6,71. The Dollar is now cheaper to buy, and we say that the Dollar has depreciated (or weakened) against the Rand. Alternatively we could say that the Rand has appreciated (or strengthened) against the Dollar.

What if we were looking at indirect exchange rates, and the exchange rate moved from $0,149 per ZAR ($=\frac{1}{6,71}$) to $0,1538 per ZAR ($=\frac{1}{6,50}$).

Well now we can see that the R1,00 cost $0,149 at the start, and then cost $0,1538 at the end. The Rand has become more expensive (in terms of Dollars), and again we can say that the Rand has appreciated.

Regardless of which exchange rate is used, we still come to the same conclusions.

In general,

- for direct exchange rates, the home currency will appreciate (depreciate) if the exchange rate falls (rises)
- For indirect exchange rates, the home currency will appreciate (depreciate) if the exchange rate rises (falls)

As with just about everything in this chapter, do not get caught up in memorising these formulae - that is only going to get confusing. Think about what you have and what you want - and it should be quite clear how to get the correct answer.

Activity :: Discussion : Foreign Exchange Rates
In groups of 5, discuss:

1. Why might we need to know exchange rates?
2. What happens if one countries currency falls drastically vs another countries currency?

3. When might you use exchange rates?

8.2.2 Cross Currency Exchange Rates

We know that the exchange rates are the value of one currency expressed in terms of another currency, and we can quote exchange rates against any other currency. The Rand exchange rates we see on the news are usually expressed against the major currency, USD, GBP and EUR.

So if for example, the Rand exchange rates were given as 6,71 ZAR/USD and 12,71 ZAR/GBP, does this tell us anything about the exchange rate between USD and GBP?

Well I know that if \$1 will buy me R6,71, and if £1.00 will buy me R12,71, then surely the GBP is stronger than the USD because you will get more Rands for one unit of the currency, and we can work out the USD/GBP exchange rate as follows:

Before we plug in any numbers, how can we get a USD/GBP exchange rate from the ZAR/USD and ZAR/GBP exchange rates?

Well,

$$\text{USD/GBP} = \text{USD/ZAR} \times \text{ZAR/GBP}.$$

Note that the ZAR in the numerator will cancel out with the ZAR in the denominator, and we are left with the USD/GBP exchange rate.

Although we do not have the USD/ZAR exchange rate, we know that this is just the reciprocal of the ZAR/USD exchange rate.

$$\text{USD/ZAR} = \frac{1}{\text{ZAR/USD}}$$

Now plugging in the numbers, we get:

$$\begin{aligned} \text{USD/GBP} &= \text{USD/ZAR} \times \text{ZAR/GBP} \\ &= \frac{1}{\text{ZAR/USD}} \times \text{ZAR/GBP} \\ &= \frac{1}{6{,}71} \times 12{,}71 \\ &= 1{,}894 \end{aligned}$$

Important: Sometimes you will see exchange rates in the real world that do not appear to work exactly like this. This is usually because some financial institutions add other costs to the exchange rates, which alter the results. However, if you could remove the effect of those extra costs, the numbers would balance again.

Worked Example 8: Cross Exchange Rates

Question: If \$1 = R 6,40, and £1 = R11,58 what is the \$/£ exchange rate (i.e. the number of US\$ per £)?

Answer

Step 1 : Determine what is given and what is required

The following are given:

- ZAR/USD rate = R6,40
- ZAR/GBP rate = R11,58

The following is required:

- USD/GBP rate

Step 2 : Determine how to approach the problem

We know that:

$$\text{USD/GBP} = \text{USD/ZAR} \times \text{ZAR/GBP}.$$

Step 3 : Solve the problem

$$\begin{aligned} \text{USD/GBP} &= \text{USD/ZAR} \times \text{ZAR/GBP} \\ &= \frac{1}{\text{ZAR/USD}} \times \text{ZAR/GBP} \\ &= \frac{1}{6{,}40} \times 11{,}58 \\ &= 1{,}8094 \end{aligned}$$

Step 4 : Write the final answer

\$1,8094 can be bought for £1.

Activity :: Investigation : Cross Exchange Rates - Alternate Method

If \$1 = R 6,40, and £1 = R11,58 what is the \$/£ exchange rate (i.e. the number of US\$ per £)?

Overview of problem

You need the \$/£ exchange rate, in other words how many dollars must you pay for a pound. So you need £1. From the given information we know that it would cost you R11,58 to buy £1 and that \$ 1 = R6,40.

Use this information to:

1. calculate how much R1 is worth in \$.
2. calculate how much R11,58 is worth in \$.

Do you get the same answer as in the worked example?

8.2.3 Enrichment: Fluctuating exchange rates

If everyone wants to buy houses in a certain suburb, then house prices are going to go up - because the buyers will be competing to buy those houses. If there is a suburb where all residents want to move out, then there are lots of sellers and this will cause house prices in the area to fall - because the buyers would not have to struggle as much to find an eager seller.

This is all about supply and demand, which is a very important section in the study of Economics. You can think about this is many different contexts, like stamp-collecting for example. If there is a stamp that lots of people want (high demand) and few people own (low supply) then that stamp is going to be expensive.

And if you are starting to wonder why this is relevant - think about currencies. If you are going to visit London, then you have Rands but you need to "buy" Pounds. The exchange rate is the price you have to pay to buy those Pounds.

Think about a time where lots of South Africans are visiting the United Kingdom, and other South Africans are importing goods from the United Kingdom. That means there are lots of Rands (high supply) trying to buy Pounds. Pounds will start to become more expensive (compare this to the house price example at the start of this section if you are not convinced), and the exchange rate will change. In other words, for R1 000 you will get fewer Pounds than you would have before the exchange rate moved.

Another context which might be useful for you to understand this: consider what would happen if people in other countries felt that South Africa was becoming a great place to live, and that more people were wanting to invest in South Africa - whether in properties, businesses - or just buying more goods from South Africa. There would be a greater demand for Rands - and the "price of the Rand" would go up. In other words, people would need to use more Dollars, or Pounds, or Euros ... to buy the same amount of Rands. This is seen as a movement in exchange rates.

Although it really does come down to supply and demand, it is interesting to think about what factors might affect the supply (people wanting to "sell" a particular currency) and the demand (people trying to "buy" another currency). This is covered in detail in the study of Economics, but let us look at some of the basic issues here.

There are various factors affect exchange rates, some of which have more economic rationale than others:

- economic factors (such as inflation figures, interest rates, trade deficit information, monetary policy and fiscal policy)
- political factors (such as uncertain political environment, or political unrest)
- market sentiments and market behaviour (for example if foreign exchange markets perceived a currency to be overvalued and starting selling the currency, this would cause the currency to fall in value - a self fulfilling expectation).

?

Exercise: Foreign Exchange

1. I want to buy an IPOD that costs £100, with the exchange rate currently at $\pounds 1 = R14$. I believe the exchange rate will reach $R12$ in a month.
   (a) How much will the MP3 player cost in Rands, if I buy it now?
   (b) How much will I save if the exchange rate drops to $R12$?
   (c) How much will I lose if the exchange rate moves to $R15$?
2. Study the following exchange rate table:

Country	Currency	Exchange Rate
United Kingdom (UK)	Pounds(£)	$R14{,}13$
United States (USA)	Dollars ($)	$R7{,}04$

   (a) In South Africa the cost of a new Honda Civic is $R173\,400$. In England the same vehicle costs $\pounds 12\,200$ and in the USA $ $21\,900$. In which country is the car the cheapest if you compare it to the South African Rand ?
   (b) Sollie and Arinda are waiters in a South African Restaurant attracting many tourists from abroad. Sollie gets a $\pounds 6$ tip from a tourist and Arinda gets $ 12. How many South African Rand did each one get ?

8.3 Being Interested in Interest

If you had R1 000, you could either keep it in your wallet, or deposit it in a bank account. If it stayed in your wallet, you could spend it any time you wanted. If the bank looked after it for you, then they could spend it, with the plan of making profit off it. The bank usually "pays" you to deposit it into an account, as a way of encouraging you to bank it with them, This payment is like a reward, which provides you with a reason to leave it with the bank for a while, rather than keeping the money in your wallet.

We call this reward "interest".

If you deposit money into a bank account, you are effectively lending money to the bank - and you can expect to receive interest in return. Similarly, if you borrow money from a bank (or from a department store, or a car dealership, for example) then you can expect to have to pay interest on the loan. That is the price of borrowing money.

The concept is simple, yet it is core to the world of finance. Accountants, actuaries and bankers, for example, could spend their entire working career dealing with the effects of interest on financial matters.

In this chapter you will be introduced to the concept of financial mathematics - and given the tools to cope with even advanced concepts and problems.

Important: Interest

The concepts in this chapter are simple - we are just looking at the same idea, but from many different angles. The best way to learn from this chapter is to do the examples yourself, as you work your way through. Do not just take our word for it!

8.4 Simple Interest

Definition: Simple Interest
Simple interest is where you earn interest on the initial amount that you invested, but not interest on interest.

As an easy example of simple interest, consider how much you will get by investing R1 000 for 1 year with a bank that pays you 5% simple interest. At the end of the year, you will get an interest of:

$$\begin{aligned}
\text{Interest} &= \text{R1 000} \times 5\% \\
&= \text{R1 000} \times \frac{5}{100} \\
&= \text{R1 000} \times 0{,}05 \\
&= \text{R50}
\end{aligned}$$

So, with an "opening balance" of R1 000 at the start of the year, your "closing balance" at the end of the year will therefore be:

$$\begin{aligned}
\text{Closing Balance} &= \text{Opening Balance} + \text{Interest} \\
&= \text{R1 000} + \text{R50} \\
&= \text{R1 050}
\end{aligned}$$

We sometimes call the opening balance in financial calculations *Principal*, which is abbreviated as P (R1 000 in the example). The interest rate is usually labelled i (5% in the example), and the interest amount (in Rand terms) is labelled I (R50 in the example).

So we can see that:

$$I = P \times i \tag{8.1}$$

and

$$\begin{aligned}
\text{Closing Balance} &= \text{Opening Balance} + \text{Interest} \\
&= P + I \\
&= P + (P \times i) \\
&= P(1 + i)
\end{aligned}$$

This is how you calculate simple interest. It is not a complicated formula, which is just as well because you are going to see a lot of it!

Not Just One

You might be wondering to yourself:

1. how much interest will you be paid if you only leave the money in the account for 3 months, or

2. what if you leave it there for 3 years?

It is actually quite simple - which is why they call it **Simple Interest**.

1. Three months is 1/4 of a year, so you would only get 1/4 of a full year's interest, which is: $1/4 \times (P \times i)$. The closing balance would therefore be:

$$\begin{aligned} \text{Closing Balance} &= P + 1/4 \times (P \times i) \\ &= P(1 + (1/4)i) \end{aligned}$$

2. For 3 years, you would get three years' worth of interest, being: $3 \times (P \times i)$. The closing balance at the end of the three year period would be:

$$\begin{aligned} \text{Closing Balance} &= P + 3 \times (P \times i) \\ &= P \times (1 + (3)i) \end{aligned}$$

If you look carefully at the similarities between the two answers above, we can generalise the result. In other words, if you invest your money (P) in an account which pays a rate of interest (i) for a period of time (n years), then, using the symbol (A) for the Closing Balance:

$$\text{Closing Balance,(A)} = P(1 + i \cdot n) \tag{8.2}$$

As we have seen, this works when n is a fraction of a year and also when n covers several years.

Important: Interest Calculation

Annual Rates means Yearly rates. and p.a.(per annum) = per year

Worked Example 9: Simple Interest

Question: If I deposit R1 000 into a special bank account which pays a Simple Interest of 7% for 3 years, how much will I get back at the end?

Answer

Step 1 : Determine what is given and what is required

- opening balance, $P = \text{R}1\ 000$
- interest rate, $i = 7\%$
- period of time, $n = 3$ years

We are required to find the closing balance (A).

Step 2 : Determine how to approach the problem

We know from (8.2) that:

$$\text{Closing Balance,(A)} = P(1 + i \cdot n)$$

Step 3 : Solve the problem

$$\begin{aligned} \text{A} &= P(1 + i \cdot n) \\ &= R1\ 000(1 + 3 \times 7\%) \\ &= R1\ 210 \end{aligned}$$

Step 4 : Write the final answer

The closing balance after 3 years of saving R1 000 at an interest rate of 7% is R1 210.

Worked Example 10: Calculating n

Question: If I deposit R30 000 into a special bank account which pays a Simple Interest of 7.5% ,for how many years must I invest this amount to generate R45 000

Answer

Step 1 : Determine what is given and what is required

- opening balance, $P = \text{R}30\ 000$
- interest rate, $i = 7{,}5\%$
- closing balance, $A = \text{R}45\ 000$

We are required to find the number of years.

Step 2 : Determine how to approach the problem

We know from (8.2) that:

$$\text{Closing Balance (A)} = P(1 + i \cdot n)$$

Step 3 : Solve the problem

$$\begin{aligned} \text{Closing Balance (A)} &= P(1 + i \cdot n) \\ R45\ 000 &= R30\ 000(1 + n \times 7{,}5\%) \\ (1 + 0{,}075 \times n) &= \frac{45000}{30000} \\ 0{,}075 \times n &= 1{,}5 - 1 \\ n &= \frac{0{,}5}{0{,}075} \\ n &= 6{,}6666667 \end{aligned}$$

Step 4 : Write the final answer

n has to be a whole number, therefore $n = 7$.

The period is 7 years for R30 000 to generate R45 000 at a simple interest rate of 7,5%.

8.4.1 Other Applications of the Simple Interest Formula

Worked Example 11: Hire-Purchase

Question: Troy is keen to buy an addisional hard drive for his laptop advertised for R 2 500 on the internet. There is an option of paying a 10% deposit then making 24 monthly payments using a hire-purchase agreement where interest is calculated at 7,5% p.a. simple interest. Calculate what Troy's monthly payments will be.

Answer

Step 1 : Determine what is given and what is required

A new opening balance is required, as the 10% deposit is paid in cash.

- 10% of R 2 500 = R250
- new opening balance, $P = \text{R2 500} - \text{R250} = \text{R2 250}$
- interest rate, $i = 7{,}5\% = 0{,}075\text{pa}$
- period of time, $n = 2 \text{ years}$

We are required to find the closing balance (A) and then the montly payments.

Step 2 : Determine how to approach the problem

We know from (8.2) that:

$$\text{Closing Balance,(A)} = P(1 + i \cdot n)$$

Step 3 : Solve the problem

$$\begin{aligned} \text{A} &= P(1 + i \cdot n) \\ &= R2\ 250(1 + 2 \times 7{,}5\%) \\ &= R2\ 587{,}50 \\ \text{Monthly payment} &= 2587{,}50 \div 24 \\ &= R107{,}81 \end{aligned}$$

Step 4 : Write the final answer

Troy's monthly payments = R 107,81

Worked Example 12: Depreciation

Question: Seven years ago, Tjad's drum kit cost him R12 500. It has now been valued at R2 300. What rate of simple depreciation does this represent ?

Answer

Step 1 : Determine what is given and what is required

- opening balance, $P = \text{R12 500}$
- period of time, $n = 7 \text{ years}$
- closing balance, $A = \text{R2 300}$

We are required to find the rate(i).

Step 2 : Determine how to approach the problem

We know from (8.2) that:

$$\text{Closing Balance,(A)} = P(1 + i \cdot n)$$

Therefore, for **depreciation** the formula will change to:

$$\text{Closing Balance,(A)} = P(1 - i \cdot n)$$

Step 3 : Solve the problem

$$\begin{aligned} \text{A} &= P(1 - i \cdot n) \\ R2\ 300 &= R12\ 500(1 - 7 \times i) \\ i &= 0{,}11657... \end{aligned}$$

Step 4 : Write the final answer
Therefore the rate of depreciation is $11{,}66\%$

Exercise: Simple Interest

1. An amount of R3 500 is invested in a savings account which pays simple interest at a rate of 7,5% per annum. Calculate the balance accumulated by the end of 2 years.
2. Calculate the simple interest for the following problems.
   (a) A loan of R300 at a rate of 8% for l year.
   (b) An investment of R225 at a rate of 12,5% for 6 years.
3. I made a deposit of R5 000 in the bank for my 5 year old son's 21st birthday. I have given him the amount of R 18 000 on his birtday. At what rate was the money invested, if simple interest was calculated ?
4. Bongani buys a dining room table costing R 8 500 on Hire Purchase. He is charged simple interest at 17,5% per annum over 3 years.
   (a) How much will Bongani pay in total ?
   (b) How much interest does he pay ?
   (c) What is his montly installment ?

8.5 Compound Interest

To explain the concept of compound interest, the following example is discussed:

Worked Example 13: Using Simple Interest to lead to the concept Compound Interest

Question: If I deposit R1 000 into a special bank account which pays a Simple Interest of 7%. What if I empty the bank account after a year, and then take the principal and the interest and invest it back into the same account again. Then I take it all out at the end of the second year, and then put it all back in again? And then I take it all out at the end of 3 years?

Answer

Step 1 : Determine what is given and what is required

- opening balance, $P = \text{R}1\ 000$
- interest rate, $i = 7\%$

- period of time, 1 year at a time, for 3 years

We are required to find the closing balance at the end of three years.

Step 2 : Determine how to approach the problem

We know that:

$$\text{Closing Balance} = P(1 + i \cdot n)$$

Step 3 : Determine the closing balance at the end of the first year

$$\begin{aligned}\text{Closing Balance} &= P(1 + i \cdot n)\\ &= \text{R1 000}(1 + 1 \times 7\%)\\ &= \text{R1 070}\end{aligned}$$

Step 4 : Determine the closing balance at the end of the second year

After the first year, we withdraw all the money and re-deposit it. The opening balance for the second year is therefore R1 070, because this is the balance after the first year.

$$\begin{aligned}\text{Closing Balance} &= P(1 + i \cdot n)\\ &= \text{R1 070}(1 + 1 \times 7\%)\\ &= \text{R1 144,90}\end{aligned}$$

Step 5 : Determine the closing balance at the end of the third year

After the second year, we withdraw all the money and re-deposit it. The opening balance for the third year is therefore R1 144,90, because this is the balance after the first year.

$$\begin{aligned}\text{Closing Balance} &= P(1 + i \cdot n)\\ &= \text{R1 144,90}(1 + 1 \times 7\%)\\ &= \text{R1 225,04}\end{aligned}$$

Step 6 : Write the final answer

The closing balance after withdrawing the all the money and re-depositing each year for 3 years of saving R1 000 at an interest rate of 7% is R1 225,04.

In the two worked examples using simple interest, we have basically the same problem because P=R1 000, i=7% and n=3 years for both problems. Except in the second situation, we end up with R1 225,04 which is more than R1 210 from the first example. What has changed?

In the first example I earned R70 interest each year - the same in the first, second and third year. But in the second situation, when I took the money out and then re-invested it, I was actually earning interest in the second year on my interest (R70) from the first year. (And interest on the interest on my interest in the third year!)

This more realistically reflects what happens in the real world, and is known as Compound Interest. It is this concept which underlies just about everything we do - so we will look at more closely next.

Definition: Compound Interest

Compound interest is the interest payable on the principal and its accumulated interest.

Compound interest is a double edged sword, though - great if you are earning interest on cash you have invested, but crippling if you are stuck having to pay interest on money you have borrowed!

In the same way that we developed a formula for Simple Interest, let us find one for Compound Interest.

If our opening balance is P and we have an interest rate of i then, the closing balance at the end of the first year is:

$$\text{Closing Balance after 1 year} = P(1+i)$$

This is the same as Simple Interest because it only covers a single year. Then, if we take that out and re-invest it for another year - just as you saw us doing in the worked example above - then the balance after the second year will be:

$$\begin{aligned}\text{Closing Balance after 2 years} &= [P(1+i)] \times (1+i) \\ &= P(1+i)^2\end{aligned}$$

And if we take that money out, then invest it for another year, the balance becomes:

$$\begin{aligned}\text{Closing Balance after 3 years} &= [P(1+i)^2] \times (1+i) \\ &= P(1+i)^3\end{aligned}$$

We can see that the power of the term $(1+i)$ is the same as the number of years. Therefore,

$$\text{Closing Balance after } n \text{ years} = P(1+i)^n \tag{8.3}$$

8.5.1 Fractions add up to the Whole

It is easy to show that this formula works even when n is a fraction of a year. For example, let us invest the money for 1 month, then for 4 months, then for 7 months.

$$\begin{aligned}\text{Closing Balance after 1 month} &= P(1+i)^{\frac{1}{12}} \\ \text{Closing Balance after 5 months} &= \text{Closing Balance after 1 month invested for 4 months more} \\ &= [P(1+i)^{\frac{1}{12}}]^{\frac{4}{12}} \\ &= P(1+i)^{\frac{1}{12}+\frac{4}{12}} \\ &= P(1+i)^{\frac{5}{12}} \\ \text{Closing Balance after 12 months} &= \text{Closing Balance after 5 month invested for 7 months more} \\ &= [P(1+i)^{\frac{5}{12}}]^{\frac{7}{12}} \\ &= P(1+i)^{\frac{5}{12}+\frac{7}{12}} \\ &= P(1+i)^{\frac{12}{12}} \\ &= P(1+i)^1\end{aligned}$$

which is the same as investing the money for a year.

Look carefully at the long equation above. It is not as complicated as it looks! All we are doing is taking the opening amount (P), then adding interest for just 1 month. Then we are taking that new balance and adding interest for a further 4 months, and then finally we are taking the new balance after a total of 5 months, and adding interest for 7 more months. Take a look again, and check how easy it really is.

Does the final formula look familiar? Correct - it is the same result as you would get for simply investing P for one full year. This is exactly what we would expect, because:

$$1 \text{ month} + 4 \text{ months} + 7 \text{ months} = 12 \text{ months},$$

which is a year. Can you see that? Do not move on until you have understood this point.

8.5.2 The Power of Compound Interest

To see how important this "interest on interest" is, we shall compare the difference in closing balances for money earning simple interest and money earning compound interest. Consider an amount of R10 000 that you have to invest for 10 years, and assume we can earn interest of 9%. How much would that be worth after 10 years?

The closing balance for the money earning simple interest is:

$$\begin{aligned}\text{Closing Balance} &= P(1 + i \cdot n)\\ &= \text{R10 000}(1 + 9\% \times 10)\\ &= \text{R19 000}\end{aligned}$$

The closing balance for the money earning compound interest is:

$$\begin{aligned}\text{Closing Balance} &= P(1 + i)^n)\\ &= \text{R10 000}(1 + 9\%)^{10}\\ &= \text{R23 673,64}\end{aligned}$$

So next time someone talks about the "magic of compound interest", not only will you know what they mean - but you will be able to prove it mathematically yourself!

Again, keep in mind that this is good news and bad news. When you are earning interest on money you have invested, compound interest helps that amount to increase exponentially. But if you have borrowed money, the build up of the amount you owe will grow exponentially too.

Worked Example 14: Taking out a Loan

Question: Mr Lowe wants to take out a loan of R 350 000. He does not want to pay back more than R625 000 altogether on the loan. If the interest rate he is offered is 13%, over what period should he take the loan

Answer

Step 1 : Determine what has been provided and what is required

- opening balance, $P = \text{R350 000}$
- closing balance, $A = \text{R625 000}$
- interest rate, $i = 13\% \text{ peryear}$

We are required to find the time period(n).

Step 2 : Determine how to approach the problem

We know from (8.3) that:

$$\text{Closing Balance,(A)} = P(1 + i)^n$$

We need to find n.

Therefore we covert the formula to:

$$\frac{\text{A}}{\text{P}} = (1 + i)^n$$

and then find n by trial and error.

Step 3 : Solve the problem

$$\begin{aligned}\frac{\text{A}}{\text{P}} &= (1 + i)^n\\ \frac{625000}{350000} &= (1 + 0{,}13)^n\\ 1{,}785... &= (1{,}13)^n\end{aligned}$$

$$\begin{aligned}\text{Try n} &= 3: \quad (1{,}13)^3 = 1{,}44...\\ \text{Try n} &= 4: \quad (1{,}13)^4 = 1{,}63...\\ \text{Try n} &= 5: \quad (1{,}13)^5 = 1{,}84...\end{aligned}$$

Step 4 : Write the final answer

Mr Lowe should take the loan over four years

8.5.3 Other Applications of Compound Growth

Worked Example 15: Population Growth

Question: South Africa's population is increasing by 2,5% per year. If the current population is 43 million, how many more people will there be in South Africa in two year's time ?

Answer

Step 1 : Determine what has been provided and what is required

- opening balance, $P = 43\ 000\ 000$
- period of time, $n = 2$ year
- interest rate, $i = 2{,}5\%$ peryear

We are required to find the closing balance(A).

Step 2 : Determine how to approach the problem

We know from (8.3) that:

$$\text{Closing Balance,(A)} = P(1+i)^n$$

Step 3 : Solve the problem

$$\begin{aligned} \text{A} &= P(1+i)^n \\ &= 43\ 000\ 000(1+0{,}025)^2 \\ &= 45\ 176\ 875 \end{aligned}$$

Step 4 : Write the final answer

There are $45\ 176\ 875 - 43\ 000\ 000 = 2\ 176\ 875$ more people in 2 year's time

Worked Example 16: Compound Decrease

Question: A swimming pool is being treated for a build-up of algae. Initially, $50m^2$ of the pool is covered by algae. With each day of treatment, the algae reduces by 5%. What area is covered by algae after 30 days of treatment ?

Answer

Step 1 : Determine what has been provided and what is required

- opening balance, $P = 50\text{m}^2$
- period of time, $n = 30$ days
- interest rate, $i = 5\%$ perday

We are required to find the closing balance(A).

Step 2 : Determine how to approach the problem

We know from (8.3) that:

$$\text{Closing Balance,(A)} = P(1+i)^n$$

But this is compound **decrease** so we can use the formula:

$$\text{Closing Balance,(A)} = P(1-i)^n$$

Step 3 : Solve the problem

$$\begin{aligned} \text{A} &= P(1-i)^n \\ &= 50(1-0{,}05)^{30} \\ &= 10{,}73m^2 \end{aligned}$$

Step 4 : Write the final answer
Therefore the area still covered with algae is $10{,}73m^2$

?

Exercise: Compound Interest

1. An amount of R3 500 is invested in a savings account which pays compound interest at a rate of 7,5% per annum. Calculate the balance accumulated by the end of 2 years.
2. If the average rate of inflation for the past few years was 7,3% and your water and electricity account is R 1 425 on average, what would you expect to pay in 6 years time ?
3. Shrek wants to invest some money at 11% per annum compound interest. How much money (to the nearest rand) should he invest if he wants to reach a sum of R 100 000 in five year's time ?

8.6 Summary

As an easy reference, here are the key formulae that we derived and used during this chapter. While memorising them is nice (there are not many), it is the application that is useful. Financial experts are not paid a salary in order to recite formulae, they are paid a salary to use the right methods to solve financial problems.

8.6.1 Definitions

P Principal (the amount of money at the starting point of the calculation)
i interest rate, normally the effective rate per annum
n period for which the investment is made

8.6.2 Equations

$$\left.\begin{array}{l}\text{Closing Balance - simple interest}\\ \text{Solve for } i\\ \text{Solve for } n\end{array}\right\} = P(1+i\cdot n)$$

$$\left.\begin{array}{l}\text{Closing Balance - compound interest}\\ \text{Solve for } i\\ \text{Solve for } n\end{array}\right\} = P(1+i)^n$$

! **Important:** Always keep the interest and the time period in the same units of time (e.g. both in years, or both in months etc.).

8.7 End of Chapter Exercises

1. You are going on holiday to Europe. Your hotel will cost €200 per night. How much will you need in Rands to cover your hotel bill, if the exchange rate is €1 = R9,20.
2. Calculate how much you will earn if you invested R500 for 1 year at the following interest rates:
   (a) 6,85% simple interest
   (b) 4,00% compound interest
3. Bianca has R1 450 to invest for 3 years. Bank A offers a savings account which pays simple interest at a rate of 11% per annum, whereas Bank B offers a savings account paying compound interest at a rate of 10,5% per annum. Which account would leave Bianca with the highest accumulated balance at the end of the 3 year period?
4. How much simple interest is payable on a loan of R2 000 for a year, if the interest rate is 10%?
5. How much compound interest is payable on a loan of R2 000 for a year, if the interest rate is 10%?
6. Discuss:
   (a) Which type of interest would you like to use if you are the borrower?
   (b) Which type of interest would you like to use if you were the banker?
7. Calculate the compound interest for the following problems.
   (a) A R2 000 loan for 2 years at 5%.
   (b) A R1 500 investment for 3 years at 6%.
   (c) An R800 loan for l year at 16%.
8. If the exchange rate 100 Yen = R 6,2287 and 1 AUD = R 5,1094 , determine the exchange rate between the Australian Dollar and the Japanese Yen.
9. Bonnie bought a stove for R 3 750. After 3 years she paid for it and the R 956,25 interest that was charged for hire-purchase. Determine the simple rate of interest that was charged.

Chapter 9

Products and Factors - Grade 10

9.1 Introduction

In this chapter you will learn how to work with algebraic expressions. You will recap some of the work on factorisation and multiplying out expressions that you learnt in earlier grades. This work will then be extended upon for Grade 10.

9.2 Recap of Earlier Work

The following should be familiar. Examples are given as reminders.

9.2.1 Parts of an Expression

Mathematical expressions are just like sentences and their parts have special names. You should be familiar with the following names used to describe the parts of an mathematical expression.

$$a \cdot x^k + b \cdot x + c^m = 0 \quad (9.1)$$

$$d \cdot y^p + e \cdot y + f \leq 0 \quad (9.2)$$

Name	Examples (separated by commas)
term	$a \cdot x^k$,$b \cdot x$, c^m, $d \cdot y^p$, $e \cdot y$, f
expression	$a \cdot x^k + b \cdot x + c^m$, $d \cdot y^p + e \cdot y + f$
coefficient	a, b, d, e
exponent (or index)	k, p
base	x, y, c
constant	a, b, c, d, e, f
variable	x, y
equation	$a \cdot x^k + b \cdot x + c^m = 0$
inequality	$d \cdot y^p + e \cdot y + f \leq 0$
binomial	expression with two terms
trinomial	expression with three terms

9.2.2 Product of Two Binomials

A *binomial* is a mathematical expression with two terms, e.g. $(ax + b)$ and $(cx + d)$. If these two binomials are multiplied, the following is the result:

$$\begin{aligned}(a\cdot x+b)(c\cdot x+d) &= (ax)(c\cdot x+d)+b(c\cdot x+d)\\ &= (ax)(cx)+(ax)d+b(cx)+b\cdot d\end{aligned}$$

Worked Example 17: Product of two Binomials

Question: Find the product of $(3x-2)(5x+8)$
Answer

$$\begin{aligned}(3x-2)(5x+8) &= (3x)(5x)+(3x)(8)+(-2)(5x)+(-2)(8)\\ &= 15x^2+24x-10x-16\\ &= 15x^2+14x-16\end{aligned}$$

The product of two identical binomials is known as the *square of the binomials* and is written as:

$$(ax+b)^2 = a^2x^2+2abx+b^2$$

If the two terms are $ax+b$ and $ax-b$ then their product is:

$$(ax+b)(ax-b) = a^2x^2-b^2$$

This is known as the *difference of squares*.

9.2.3 Factorisation

Factorisation is the opposite of expanding brackets. For example expanding brackets would require $2(x+1)$ to be written as $2x+2$. Factorisation would be to start with $2x+2$ and to end up with $2(x+1)$. In previous grades you factorised based on common factors and on difference of squares.

Common Factors

Factorising based on common factors relies on there being common factors between your terms. For example, $2x-6x^2$ can be factorised as follows:

$$2x-6x^2 = 2x(1-3x)$$

Activity :: Investigation : Common Factors
Find the highest common factors of the following pairs of terms:

(a) $6y; 18x$	(b) $12mn; 8n$	(c) $3st; 4su$	(d) $18kl; 9kp$	(e) $abc; ac$
(f) $2xy; 4xyz$	(g) $3uv; 6u$	(h) $9xy; 15xz$	(i) $24xyz; 16yz$	(j) $3m; 45n$

Difference of Squares

We have seen that:

$$(ax+b)(ax-b) = a^2x^2 - b^2 \tag{9.3}$$

Since 9.3 is an equation, both sides are always equal. This means that an expression of the form:

$$a^2x^2 - b^2$$

can be factorised to

$$(ax+b)(ax-b)$$

Therefore,

$$a^2x^2 - b^2 = (ax+b)(ax-b)$$

For example, $x^2 - 16$ can be written as $(x^2 - 4^2)$ which is a difference of squares. Therefore the factors of $x^2 - 16$ are $(x-4)$ and $(x+4)$.

Worked Example 18: Factorisation

Question: Factorise completely: $b^2y^5 - 3aby^3$

Answer

$$b^2y^5 - 3aby^3 = by^3(by^2 - 3a)$$

Worked Example 19: Factorising binomials with a common bracket

Question: Factorise completely: $3a(a-4) - 7(a-4)$

Answer

Step 1 : bracket $(a-4)$ is the common factor

$$3a(a-4) - 7(a-4) = (a-4)(3a-7)$$

Worked Example 20: Factorising using a switch around in brackets

Question: Factorise $5(a-2) - b(2-a)$

Answer

Step 1 : Note that $(2-a) = -(a-2)$

$$\begin{aligned} 5(a-2) - b(2-a) &= 5(a-2) - [-b(a-2)] \\ &= 5(a-2) + b(a-2) \\ &= (a-2)(5+b) \end{aligned}$$

?

Exercise: Recap

1. Find the products of:

(a) $2y(y+4)$	(b) $(y+5)(y+2)$	(c) $(y+2)(2y+1)$
(d) $(y+8)(y+4)$	(e) $(2y+9)(3y+1)$	(f) $(3y-2)(y+6)$

2. Factorise:
   (a) $2l+2w$
   (b) $12x+32y$
   (c) $6x^2+2x+10x^3$
   (d) $2xy^2+xy^2z+3xy$
   (e) $-2ab^2-4a^2b$

3. Factorise completely:

(a) $7a+4$	(b) $20a-10$	(c) $18ab-3bc$
(d) $12kj+18kq$	(e) $16k^2-4k$	(f) $3a^2+6a-18$
(g) $-6a-24$	(h) $-2ab-8a$	(i) $24kj-16k^2j$
(j) $-a^2b-b^2a$	(k) $12k^2j+24k^2j^2$	(l) $72b^2q-18b^3q^2$
(m) $4(y-3)+k(3-y)$	(n) $a(a-1)-5(a-1)$	(o) $bm(b+4)-6m(b+4)$
(p) $a^2(a+7)+a(a+7)$	(q) $3b(b-4)-7(4-b)$	(r) $a^2b^2c^2-1$

9.3 More Products

We have seen how to multiply two binomials in section 9.2.2. In this section we learn how to multiply a binomial (expression with two terms) by a trinomial (expression with three terms). Fortunately, we use the same methods we used to multiply two binomials to multiply a binomial and a trinomial.

For example, multiply $2x+1$ by x^2+2x+1.

$$\begin{aligned}
& (2x+1)(x^2+2x+1) \\
= \; & 2x(x^2+2x+1)+1(x^2+2x+1) \quad \text{(apply distributive law)} \\
= \; & [2x(x^2)+2x(2x)+2x(1)]+[1(x^2)+1(2x)+1(1)] \\
= \; & 4x^3+4x^2+2x+x^2+2x+1 \quad \text{(expand the brackets)} \\
= \; & 4x^3+(4x^2+x^2)+(2x+2x)+1 \quad \text{(group like terms to simplify)} \\
= \; & 4x^3+5x^2+4x+1 \quad \text{(simplify to get final answer)}
\end{aligned}$$

!

Important: Multiplication of Binomial with Trinomial

If the binomial is $A+B$ and the trinomial is $C+D+E$, then the very first step is to apply the distributive law:

$$(A+B)(C+D+E) = A(C+D+E)+B(C+D+E) \quad (9.4)$$

If you remember this, you will never go wrong!

Worked Example 21: Multiplication of Binomial with Trinomial

Question: Multiply $x-1$ with x^2-2x+1.
Answer
Step 1 : Determine what is given and what is required
We are given two expressions: a binomial, $x-1$, and a trinomial, x^2-2x+1. We need to multiply them together.
Step 2 : Determine how to approach the problem
Apply the distributive law and then simplify the resulting expression.
Step 3 : Solve the problem

$$\begin{aligned}
& (x-1)(x^2-2x+1) \\
= \ & x(x^2-2x+1)-1(x^2-2x+1) \quad \text{(apply distributive law)} \\
= \ & [x(x^2)+x(-2x)+x(1)]+[-1(x^2)-1(-2x)-1(1)] \\
= \ & x^3-2x^2+x-x^2+2x-1 \quad \text{(expand the brackets)} \\
= \ & x^3+(-2x^2-x^2)+(x+2x)-1 \quad \text{(group like terms to simplify)} \\
= \ & x^3-3x^2+3x-1 \quad \text{(simplify to get final answer)}
\end{aligned}$$

Step 4 : Write the final answer
The product of $x-1$ and x^2-2x+1 is x^3-3x^2+3x-1.

Worked Example 22: Sum of Cubes

Question: Find the product of $x+y$ and x^2-xy+y^2.
Answer
Step 1 : Determine what is given and what is required
We are given two expressions: a binomial, $x+y$, and a trinomial, x^2-xy+y^2. We need to multiply them together.
Step 2 : Determine how to approach the problem
Apply the distributive law and then simplify the resulting expression.
Step 3 : Solve the problem

$$\begin{aligned}
& (x+y)(x^2-xy+y^2) \\
= \ & x(x^2-xy+y^2)+y(x^2-xy+y^2) \quad \text{(apply distributive law)} \\
= \ & [x(x^2)+x(-xy)+x(y^2)]+[y(x^2)+y(-xy)+y(y^2)] \\
= \ & x^3-x^2y+xy^2+yx^2-xy^2+y^3 \quad \text{(expand the brackets)} \\
= \ & x^3+(-x^2y+yx^2)+(xy^2-xy^2)+y^3 \quad \text{(group like terms to simplify)} \\
= \ & x^3+y^3 \quad \text{(simplify to get final answer)}
\end{aligned}$$

Step 4 : Write the final answer
The product of $x+y$ and x^2-xy+y^2 is x^3+y^3.

Important: We have seen that:

$$(x+y)(x^2-xy+y^2)=x^3+y^3$$

This is known as a *sum of cubes*.

Activity :: Investigation : Difference of Cubes
Show that the difference of cubes (x^3-y^3) is given by the product of $x-y$ and x^2+xy+y^2.

Exercise: Products

1. Find the products of:

   (a) $(-2y^2-4y+11)(5y-12)$
   (b) $(-11y+3)(-10y^2-7y-9)$
   (c) $(4y^2+12y+10)(-9y^2+8y+2)$
   (d) $(7y^2-6y-8)(-2y+2)$
   (e) $(10y^5+3)(-2y^2-11y+2)$
   (f) $(-12y-3)(12y^2-11y+3)$
   (g) $(-10)(2y^2+8y+3)$
   (h) $(2y^6+3y^5)(-5y-12)$
   (i) $(6y^7-8y^2+7)(-4y-3)(-6y^2-7y-11)$
   (j) $(-9y^2+11y+2)(8y^2+6y-7)$
   (k) $(8y^5+3y^4+2y^3)(5y+10)(12y^2+6y+6)$
   (l) $(-7y+11)(-12y+3)$
   (m) $(4y^3+5y^2-12y)(-12y-2)(7y^2-9y+12)$
   (n) $(7y+3)(7y^2+3y+10)$
   (o) $(9)(8y^2-2y+3)$
   (p) $(-12y+12)(4y^2-11y+11)$
   (q) $(-6y^4+11y^2+3y)(10y+4)(4y-4)$
   (r) $(-3y^6-6y^3)(11y-6)(10y-10)$
   (s) $(-11y^5+11y^4+11)(9y^3-7y^2-4y+6)$
   (t) $(-3y+8)(-4y^3+8y^2-2y+12)$

2. Remove the brackets and simplify:$(2h+3)(4h^2-6h+9)$

9.4 Factorising a Quadratic

Finding the factors of a quadratic is quite easy, and some are easier than others.

The simplest quadratic has the form ax^2, which factorises to $(x)(ax)$. For example, $25x^2$ factorises to $(5x)(5x)$ and $2x^2$ factorises to $(2x)(x)$.

The second simplest quadratic is of the form ax^2+bx. We can see here that x is a common factor of both terms. Therefore, ax^2+bx factorises to $x(ax+b)$. For example, $8y^2+4y$ factorises to $4y(2y+1)$.

The third simplest quadratic is made up of the difference of squares. We know that:

$$(a+b)(a-b)=a^2-b^2.$$

This is true for any values of a and b, and more importantly since it is an equality, we can also write:

$$a^2-b^2=(a+b)(a-b).$$

This means that if we ever come across a quadratic that is made up of a difference of squares, we can immediately write down what the factors are.

Worked Example 23: Difference of Squares

Question: Find the factors of $9x^2 - 25$.

Answer

Step 1 : Examine the quadratic

We see that the quadratic is a difference of squares because:

$$(3x)^2 = 9x^2$$

and

$$5^2 = 25.$$

Step 2 : Write the quadratic as the difference of squares

$$9x^2 - 25 = (3x)^2 - 5^2$$

Step 3 : Write the factors

$$(3x)^2 - 5^2 = (3x - 5)(3x + 5)$$

Step 4 : Write the final answer

The factors of $9x^2 - 25$ are $(3x - 5)(3x + 5)$.

The three types of quadratic that we have seen are very simple to factorise. However, many quadratics do not fall into these categories, and we need a more general method to factorise quadratics like $x^2 - x - 2$?

We can learn about how to factorise quadratics by looking at how two binomials are multiplied to get a quadratic. For example, $(x + 2)(x + 3)$ is multiplied out as:

$$\begin{aligned}(x+2)(x+3) &= x(x+3) + 2(x+3) \\ &= (x)(x) + 3x + 2x + (2)(3) \\ &= x^2 + 5x + 6.\end{aligned}$$

We see that the x^2 term in the quadratic is the product of the x-terms in each bracket. Similarly, the 6 in the quadratic is the product of the 2 and 3 the brackets. Finally, the middle term is the sum of two terms.

So, how do we use this information to factorise the quadratic?

Let us start with factorising $x^2 + 5x + 6$ and see if we can decide upon some general rules. Firstly, write down two brackets with an x in each bracket and space for the remaining terms.

$$(\quad x \qquad)(\quad x \qquad)$$

Next decide upon the factors of 6. Since the 6 is positive, these are:

Factors of 6	
1	6
2	3
-1	-6
-2	-3

Therefore, we have four possibilities:

Option 1	Option 2	Option 3	Option 4
$(x+1)(x+6)$	$(x-1)(x-6)$	$(x+2)(x+3)$	$(x-2)(x-3)$

Next we expand each set of brackets to see which option gives us the correct middle term.

Option 1	Option 2	Option 3	Option 4
$(x+1)(x+6)$	$(x-1)(x-6)$	$(x+2)(x+3)$	$(x-2)(x-3)$
x^2+7x+6	x^2-7x+6	$\underline{x^2+5x+6}$	x^2-5x+6

We see that Option 3 (x+2)(x+3) is the correct solution. As you have seen that the process of factorising a quadratic is mostly trial and error, however the is some information that can be used to simplify the process.

Method: Factorising a Quadratic

1. First divide the entire equation by any common factor of the coefficients, so as to obtain an equation of the form $ax^2 + bx + c = 0$ where a, b and c have no common factors and a is positive.
2. Write down two brackets with an x in each bracket and space for the remaining terms.

$$(\quad x \qquad)(\quad x \qquad) \tag{9.5}$$

3. Write down a set of factors for a and c.
4. Write down a set of options for the possible factors for the quadratic using the factors of a and c.
5. Expand all options to see which one gives you the correct answer.

There are some tips that you can keep in mind:

- If c is positive, then the factors of c must be either both positive or both negative. The factors are both negative if b is negative, and are both positive if b is positive. If c is negative, it means only one of the factors of c is negative, the other one being positive.
- Once you get an answer, multiply out your brackets again just to make sure it really works.

Worked Example 24: Factorising a Quadratic

Question: Find the factors of $3x^2 + 2x - 1$.

Answer

Step 1 : Check whether the quadratic is in the form $ax^2 + bx + c = 0$ with a positive.

The quadratic is in the required form.

Step 2 : Write down two brackets with an x in each bracket and space for the remaining terms.

$$(\quad x \qquad)(\quad x \qquad) \tag{9.6}$$

Write down a set of factors for a and c. The possible factors for a are: (1,3). The possible factors for c are: (-1,1) or (1,-1).

Write down a set of options for the possible factors for the quadratic using the factors of a and c. Therefore, there are two possible options.

Option 1	Option 2
$(x-1)(3x+1)$	$(x+1)(3x-1)$
$3x^2-2x-1$	$\underline{3x^2+2x-1}$

Step 3 : Check your answer

$$\begin{aligned}(x+1)(3x-1) &= x(3x-1)+1(3x-1)\\ &= (x)(3x)+(x)(-1)+(1)(3x)+(1)(-1)\\ &= 3x^2-x+3x-1\\ &= x^2+2x-1.\end{aligned}$$

Step 4 : Write the final answer
The factors of $3x^2+2x-1$ are $(x+1)$ and $(3x-1)$.

?

Exercise: Factorising a Trinomial

1. Factorise the following:

(a) $x^2+8x+15$	(b) $x^2+10x+24$	(c) x^2+9x+8
(d) $x^2+9x+14$	(e) $x^2+15x+36$	(f) $x^2+13x+36$

2. Factorise the following:
   (a) $x^2-2x-15$
   (b) x^2+2x-3
   (c) x^2+2x-8
   (d) x^2+x-20
   (e) x^2-x-20
3. Find the factors for the following quadratic expressions:
   (a) $2x^2+11x+5$
   (b) $3x^2+19x+6$
   (c) $6x^2+7x+2$
   (d) $12x^2+7x+1$
   (e) $8x^2+6x+1$
4. Find the factors for the following trinomials:
   (a) $3x^2+17x-6$
   (b) $7x^2-6x-1$
   (c) $8x^2-6x+1$
   (d) $2x^2-5x-3$

9.5 Factorisation by Grouping

One other method of factorisation involves the use of common factors. We know that the factors of $3x+3$ are 3 and $(x+1)$. Similarly, the factors of $2x^2+2x$ are $2x$ and $(x+1)$. Therefore, if we have an expression:

$$2x^2+2x+3x+3$$

then we can factorise as:

$$2x(x+1)+3(x+1).$$

You can see that there is another common factor: $x+1$. Therefore, we can now write:

$$(x+1)(2x+3).$$

We get this by taking out the $x+1$ and see what is left over. We have a $+2x$ from the first term and a $+3$ from the second term. This is called *factorisation by grouping*.

Worked Example 25: Factorisation by Grouping

Question: Find the factors of $7x + 14y + bx + 2by$ by grouping
Answer
Step 1 : Determine if there are common factors to all terms
There are no factors that are common to all terms.
Step 2 : Determine if there are factors in common between some terms
7 is a common factor of the first two terms and b is a common factor of the second two terms.
Step 3 : Re-write expression taking the factors into account

$$7x + 14y + bx + 2by = 7(x + 2y) + b(x + 2y)$$

Step 4 : Determine if there are more common factors
$x + 2y$ is a common factor.
Step 5 : Re-write expression taking the factors into account

$$7(x + 2y) + b(x + 2y) = (x + 2y)(7 + b)$$

Step 6 : Write the final answer
The factors of $7x + 14y + bx + 2by$ are $(7 + b)$ and $(x + 2y)$.

Exercise: Factorisation by Grouping

1. Factorise by grouping: $6x + 9 + 2ax + 3$
2. Factorise by grouping: $x^2 - 6x + 5x - 30$
3. Factorise by grouping: $5x + 10y - ax - 2ay$
4. Factorise by grouping: $a^2 - 2a - ax + 2x$
5. Factorise by grouping: $5xy - 3y + 10x - 6$

9.6 Simplification of Fractions

In some cases of simplifying an algebraic expression, the expression will be a fraction. For example,

$$\frac{x^2 + 3x}{x + 3}$$

has a quadratic in the numerator and a binomial in the denominator. You can apply the different factorisation methods to simplify the expression.

$$\begin{aligned} & \frac{x^2 + 3x}{x + 3} \\ = & \frac{x(x + 3)}{x + 3} \\ = & x \quad \text{provided } x \neq -3 \end{aligned}$$

Worked Example 26: Simplification of Fractions

Question: Simplify: $\frac{2x-b+x-ab}{ax^2-abx}$

Answer

Step 1 : Factorise numerator and denominator

Use *grouping* for numerator and *common factor* for denominator in this example.

$$\begin{aligned} &= \frac{(ax-ab)+(x-b)}{ax^2-abx} \\ &= \frac{a(x-b)+(x-b)}{ax(x-b)} \\ &= \frac{(x-b)(a+1)}{ax(x-b)} \end{aligned}$$

Step 2 : Cancel out same factors

The simplified answer is:

$$= \frac{a+1}{ax}$$

Worked Example 27: Simplification of Fractions

Question: Simplify: $\frac{x^2-x-2}{x^2-4} \div \frac{x^2+x}{x^2+2x}$

Answer

Step 1 : Factorise numerators and denominators

$$= \frac{(x+1)(x-2)}{(x+2)(x-2)} \div \frac{x(x+1)}{x(x+2)}$$

Step 2 : Multiply by factorised reciprocal

$$= \frac{(x+1)(x-2)}{(x+2)(x-2)} \times \frac{x(x+2)}{x(x+1)}$$

Step 3 : Cancel out same factors

The simplified answer is

$$= 1$$

Exercise: Simplification of Fractions

1. Simplify:

(a) $\frac{3a}{15}$
(b) $\frac{2a+10}{4}$
(c) $\frac{5a+20}{a+4}$
(d) $\frac{a^2-4a}{a-4}$
(e) $\frac{3a^2-9a}{2a-6}$
(f) $\frac{9a+27}{9a+18}$
(g) $\frac{6ab+2a}{2b}$
(h) $\frac{16x^2y-8xy}{12x-6}$
(i) $\frac{4xyp-8xp}{12xy}$
(j) $\frac{3a+9}{14} \div \frac{7a+21}{a+3}$
(k) $\frac{a^2-5a}{2a+10} \div \frac{3a+15}{4a}$
(l) $\frac{3xp+4p}{8p} \div \frac{12p^2}{3x+4}$
(x) $\frac{16}{2xp+4x} \div \frac{6x^2+8x}{12}$
(y) $\frac{24a-8}{12} \div \frac{9a-3}{6}$
(o) $\frac{a^2+2a}{5} \div \frac{2a+4}{20}$
(p) $\frac{p^2+pq}{7p} \div \frac{8p+8q}{21q}$
(q) $\frac{5ab-15b}{4a-12} \div \frac{6b^2}{a+b}$
(r) $\frac{f^2a-fa^2}{f-a}$

2. Simplify: $\frac{x^2-1}{3} \times \frac{1}{x-1} - \frac{1}{2}$

9.7 End of Chapter Exercises

1. Factorise:

(a) $a^2 - 9$
(b) $m^2 - 36$
(c) $9b^2 - 81$
(d) $16b^6 - 25a^2$
(e) $m^2 - (1/9)$
(f) $5 - 5a^2b^6$
(g) $16ba^4 - 81b$
(h) $a^2 - 10a + 25$
(i) $16b^2 + 56b + 49$
(j) $2a^2 - 12ab + 18b^2$
(k) $-4b^2 - 144b^8 + 48b^5$
(l) $a^3 - 27$
(m) $125a^3 + b^3$
(n) $128b^7 - 250ba^6$
(o) $c^3 + 27$
(p) $64b^3 + 1$
(q) $5a^3 - 40c^3$
(r) $2b^4 - 128b$

2. Show that $(2x-1)^2 - (x-3)^2$ can be simplified to $(x+2)(3x-4)$

3. What must be added to $x^2 - x + 4$ to make it equal to $(x+2)^2$

Chapter 10

Equations and Inequalities - Grade 10

10.1 Strategy for Solving Equations

This chapter is all about solving different types of equations for one or two variables. In general, we want to get the unknown variable alone on the left hand side of the equation with all the constants on the right hand side of the equation. For example, in the equation $x - 1 = 0$, we want to be able to write the equation as $x = 1$.

As we saw in section 2.9 (page 13), an equation is like a set of weighing scales, that must always be balanced. When we solve equations, we need to keep in mind that what is done to one side must be done to the other.

Method: Rearranging Equations

You can add, subtract, multiply or divide both sides of an equation by any number you want, as long as you always do it to both sides.

For example, in the equation $x+5-1 = -6$, we want to get x alone on the left hand side of the equation. This means we need to subtract 5 and add 1 on the left hand side. However, because we need to keep the equation balanced, we also need to subtract 5 and add 1 on the right hand side.

$$\begin{aligned} x+5-1 &= -6 \\ x+5-5-1+1 &= -6-5+1 \\ x+0+0 &= -11+1 \\ x &= -10 \end{aligned}$$

In another example, $\frac{2}{3}x = 8$, we must divide by 2 and multiply by 3 on the left hand side in order to get x alone. However, in order to keep the equation balanced, we must also divide by 2 and multiply by 3 on the right hand side.

$$\begin{aligned} \frac{2}{3}x &= 8 \\ \frac{2}{3}x \div 2 \times 3 &= 8 \div 2 \times 3 \\ \frac{2}{2} \times \frac{3}{3} \times x &= \frac{8 \times 3}{2} \\ 1 \times 1 \times x &= 12 \\ x &= 12 \end{aligned}$$

These are the basic rules to apply when simplifying any equation. In most cases, these rules have to be applied more than once, before we have the unknown variable on the left hand side of the equation.

We are now ready to solve some equations!

Important: The following must also be kept in mind:

1. Division by 0 is undefined.
2. If $\frac{x}{y} = 0$, then $x = 0$ and $y \neq 0$, because division by 0 is undefined.

Activity :: Investigation : Strategy for Solving Equations
In the following, identify what is wrong.

$$\begin{aligned} 4x - 8 &= 3(x-2) \\ 4(x-2) &= 3(x-2) \\ \frac{4(x-2)}{(x-2)} &= \frac{3(x-2)}{(x-2)} \\ 4 &= 3 \end{aligned}$$

10.2 Solving Linear Equations

The simplest equation to solve is a linear equation. A linear equation is an equation where the power on the variable(letter, e.g. x) is 1(one). The following are examples of linear equations.

$$\begin{aligned} 2x + 2 &= 1 \\ \frac{2-x}{3x+1} &= 2 \\ \frac{4}{3}x - 6 &= 7x + 2 \end{aligned}$$

In this section, we will learn how to find the value of the variable that makes both sides of the linear equation true. For example, what value of x makes both sides of the very simple equation, $x + 1 = 1$ true.

Since the highest power on the variable is one(1) in a linear equation, there is at most *one solution* or *root* for the equation.

This section relies on all the methods we have already discussed: multiplying out expressions, grouping terms and factorisation. Make sure that you are comfortable with these methods, before trying out the work in the rest of this chapter.

$$\begin{aligned} 2x + 2 &= 1 \\ 2x &= 1 - 2 \quad \text{(like terms together)} \\ 2x &= -1 \quad \text{(simplified as much a possible)} \end{aligned}$$

Now we see that $2x = -1$. This means if we divide both sides by 2, we will get:

$$x = -\frac{1}{2}$$

If we substitute $x = -\frac{1}{2}$, back into the original equation, we get:

$$\begin{aligned} & 2x + 2 \\ = & \ 2(-\frac{1}{2}) + 2 \\ = & \ -1 + 2 \\ = & \ 1 \end{aligned}$$

That is all that there is to solving linear equations.

Important: Solving Equations

When you have found the solution to an equation, substitute the solution into the original equation, to check your answer.

Method: Solving Equations

The general steps to solve equations are:

1. Expand(Remove) all brackets.
2. "Move" all terms with the variable to the left hand side of equation, and all constant terms (the numbers) to the right hand side of the equal to-sign. Bearing in mind that the sign of the terms will chance(from (+) to (-) or vice versa, as they "cross over" the equal to sign.
3. Group all like terms together and simplify as much as possible.
4. Factorise if necessary.
5. Find the solution.
6. Substitute solution into **original** equation to check answer.

Worked Example 28: Solving Linear Equations

Question: Solve for x: $4 - x = 4$
Answer
Step 1 : Determine what is given and what is required
We are given $4 - x = 4$ and are required to solve for x.
Step 2 : Determine how to approach the problem
Since there are no brackets, we can start with grouping like terms and then simplifying.
Step 3 : Solve the problem

$$\begin{aligned} 4 - x &= 4 \\ -x &= 4 - 4 \quad \text{(move all constant terms (numbers) to the RHS (right hand side))} \\ -x &= 0 \quad \text{(group like terms together)} \\ -x &= 0 \quad \text{(simplify grouped terms)} \\ -x &= 0 \\ \therefore \quad x &= 0 \end{aligned}$$

Step 4 : Check the answer

Substitute solution into original equation:

$$4 - 0 = 4$$

$$4 = 4$$

Since both sides are equal, the answer is correct.

Step 5 : Write the Final Answer

The solution of $4 - x = 4$ is $x = 0$.

Worked Example 29: Solving Linear Equations

Question: Solve for x: $4(2x - 9) - 4x = 4 - 6x$

Answer

Step 1 : Determine what is given and what is required

We are given $4(2x - 9) - 4x = 4 - 6x$ and are required to solve for x.

Step 2 : Determine how to approach the problem

We start with expanding the brackets, then grouping like terms and then simplifying.

Step 3 : Solve the problem

$$\begin{aligned}
4(2x-9)-4x &= 4-6x \\
8x-36-4x &= 4-6x \quad \text{(expand the brackets)} \\
8x-4x+6x &= 4+36 \quad \text{(move all terms with x to the LHS and all constant terms to the RHS of the =)} \\
(8x-4x+6x) &= (4+36) \quad \text{(group like terms together)} \\
10x &= 40 \quad \text{(simplify grouped terms)} \\
\frac{10}{10}x &= \frac{40}{10} \quad \text{(divide both sides by 10)} \\
x &= 4
\end{aligned}$$

Step 4 : Check the answer

Substitute solution into original equation:

$$\begin{aligned}
4(2(4)-9)-4(4) &= 4-6(4) \\
4(8-9)-16 &= 4-24 \\
4(-1)-16 &= -20 \\
-4-16 &= -20 \\
-20 &= -20
\end{aligned}$$

Since both sides are equal to -20, the answer is correct.

Step 5 : Write the Final Answer

The solution of $4(2x - 9) - 4x = 4 - 6x$ is $x = 4$.

Worked Example 30: Solving Linear Equations

Question: Solve for x: $\frac{2-x}{3x+1} = 2$

Answer

Step 1 : Determine what is given and what is required

We are given $\frac{2-x}{3x+1} = 2$ and are required to solve for x.

Step 2 : Determine how to approach the problem

Since there is a denominator of (3x+1), we can start by multiplying both sides of the equation by (3x+1). But because division by 0 is not permissible, there is a restriction on a value for x. ($x \neq \frac{-1}{3}$)

Step 3 : Solve the problem

$$\begin{aligned}
\frac{2-x}{3x+1} &= 2 \\
(2-x) &= 2(3x+1) \\
2-x &= 6x+2 \quad \text{(remove/expand brackets)} \\
-x-6x &= 2-2 \quad \text{(move all terms containing x to the LHS and all constant terms (numbers) to the RHS.} \\
-7x &= 0 \qquad \text{(simplify grouped terms)} \\
x &= 0 \div (-7) \\
therefore \quad x &= 0 \qquad \text{zero divide by any number is 0}
\end{aligned}$$

Step 4 : Check the answer

Substitute solution into original equation:

$$\begin{aligned}
\frac{2-(0)}{3(0)+1} &= 2 \\
\frac{2}{1} &= 2
\end{aligned}$$

Since both sides are equal to 2, the answer is correct.

Step 5 : Write the Final Answer

The solution of $\frac{2-x}{3x+1} = 2$ is $x = 0$.

Worked Example 31: Solving Linear Equations

Question: Solve for x: $\frac{4}{3}x - 6 = 7x + 2$

Answer

Step 1 : Determine what is given and what is required

We are given $\frac{4}{3}x - 6 = 7x + 2$ and are required to solve for x.

Step 2 : Determine how to approach the problem

We start with multiplying each of the terms in the equation by 3, then grouping like terms and then simplifying.

Step 3 : Solve the problem

$$\begin{aligned}
\frac{4}{3}x - 6 &= 7x + 2 \\
4x - 18 &= 21x + 6 \quad \text{(each term is multiplied by 3} \\
4x - 21x &= 6 + 18 \quad \text{(move all terms with } x \text{ to the LHS and all constant terms to the RHS of the } =) \\
-17x &= 24 \qquad \text{(simplify grouped terms)} \\
\frac{-17}{-17}x &= \frac{24}{-17} \qquad \text{(divide both sides by -17)} \\
x &= \frac{-24}{17}
\end{aligned}$$

Step 4 : Check the answer
Substitute solution into original equation:

$$\begin{aligned}
\frac{4}{3} \times \frac{-24}{17} - 6 &= 7 \times \frac{-24}{17} + 2 \\
\frac{4 \times (-8)}{(17)} - 6 &= \frac{7 \times (-24)}{17} + 2 \\
\frac{(-32)}{17} - 6 &= \frac{-168}{17} + 2 \\
\frac{-32 - 102}{17} &= \frac{(-168) + 34}{17} \\
\frac{-134}{17} &= \frac{-134}{17}
\end{aligned}$$

Since both sides are equal to $\frac{-134}{17}$, the answer is correct.

Step 5 : Write the Final Answer
The solution of $\frac{4}{3}x - 6 = 7x + 2$ is, $x = \frac{-24}{17}$.

?

Exercise: Solving Linear Equations

1. Solve for y: $2y - 3 = 7$
2. Solve for w: $-3w = 0$
3. Solve for z: $4z = 16$
4. Solve for t: $12t + 0 = 144$
5. Solve for x: $7 + 5x = 62$
6. Solve for y: $55 = 5y + \frac{3}{4}$
7. Solve for z: $5z = 3z + 45$
8. Solve for a: $23a - 12 = 6 + 2a$
9. Solve for b: $12 - 6b + 34b = 2b - 24 - 64$
10. Solve for c: $6c + 3c = 4 - 5(2c - 3)$.
11. Solve for p: $18 - 2p = p + 9$
12. Solve for q: $\frac{4}{q} = \frac{16}{24}$
13. Solve for q: $\frac{4}{1} = \frac{q}{2}$
14. Solve for r: $-(-16 - r) = 13r - 1$
15. Solve for d: $6d - 2 + 2d = -2 + 4d + 8$
16. Solve for f: $3f - 10 = 10$
17. Solve for v: $3v + 16 = 4v - 10$
18. Solve for k: $10k + 5 + 0 = -2k + -3k + 80$
19. Solve for j: $8(j - 4) = 5(j - 4)$
20. Solve for m: $6 = 6(m + 7) + 5m$

10.3 Solving Quadratic Equations

A quadratic equation is an equation where the power on the variable is at most 2. The following are examples of quadratic equations.

$$\begin{aligned} 2x^2 + 2x &= 1 \\ \frac{2-x}{3x+1} &= 2x \\ \frac{4}{3}x - 6 &= 7x^2 + 2 \end{aligned}$$

Quadratic equations differ from linear equations by the fact that a linear equation only has one solution, while a quadratic equation has *at most* two solutions. There are some special situations when a quadratic equation only has one solution.

We solve quadratic equations by factorisation, that is writing the quadratic as a product of two expressions in brackets. For example, we know that:

$$(x+1)(2x-3) = 2x^2 - x - 3.$$

In order to solve:

$$2x^2 - x - 3 = 0$$

we need to be able to write $2x^2 - x - 3$ as $(x+1)(2x-3)$, which we already know how to do.

Activity :: Investigation : Factorising a Quadratic
Factorise the following quadratic expressions:

1. $x + x^2$
2. $x^2 + 1 + 2x$
3. $x^2 - 4x + 5$
4. $16x^2 - 9$
5. $4x^2 + 4x + 1$

Being able to factorise a quadratic means that you are one step away from solving a quadratic equation. For example, $x^2 - 3x - 2 = 0$ can be written as $(x-1)(x-2) = 0$. This means that both $x - 1 = 0$ and $x - 2 = 0$, which gives $x = 1$ and $x = 2$ as the two solutions to the quadratic equation $x^2 - 3x - 2 = 0$.

Method: Solving Quadratic Equations

1. First divide the entire equation by any common factor of the coefficients, so as to obtain an equation of the form $ax^2 + bx + c = 0$ where a, b and c have no common factors. For example, $2x^2 + 4x + 2 = 0$ can be written as $x^2 + 2x + 1 = 0$ by dividing by 2.
2. Write $ax^2 + bx + c$ in terms of its factors $(rx + s)(ux + v)$.
   This means $(rx + s)(ux + v) = 0$.
3. Once writing the equation in the form $(rx + s)(ux + v) = 0$, it then follows that the two solutions are $x = -\frac{s}{r}$ or $x = -\frac{u}{v}$.

Extension: Solutions of Quadratic Equations
There are two solutions to a quadratic equation, because any **one** of the values can solve the equation.

Worked Example 32: Solving Quadratic Equations

Question: Solve for x: $3x^2 + 2x - 1 = 0$
Answer
Step 1 : Find the factors of $3x^2 + 2x - 1$
As we have seen the factors of $3x^2 + 2x - 1$ are $(x + 1)$ and $(3x - 1)$.
Step 2 : Write the equation with the factors

$$(x + 1)(3x - 1) = 0$$

Step 3 : Determine the two solutions
We have

$$x + 1 = 0$$

or

$$3x - 1 = 0$$

Therefore, $x = -1$ or $x = \frac{1}{3}$.
Step 4 : Write the final answer
$3x^2 + 2x - 1 = 0$ for $x = -1$ or $x = \frac{1}{3}$.

Worked Example 33: Solving Quadratic Equations

Question: Solve for x: $\sqrt{x + 2} = x$
Answer
Step 1 : Square both sides of the equation
Both sides of the equation should be squared to remove the square root sign.

$$x + 2 = x^2$$

Step 2 : Write equation in the form $ax^2 + bx + c = 0$

$$\begin{aligned} x + 2 &= x^2 \quad \text{(subtract } x^2 \text{ to both sides)} \\ x + 2 - x^2 &= 0 \quad \text{(divide both sides by -1)} \\ -x - 2 + x^2 &= 0 \\ x^2 - x + 2 &= 0 \end{aligned}$$

Step 3 : Factorise the quadratic

$$x^2 - x + 2$$

The factors of $x^2 - x + 2$ are $(x - 2)(x + 1)$.

Step 4 : Write the equation with the factors

$$(x - 2)(x + 1) = 0$$

Step 5 : Determine the two solutions
We have

$$x + 1 = 0$$

or

$$x - 2 = 0$$

Therefore, $x = -1$ or $x = 2$.

Step 6 : Check whether solutions are valid
Substitute $x = -1$into the original equation $\sqrt{x+2} = x$:

$$\begin{aligned} LHS &= \sqrt{(-1)+2} \\ &= \sqrt{1} \\ &= 1 \\ but & \\ RHS &= (-1) \end{aligned}$$

Therefore LHS$\neq$RHS
Therefore $x \neq -1$
Now substitute $x = 2$ into original equation $\sqrt{x+2} = x$:

$$\begin{aligned} LHS &= \sqrt{2+2} \\ &= \sqrt{4} \\ &= 2 \\ and & \\ RHS &= 2 \end{aligned}$$

Therefore LHS = RHS
Therefore $x = 2$ is the only valid solution
Step 7 : Write the final answer
$\sqrt{x+2} = x$ for $x = 2$ only.

Worked Example 34: Solving Quadratic Equations

Question: Solve the equation: $x^2 + 3x - 4 = 0$.
Answer
Step 1 : Check if the equation is in the form $ax^2 + bx + c = 0$
The equation is in the required form, with $a = 1$.
Step 2 : Factorise the quadratic
You need the factors of 1 and 4 so that the middle term is $+3$ So the factors are:
$(x - 1)(x + 4)$
Step 3 : Solve the quadratic equation

$$x^2 + 3x - 4 = (x - 1)(x + 4) = 0 \tag{10.1}$$

Therefore $x = 1$ or $x = -4$.
Step 4 : Write the final solution
Therefore the solutions are $x = 1$ or $x = -4$.

Worked Example 35: Solving Quadratic Equations

Question: Find the roots of the quadratic equation $0 = -2x^2 + 4x - 2$.
Answer
Step 1 : Determine whether the equation is in the form $ax^2 + bx + c = 0$, with no common factors.
There is a common factor: -2. Therefore, divide both sides of the equation by -2.

$$\begin{aligned} -2x^2 + 4x - 2 &= 0 \\ x^2 - 2x + 1 &= 0 \end{aligned}$$

Step 2 : Factorise $x^2 - 2x + 1$
The middle term is negative. Therefore, the factors are $(x - 1)(x - 1)$
If we multiply out $(x - 1)(x - 1)$, we get $x^2 - 2x + 1$.
Step 3 : Solve the quadratic equation

$$x^2 - 2x + 1 = (x - 1)(x - 1) = 0$$

In this case, the quadratic is a perfect square, so there is only one solution for x: $x = 1$.
Step 4 : Write the final solution
The root of $0 = -2x^2 + 4x - 2$ is $x = 1$.

Exercise: Solving Quadratic Equations

1. Solve for x: $(3x + 2)(3x - 4) = 0$
2. Solve for a: $(5a - 9)(a + 6) = 0$
3. Solve for x: $(2x + 3)(2x - 3) = 0$
4. Solve for x: $(2x + 1)(2x - 9) = 0$
5. Solve for x: $(2x - 3)(2x - 3) = 0$
6. Solve for x: $20x + 25x^2 = 0$
7. Solve for a: $4a^2 - 17a - 77 = 0$
8. Solve for x: $2x^2 - 5x - 12 = 0$
9. Solve for b: $-75b^2 + 290b - 240 = 0$
10. Solve for y: $2y = \frac{1}{3}y^2 - 3y + 14\frac{2}{3}$
11. Solve for θ: $\theta^2 - 4\theta = -4$
12. Solve for q: $-q^2 + 4q - 6 = 4q^2 - 5q + 3$
13. Solve for t: $t^2 = 3t$
14. Solve for w: $3w^2 + 10w - 25 = 0$
15. Solve for v: $v^2 - v + 3$
16. Solve for x: $x^2 - 4x + 4 = 0$
17. Solve for t: $t^2 - 6t = 7$
18. Solve for x: $14x^2 + 5x = 6$
19. Solve for t: $2t^2 - 2t = 12$
20. Solve for y: $3y^2 + 2y - 6 = y^2 - y + 2$

10.4 Exponential Equations of the form $ka^{(x+p)} = m$

examples solved by trial and error)

Exponential equations generally have the unknown variable as the power. The following are examples of exponential equations:

$$\begin{aligned} 2^x &= 1 \\ \frac{2^{-x}}{3^{x+1}} &= 2 \\ \frac{4}{3} - 6 &= 7^x + 2 \end{aligned}$$

You should already be familiar with exponential notation. Solving exponential equations are simple, if we remember how to apply the laws of exponentials.

Activity :: Investigation : Solving Exponential Equations
Solve the following equations by completing the table:

$2^x = 2$	x						
	-3	-2	-1	0	1	2	3
2^x							

$3^x = 9$	x						
	-3	-2	-1	0	1	2	3
3^x							

$2^{x+1} = 8$	x						
	-3	-2	-1	0	1	2	3
2^{x+1}							

10.4.1 Algebraic Solution

Definition: Equality for Exponential Functions
If a is a positive number such that $a > 0$, then:

$$a^x = a^y$$

if and only if:

$$x = y$$

.

This means that if we can write all terms in an equation with the same base, we can solve the exponential equations by equating the indices. For example take the equation $3^{x+1} = 9$. This can be written as:

$$3^{x+1} = 3^2.$$

Since the bases are equal (to 3), we know that the exponents must also be equal. Therefore we can write:

$$x + 1 = 2.$$

This gives:

$$x = 1.$$

Method: Solving Exponential Equations

Try to write all terms with the same base.

Activity :: Investigation : Exponential Numbers

Write the following with the same base. The base is the first in the list. For example, in the list 2, 4, 8, the base is two and we can write 4 as 2^2.

1. 2,4,8,16,32,64,128,512,1024
2. 3,9,27,81,243
3. 5,25,125,625
4. 13,169
5. $2x$, $4x^2$, $8x^3$, $49x^8$

Worked Example 36: Solving Exponential Equations

Question: Solve for x: $2^x = 2$

Answer

Step 1 : Try to write all terms with the same base.

All terms are written with the same base.

$$2^x = 2^1$$

Step 2 : Equate the indices

$$x = 1$$

Step 3 : Check your answer

$$\begin{aligned} & 2^x \\ = & \ 2^{(1)} \\ = & \ 2^1 \end{aligned}$$

Since both sides are equal, the answer is correct.

Step 4 : Write the final answer

$$x = 1$$

is the solution to $2^x = 2$.

Worked Example 37: Solving Exponential Equations

Question: Solve:

$$2^{x+4} = 4^{2x}$$

Answer

Step 1 : Try to write all terms with the same base.

$$\begin{aligned} 2^{x+4} &= 4^{2x} \\ 2^{x+4} &= 2^{2(2x)} \\ 2^{x+4} &= 2^{4x} \end{aligned}$$

Step 2 : Equate the indices

$$x + 4 = 4x$$

Step 3 : Solve for x

$$\begin{aligned} x+4 &= 4x \\ x-4x &= -4 \\ -3x &= -4 \\ x &= \frac{-4}{-3} \\ x &= \frac{4}{3} \end{aligned}$$

Step 4 : Check your answer

$$\begin{aligned} \text{LHS} &= 2^{x+4} \\ &= 2^{(\frac{4}{3}+4)} \\ &= 2^{\frac{16}{3}} \\ &= (2^{16})^{\frac{1}{3}} \\ \text{RHS} &= 4^{2x} \\ &= 4^{2(\frac{4}{3})} \\ &= 4^{\frac{8}{3}} \\ &= (4^{8})^{\frac{1}{3}} \\ &= ((2^2)^8)^{\frac{1}{3}} \\ &= (2^{16})^{\frac{1}{3}} \\ &= \text{LHS} \end{aligned}$$

Since both sides are equal, the answer is correct.

Step 5 : Write the final answer

$$x = \frac{4}{3}$$

is the solution to $2^{x+4} = 4^{2x}$.

?

Exercise: Solving Exponential Equations

1. Solve the following exponential equations.

   a. $2^{x+5} = 2^5$
   b. $3^{2x+1} = 3^3$
   c. $5^{2x+2} = 5^3$
   d. $6^{5-x} = 6^{12}$
   e. $64^{x+1} = 16^{2x+5}$
   f. $125^x = 5$

2. Solve: $3^{9x-2} = 27$

3. Solve for k: $81^{k+2} = 27^{k+4}$
4. The growth of an algae in a pond is can be modeled by the function $f(t) = 2^t$. Find the value of t such that $f(t) = 128$?
5. Solve for x: $25^{(1-2x)} = 5^4$
6. Solve for x: $27^x \times 9^{x-2} = 1$

10.5 Linear Inequalities

graphically;

Activity :: Investigation : Inequalities on a Number Line
Represent the following on number lines:

1. $x = 4$
2. $x < 4$
3. $x \leq 4$
4. $x \geq 4$
5. $x > 4$

A linear inequality is similar to a linear equation and has the power on the variable is equal to 1. The following are examples of linear inequalities.

$$\begin{aligned} 2x + 2 &\leq 1 \\ \frac{2-x}{3x+1} &\geq 2 \\ \frac{4}{3}x - 6 &< 7x + 2 \end{aligned}$$

The methods used to solve linear inequalities are identical to those used to solve linear equations. The only difference occurs when there is a multiplication or a division that involves a minus sign. For example, we know that $8 > 6$. If both sides of the inequality are divided by -2, -4 is not greater than -3. Therefore, the inequality must switch around, making $-4 < -3$.

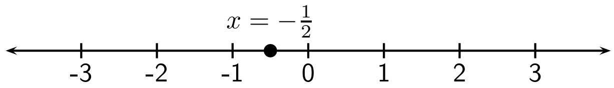
Important: When you divide or multiply both sides of an inequality by any number with a minus sign, the direction of the inequality changes.

For example, if $x < 1$, then $-x > -1$.

In order to compare am inequality to a normal equation, we shall solve an equation first. Solve $2x + 2 = 1$.

$$\begin{aligned} 2x + 2 &= 1 \\ 2x &= 1 - 2 \\ 2x &= -1 \\ x &= -\frac{1}{2} \end{aligned}$$

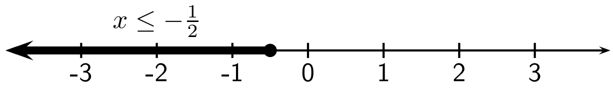
If we represent this answer on a number line, we get



Now let us solve the inequality $2x + 2 \leq 1$.

$$\begin{aligned} 2x + 2 &\leq 1 \\ 2x &\leq 1 - 2 \\ 2x &\leq -1 \\ x &\leq -\frac{1}{2} \end{aligned}$$

If we represent this answer on a number line, we get



As you can see, for the equation, there is only a single value of x for which the equation is true. However, for the inequality, there is a range of values for which the inequality is true. This is the main difference between an equation and an inequality.

Worked Example 38: Linear Inequalities

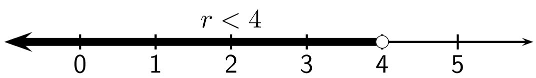
Question: Solve for r: $6 - r > 2$
Answer
Step 1 : Move all constants to the RHS

$$\begin{aligned} -r &> 2 - 6 \\ -r &> -4 \end{aligned}$$

Step 2 : Multiply both sides by -1
When you multiply by a minus sign, the direction of the inequality changes.

$$r < 4$$

Step 3 : Represent answer graphically



Worked Example 39: Linear Inequalities

Question: Solve for q: $4q + 3 < 2(q + 3)$ and represent solution on a number line.
Answer
Step 1 : Expand all brackets

$$\begin{aligned} 4q + 3 &< 2(q + 3) \\ 4q + 3 &< 2q + 6 \end{aligned}$$

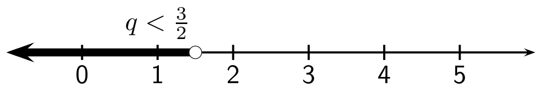
Step 2 : Move all constants to the RHS and all unknowns to the LHS

$$\begin{aligned} 4q + 3 &< 2q + 6 \\ 4q - 2q &< 6 - 3 \\ 2q &< 3 \end{aligned}$$

Step 3 : Solve inequality

$$\begin{aligned} 2q &< 3 \quad \text{Divide both sides by 2} \\ q &< \frac{3}{2} \end{aligned}$$

Step 4 : Represent answer graphically



Worked Example 40: Compound Linear Inequalities

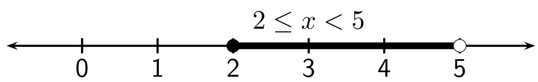
Question: Solve for x: $5 \leq x + 3 < 8$ and represent solution on a number line.

Answer

Step 1 : Subtract 3 from Left, middle and right of inequalities

$$\begin{aligned} 5 - 3 \leq \quad & x + 3 - 3 \quad < 8 - 3 \\ 2 \leq \quad & x \quad < 5 \end{aligned}$$

Step 2 : Represent answer graphically



Exercise: Linear Inequalities

1. Solve for x and represent the solution graphically:
   (a) $3x + 4 > 5x + 8$
   (b) $3(x - 1) - 2 \leq 6x + 4$
   (c) $\frac{x-7}{3} > \frac{2x-3}{2}$
   (d) $-4(x - 1) < x + 2$
   (e) $\frac{1}{2}x + \frac{1}{3}(x - 1) \geq \frac{5}{6}x - \frac{1}{3}$
2. Solve the following inequalities. Illustrate your answer on a number line if x is a real number.
   (a) $-2 \leq x - 1 < 3$

(b) $-5 < 2x - 3 \leq 7$

3. Solve for x: $7(3x + 2) - 5(2x - 3) > 7$.
Illustrate this answer on a number line.

10.6 Linear Simultaneous Equations

Thus far, all equations that have been encountered have one unknown variable, that must be solved for. When two unknown variables need to be solved for, two equations are required and these equations are known as simultaneous equations. The solutions to the system of simultaneous equations, are the values of the unknown variables which satisfy the system of equations simultaneously, that means at the same time. In general, if there are n unknown variables, then n equations are required to obtain a solution for each of the n variables.

An example of a system of simultaneous equations is:

$$\begin{aligned} 2x + 2y &= 1 \\ \frac{2-x}{3y+1} &= 2 \end{aligned} \tag{10.2}$$

10.6.1 Finding solutions

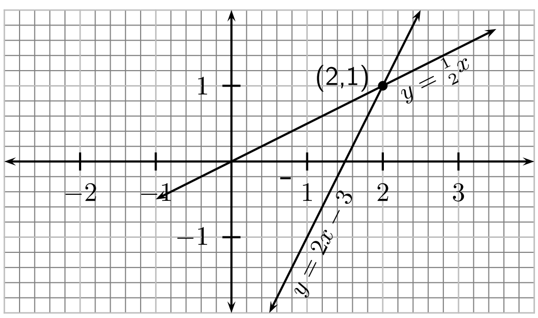
In order to find a numerical value for an unknown variable, one must have at least as many independent equations as variables. We solve simultaneous equations graphically and algebraically/

10.6.2 Graphical Solution

Simultaneous equations can also be solved graphically. If the graphs corresponding to each equation is drawn, then the solution to the system of simultaneous equations is the co-ordinate of the point at which both graphs intersect.

$$\begin{aligned} x &= 2y \\ y &= 2x - 3 \end{aligned} \tag{10.3}$$

Draw the graphs of the two equations in (10.3).



The intersection of the two graphs is (2,1). So the solution to the system of simultaneous equations in (10.3) is $y = 1$ and $x = 2$.

This can be shown algebraically as:

$$
\begin{aligned}
x &= 2y \\
\therefore \quad y &= 2(2y) - 3 \\
y - 4y &= -3 \\
-3y &= -3 \\
y &= 1 \\
\text{Substitute into the first equation:} \quad x &= 2(1) \\
&= 2
\end{aligned}
$$

Worked Example 41: Simultaneous Equations

Question: Solve the following system of simultaneous equations graphically.

$$
\begin{aligned}
4y + 3x &= 100 \\
4y - 19x &= 12
\end{aligned}
$$

Answer

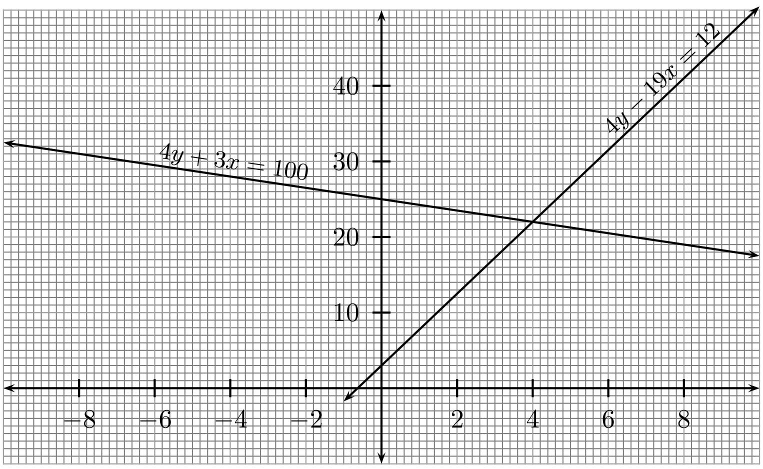
Step 1 : Draw the graphs corresponding to each equation.

For the first equation:

$$
\begin{aligned}
4y + 3x &= 100 \\
4y &= 100 - 3x \\
y &= 25 - \frac{3}{4}x
\end{aligned}
$$

and for the second equation:

$$
\begin{aligned}
4y - 19x &= 12 \\
4y &= 19x + 12 \\
y &= \frac{19}{4}x + 3
\end{aligned}
$$



Step 2 : Find the intersection of the graphs.

The graphs intersect at (4,22).

Step 3 : Write the solution of the system of simultaneous equations as given by the intersection of the graphs.

$$
\begin{aligned}
x &= 4 \\
y &= 22
\end{aligned}
$$

10.6.3 Solution by Substitution

A common algebraic technique is the substitution method: try to solve one of the equations for one of the variables and substitute the result into the other equations, thereby reducing the number of equations and the number of variables by 1. Continue until you reach a single equation with a single variable, which (hopefully) can be solved; back substitution then yields the values for the other variables.

In the example (**??**), we first solve the first equation for x:

$$x = \frac{1}{2} - y$$

and substitute this result into the second equation:

$$\begin{aligned}
\frac{2-x}{3y+1} &= 2 \\
\frac{2-(\frac{1}{2}-y)}{3y+1} &= 2 \\
2-(\frac{1}{2}-y) &= 2(3y+1) \\
2-\frac{1}{2}+y &= 6y+2 \\
y-6y &= -2+\frac{1}{2}+2 \\
-5y &= \frac{1}{2} \\
y &= -\frac{1}{10}
\end{aligned}$$

$$\begin{aligned}
\therefore \quad x &= \frac{1}{2} - y \\
&= \frac{1}{2} - (-\frac{1}{10}) \\
&= \frac{6}{10} \\
&= \frac{3}{5}
\end{aligned}$$

The solution for the system of simultaneous equations (**??**) is:

$$\begin{aligned}
x &= \frac{3}{5} \\
y &= -\frac{1}{10}
\end{aligned}$$

Worked Example 42: Simultaneous Equations

Question: Solve the following system of simultaneous equations:

$$\begin{aligned}
4y+3x &= 100 \\
4y-19x &= 12
\end{aligned}$$

Answer

Step 1 : If the question, does not explicitly ask for a graphical solution, then the system of equations should be solved algebraically.

Step 2 : Make x the subject of the first equation.

$$\begin{aligned} 4y + 3x &= 100 \\ 3x &= 100 - 4y \\ x &= \frac{100 - 4y}{3} \end{aligned}$$

Step 3 : Substitute the value obtained for x into the second equation.

$$\begin{aligned} 4y - 19(\frac{100 - 4y}{3}) &= 12 \\ 12y - 19(100 - 4y) &= 36 \\ 12y - 1900 + 76y &= 36 \\ 88y &= 1936 \\ y &= 22 \end{aligned}$$

Step 4 : Substitute into the equation for x.

$$\begin{aligned} x &= \frac{100 - 4(22)}{3} \\ &= \frac{100 - 88}{3} \\ &= \frac{12}{3} \\ &= 4 \end{aligned}$$

Step 5 : Substitute the values for x and y into both equations to check the solution.

$$\begin{aligned} 4(22) + 3(4) = 88 + 12 &= 100 \quad \checkmark \\ 4(22) - 19(4) = 88 - 76 &= 12 \quad \checkmark \end{aligned}$$

Worked Example 43: Bicycles and Tricycles

Question: A shop sells bicycles and tricycles. In total there are 7 cycles and 19 wheels. Determine how many of each there are, if a bicycle has two wheels and a tricycle has three wheels.

Answer

Step 1 : Identify what is required

The number of bicycles and the number of tricycles are required.

Step 2 : Set up the necessary equations

If b is the number of bicycles and t is the number of tricycles, then:

$$\begin{aligned} b + t &= 7 \\ 2b + 3t &= 19 \end{aligned}$$

Step 3 : Solve the system of simultaneous equations using substitution.

$$
\begin{aligned}
b &= 7-t \\
\text{Into second equation: } 2(7-t)+3t &= 19 \\
14-2t+3t &= 19 \\
t &= 5 \\
\text{Into first equation: } b &= 7-5 \\
&= 2
\end{aligned}
$$

Step 4 : Check solution by substituting into original system of equations.

$$
\begin{aligned}
2+5 &= 7 \quad \checkmark \\
2(2)+3(5)=4+15 &= 19 \quad \checkmark
\end{aligned}
$$

?

Exercise: Simultaneous Equations

1. Solve graphically and confirm your answer algebraically: $3a - 2b7 = 0$, $a - 4b + 1 = 0$
2. Solve algebraically: $15c + 11d - 132 = 0$, $2c + 3d - 59 = 0$
3. Solve algebraically: $-18e - 18 + 3f = 0$, $e - 4f + 47 = 0$
4. Solve graphically: $x + 2y = 7$, $x + y = 0$

10.7 Mathematical Models

10.7.1 Introduction

Tom and Jane are friends. Tom picked up Jane's Physics test paper, but will not tell Jane what her marks are. He knows that Jane hates maths so he decided to tease her. Tom says: "I have 2 marks more than you do and the sum of both our marks is equal to 14. How much did we get?"

Let's help Jane find out what her marks are. We have two unknowns, Tom's mark (which we shall call t) and Jane's mark (which we shall call j). Tom has 2 more marks than Jane. Therefore,

$$t = j + 2$$

Also, both marks add up to 14. Therefore,

$$t + j = 14$$

The two equations make up a set of linear (because the highest power is one) simultaneous equations, which we know how to solve! Substitute for t in the second equation to get:

$$\begin{aligned} t+j &= 14 \\ j+2+j &= 14 \\ 2j+2 &= 14 \\ 2(j+1) &= 14 \\ j+1 &= 7 \\ j &= 7-1 \\ &= 6 \end{aligned}$$

Then,

$$\begin{aligned} t &= j+2 \\ &= 6+2 \\ &= 8 \end{aligned}$$

So, we see that Tom scored 8 on his test and Jane scored 6.

This problem is an example of a simple *mathematical model*. We took a problem and we able to write a set of equations that represented the problem, mathematically. The solution of the equations then gave the solution to the problem.

10.7.2 Problem Solving Strategy

The purpose of this section is to teach you the skills that you need to be able to take a problem and formulate it mathematically, in order to solve it. The general steps to follow are:

1. Read ALL of it !
2. Find out what is requested.
3. Let the requested be a variable e.g. x.
4. Rewrite the information given in terms of x. That is, translate the words into algebraic language. This is the reponse
5. Set up an equation (i.e. a mathematical sentence or model) to solve the required variable.
6. Solve the equation algebraically to find the result.

Important: Follow the three R's and solve the problem... **Request** - **Response** - **Result**

10.7.3 Application of Mathematical Modelling

Worked Example 44: Mathematical Modelling: One variable

Question: A fruit shake costs R2,00 more than a chocolate milkshake. If three fruit shakes and 5 chocolate milkshakes cost R78,00, determine the individual prices.

Answer

Step 1 : Summarise the information in a table

	Price	number	Total
Fruit	$x+2$	3	$3(x+2)$
Chocolate	x	5	$5x$

Step 2 : Set up an algebraic equation

$$3(x+2)+5x = 78$$

Step 3 : Solve the equation

$$\begin{aligned} 3x+6+5x &= 78 \\ 8x &= 72 \\ x &= 9 \end{aligned}$$

Step 4 : Present the final answer

Chocolate milkshake costs R 9,00 and the Fruitshake costs R 11,00

Worked Example 45: Mathematical Modelling: Two variables

Question: Three rulers and two pens cost R 21,00. One ruler and one pen cost R 8,00. Find the cost of one ruler and one pen

Answer

Step 1 : Translate the problem using variables

Let the cost of one ruler be x rand and the cost of one pen be y rand.

Step 2 : Rewrite the information in terms of the variables

$$3x+2y = 21 \qquad (10.4)$$

$$x+y = 8 \qquad (10.5)$$

Step 3 : Solve the equations simultaneously

First solve the second equation for y:

$$y = 8-x$$

and substitute the result into the first equation:

$$\begin{aligned} 3x+2(8-x) &= 21 \\ 3x+16-2x &= 21 \\ x &= 5 \end{aligned}$$

therefore

$$\begin{aligned} y &= 8-5 \\ y &= 3 \end{aligned}$$

Step 4 : Present the final answers

one Ruler costs R 5,00 and one Pen costs R 3,00

?

Exercise: Mathematical Models

1. Stephen has 1 l of a mixture containing 69% of salt. How much water must Stephen add to make the mixture 50% salt? Write your answer as a fraction.
2. The diagonal of a rectangle is 25 cm more than its width. The length of the rectangle is 17 cm more than its width. What are the dimensions of the rectangle?
3. The sum of 27 and 12 is 73 more than an unknown number. Find the unknown number.
4. The two smaller angles in a right-angled triangle are in the ratio of 1:2. What are the sizes of the two angles?
5. George owns a bakery that specialises in wedding cakes. For each wedding cake, it costs George R150 for ingredients, R50 for overhead, and R5 for advertising. George's wedding cakes cost R400 each. As a percentage of George's costs, how much profit does he make for each cake sold?
6. If 4 times a number is increased by 7, the result is 15 less than the square of the number. Find the numbers that satisfy this statement, by formulating an equation and then solving it.
7. The length of a rectangle is 2 cm more than the width of the rectangle. The perimeter of the rectangle is 20 cm. Find the length and the width of the rectangle.

10.7.4 End of Chapter Exercises

1. What are the roots of the quadratic equation $x^2 - 3x + 2 = 0$?
2. What are the solutions to the equation $x^2 + x = 6$?
3. In the equation $y = 2x^2 - 5x - 18$, which is a value of x when $y = 0$?
4. Manuel has 5 more CDs than Pedro has. Bob has twice as many CDs as Manuel has. Altogether the boys have 63 CDs. Find how many CDs each person has.
5. Seven-eighths of a certain number is 5 more than one-third of the number. Find the number.
6. A man runs to a telephone and back in 15 minutes. His speed on the way to the telephone is 5 m/s and his speed on the way back is 4 m/s. Find the distance to the telephone.
7. Solve the inequality and then answer the questions:
   $\frac{x}{3} - 14 > 14 - \frac{x}{4}$
   (a) If $x \epsilon R$, write the solution in interval notation.
   (b) if $x \epsilon Z$ and $x < 51$, write the solution as a set of integers.
8. Solve for a: $\frac{1-a}{2} - \frac{2-a}{3} > 1$
9. Solve for x: $x - 1 = \frac{42}{x}$
10. Solve for x and y: $7x + 3y = 13$ and $2x - 3y = -4$

chapterFunctions and Graphs - Grade 10

10.8 Introduction to Functions and Graphs

Functions are mathematical building blocks for designing machines, predicting natural disasters, curing diseases, understanding world economies and for keeping aeroplanes in the air. Functions can take input from many variables, but always give the same answer, unique to that function. It is the fact that you always get the same answer from a set of inputs, which is what makes functions special.

A major advantage of functions is that they allow us to *visualise* equations in terms of a *graph*. A graph is an accurate drawing of a function and is much easier to read than lists of numbers. In this chapter we will learn how to understand and create real valued functions, how to read graphs and how to draw them.

Despite their use in the problems facing humanity, functions also appear on a day-to-day level, so they are worth learning about. A function is always *dependent* on one or more variables, like time, distance or a more abstract quantity.

10.9 Functions and Graphs in the Real-World

Some typical examples of functions you may already have met include:-

- how much money you have, as a function of time. You never have more than one amount of money at any time because you can always add everything to give one number. By understanding how your money changes over time, you can plan to spend your money sensibly. Businesses find it very useful to *plot the graph* of their money over time so that they can see when they are spending too much. Such observations are not always obvious from looking at the numbers alone.
- the temperature is a very complicated function because it has so many inputs, including; the time of day, the season, the amount of clouds in the sky, the strength of the wind, where you are and many more. But the important thing is that there is only one temperature when you measure it. By understanding how the temperature is effected by these things, you can plan for the day.
- where you are is a function of time, because you cannot be in two places at once! If you were to *plot the graphs* of where two people are as a function of time, if the lines cross it means that the two people meet each other at that time. This idea is used in *logistics*, an area of mathematics that tries to plan where people and items are for businesses.
- your weight is a function of how much you eat and how much exercise you do, but everybody has a different function so that is why people are all different sizes.

10.10 Recap

The following should be familiar.

10.10.1 Variables and Constants

In section 2.4 (page 8), we were introduced to variables and constants. To recap, a *variable* can take any value in some set of numbers, so long is the equation is consistent. Most often, a variable will be written as a letter.

A *constant* has a fixed value. The number 1 is a constant. Sometimes letters are used to represent constants, as its easier to work with.

Activity :: Investigation : Variables and Constants
In the following expressions, identify the variables and the constants:

1. $2x^2 = 1$
2. $3x + 4y = 7$
3. $y = \frac{-5}{x}$
4. $y = 7^x - 2$

10.10.2 Relations and Functions

In earlier grades, you saw that variables can be *related* to each other. For example, Alan is two years older than Nathan. Therefore the relationship between the ages of Alan and Nathan can be written as $A = N + 2$, where A is Alan's age and N is Nathan's age.

In general, a *relation* is an equation which relates two variables. For example, $y = 5x$ and $y^2 + x^2 = 5$ are relations. In both examples x and y are variables and 5 is a constant, but for a given value of x the value of y will be very different in each relation.

Besides writing relations as equations, they can also be represented as words, tables and graphs. Instead of writing $y = 5x$, we could also say "y is always five times as big as x". We could also give the following table:

x	$y = 5x$
2	10
6	30
8	40
13	65
15	75

Activity :: Investigation : Relations and Functions
Complete the following table for the given functions:

x	$y = x$	$y = 2x$	$y = x + 2$
1			
2			
3			
50			
100			

10.10.3 The Cartesian Plane

When working with real valued functions, our major tool is drawing graphs. In the first place, if we have two real variables, x and y, then we can assign values to them simultaneously. That is, we can say "let x be 5 and y be 3". Just as we write "let $x = 5$" for "let x be 5", we have the shorthand notation "let $(x, y) = (5, 3)$" for "let x be 5 and y be 3". We usually think of the real numbers as an infinitely long line, and picking a number as putting a dot on that line. If we want to pick *two* numbers at the same time, we can do something similar, but now we must use two dimensions. What we do is use two lines, one for x and one for y, and rotate the one for y, as in Figure 10.1. We call this the *Cartesian plane*.

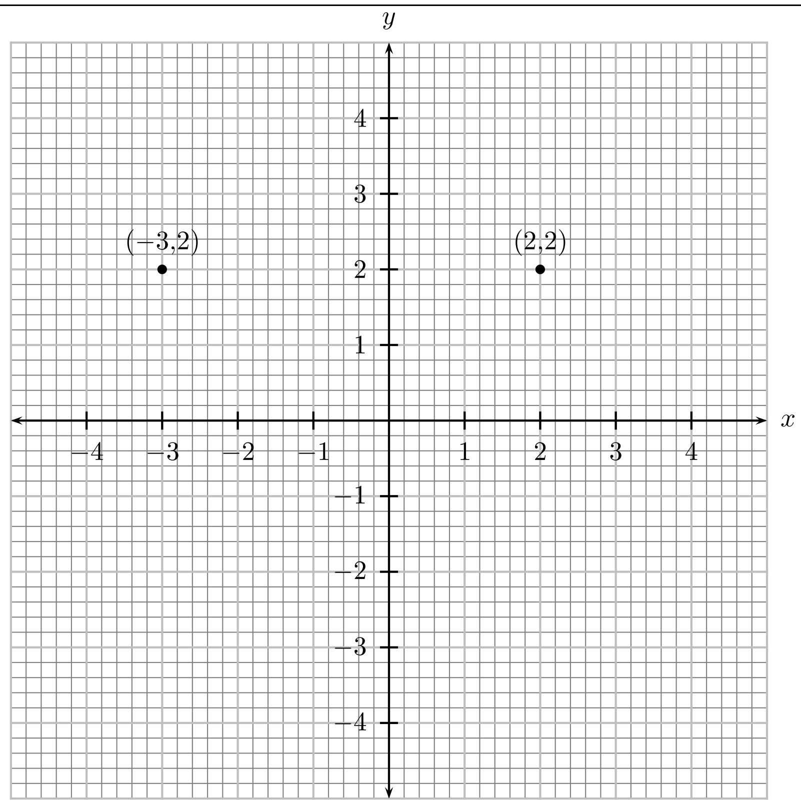


Figure 10.1: The Cartesian plane is made up of an $x-$axis (horizontal) and a $y-$axis (vertical).

10.10.4 Drawing Graphs

In order to draw the graph of a function, we need to calculate a few points. Then we plot the points on the Cartesian Plane and join the points with a smooth line.

The great beauty of doing this is that it allows us to "draw" functions, in a very abstract way. Assume that we were investigating the properties of the function $f(x) = 2x$. We could then consider all the points (x, y) such that $y = f(x)$, i.e. $y = 2x$. For example, $(1, 2), (2.5, 5)$, and $(3, 6)$ would all be such points, whereas $(3, 5)$ would not since $5 \neq 2 \times 3$. If we put a dot at each of those points, and then at every similar one for all possible values of x, we would obtain the graph shown in

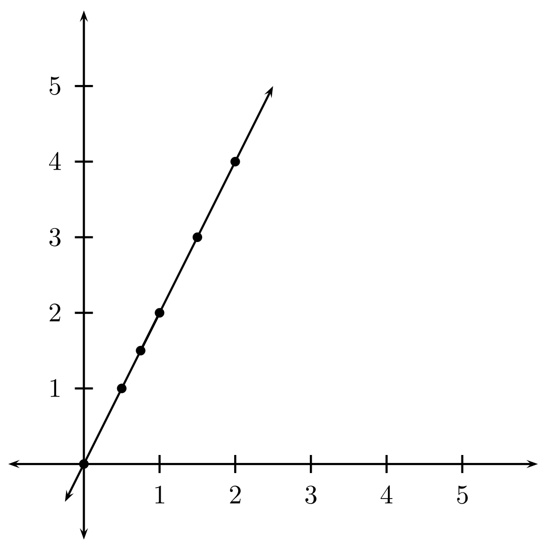


Figure 10.2: Graph of $f(x) = 2x$

The form of this graph is very pleasing – it is a simple straight line through the middle of the plane. The technique of "plotting", which we have followed here, is *the* key element in understanding functions.

Activity :: Investigation : Drawing Graphs and the Cartesian Plane
Plot the following points and draw a smooth line through them. (-6; -8),(-2; 0), (2; 8), (6; 16)

10.10.5 Notation used for Functions

Thus far you would have seen that we can use $y = 2x$ to represent a function. This notation however gets confusing when you are working with more than one function. A more general form of writing a function is to write the function as $f(x)$, where f is the function name and x is the independent variable. For example, $f(x) = 2x$ and $g(t) = 2t + 1$ are two functions.

Both notations will be used in this book.

Worked Example 46: Function notation

Question: If $f(n) = n^2 - 6n + 9$, find $f(k - 1)$ in terms of k.
Answer
Step 1 : Replace n with $k - 1$

$$\begin{aligned} f(n) &= n^2 - 6n + 9 \\ f(k-1) &= (k-1)^2 - 6(k-1) + 9 \end{aligned}$$

Step 2 : Remove brackets on RHS and simplify

$$\begin{aligned} &= k^2 - 2k + 1 - 6k + 6 + 9 \\ &= k^2 - 8k + 16 \end{aligned}$$

Worked Example 47: Function notation

Question: If $f(x) = x^2 - 4$, calculate b if $f(b) = 45$.
Answer
Step 1 : Replace x with b

$$\begin{aligned} f(b) &= b^2 - 4 \\ but f(b) &= 45 \end{aligned}$$

Step 2 : f(b) = f(b)

$$\begin{aligned} b^2 - 4 &= 45 \\ b^2 - 49 &= 0 \\ b &= +7 or - 7 \end{aligned}$$

{ExerciseRecap

1. Guess the function in the form $y = \ldots$ that has the values listed in the table.

x	1	2	3	40	50	600	700	800	900	1000
y	1	2	3	40	50	600	700	800	900	1000

2. Guess the function in the form $y = \ldots$ that has the values listed in the table.

x	1	2	3	40	50	600	700	800	900	1000
y	2	4	6	80	100	1200	1400	1600	1800	2000

3. Guess the function in the form $y = \ldots$ that has the values listed in the table.

x	1	2	3	40	50	600	700	800	900	1000
y	10	20	30	400	500	6000	7000	8000	9000	10000

4. On a Cartesian plane, plot the following points: (1,2), (2,4), (3,6), (4,8), (5,10). Join the points. Do you get a straight-line?

5. If $f(x) = x + x^2$, write out:
   (a) $f(t)$
   (b) $f(a)$
   (c) $f(1)$
   (d) $f(3)$

6. If $g(x) = x$ and $f(x) = 2x$, write out:
   (a) $f(t) + g(t)$
   (b) $f(a) - g(a)$
   (c) $f(1) + g(2)$
   (d) $f(3) + g(s)$

7. A car drives by you on a straight highway. The car is travelling 10 m every second. Complete the table below by filling in how far the car has travelled away from you after 5, 10 and 20 seconds.

Time (s)	0	1	2	5	10	20
Distance (m)	0	10	20			

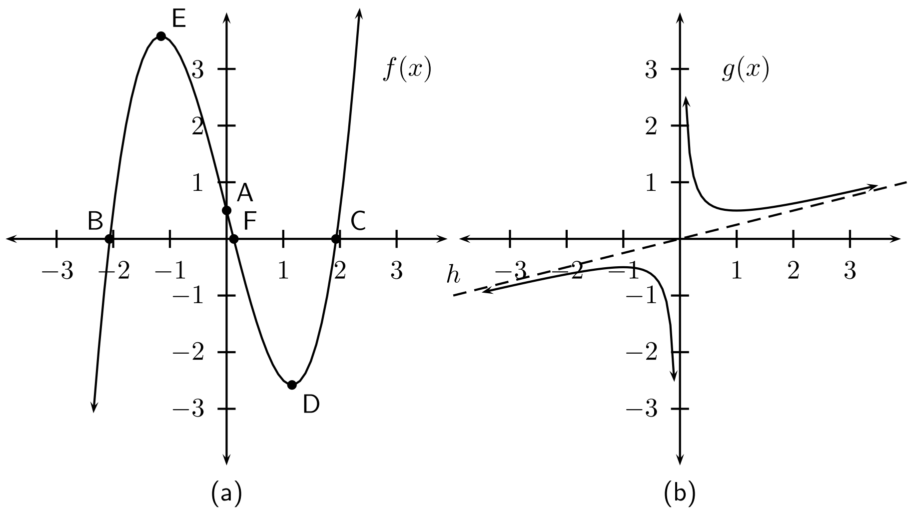
Use the values in the table and draw a graph of distance on the y-axis and time on the x-axis.

10.11 Characteristics of Functions - All Grades

There are many characteristics of graphs that help describe the graph of any function. These properties are:

1. dependent and independent variables
2. domain and range
3. intercepts with axes
4. turning points
5. asymptotes
6. lines of symmetry
7. intervals on which the function increases/decreases
8. continuous nature of the function

Some of these words may be unfamiliar to you, but each will be clearly described. Examples of these properties are shown in Figure 10.3.



A	y-intercept
B, C, F	x-intercept
D, E	turning points

Figure 10.3: (a) Example graphs showing the characteristics of a function. (b) Example graph showing asymptotes of a function.

10.11.1 Dependent and Independent Variables

Thus far, all the graphs you have drawn have needed two values, an x-value and a y-value. The y-value is usually determined from some relation based on a given or chosen x-value. These values are given special names in mathematics. The given or chosen x-value is known as the *independent* variable, because its value can be chosen freely. The calculated y-value is known as the *dependent* variable, because its value depends on the chosen x-value.

10.11.2 Domain and Range

The *domain* of a relation is the set of all the x values for which there exists at least one y value according to that relation. The *range* is the set of all the y values, which can be obtained using at least one x value.

If the relation is of height to people, then the domain is all living people, while the range would be about 0.1 to 3 metres — no living person can have a height of 0m, and while strictly its not impossible to be taller than 3 metres, no one alive is. An important aspect of this range is that it does not contain *all* the numbers between 0.1 and 3, but only six billion of them (as many as there are people).

As another example, suppose x and y are real valued variables, and we have the relation $y = 2^x$. Then for *any* value of x, there is a value of y, so the domain of this relation is the whole set of real numbers. However, we know that no matter what value of x we choose, 2^x can never be less than or equal to 0. Hence the range of this function is all the real numbers strictly greater than zero.

These are two ways of writing the domain and range of a function, *set notation* and *interval notation*. Both notations are used in mathematics, so you should be familiar with each.

Set Notation

A set of certain x values has the following form:

$$\{x : \text{conditions, more conditions}\} \tag{10.6}$$

We read this notation as "the set of all x values where all the conditions are satisfied". For example, the set of all positive real numbers can be written as $\{x : x \in \mathbb{R}, x > 0\}$ which reads as "the set of all x values where x is a real number and is greater than zero".

Interval Notation

Here we write an interval in the form '*lower bracket, lower number, comma, upper number, upper bracket*'. We can use two types of brackets, square ones $[,]$ or round ones $(,)$. A square bracket means including the number at the end of the interval whereas a round bracket means excluding the number at the end of the interval. It is important to note that this notation can only be used for all real numbers in an interval. It cannot be used to describe integers in an interval or rational numbers in an interval.

So if x is a real number greater than 2 and less than or equal to 8, then x is any number in the interval

$$(2{,}8] \tag{10.7}$$

It is obvious that 2 is the lower number and 8 the upper number. The round bracket means 'excluding 2', since x is greater than 2, and the square bracket means 'including 8' as x is less than or equal to 8.

10.11.3 Intercepts with the Axes

The *intercept* is the point at which a graph intersects an axis. The x-intercepts are the points at which the graph cuts the x-axis and the y-intercepts are the points at which the graph cuts the y-axis.

In Figure 10.3(a), the A is the y-intercept and B, C and F are x-intercepts.

You **will** usually need to calculate the intercepts. The two most important things to remember is that at the x-intercept, $y = 0$ and at the y-intercept, $x = 0$.

For example, calculate the intercepts of $y = 3x + 5$. For the y-intercept, $x = 0$. Therefore the y-intercept is $y_{int} = 3(0) + 5 = 5$. For the x-intercept, $y = 0$. Therefore the x-intercept is found from $0 = 3x_{int} + 5$, giving $x_{int} = -\frac{5}{3}$.

10.11.4 Turning Points

Turning points only occur for graphs of functions that whose highest power is greater than 1. For example, graphs of the following functions will have turning points.

$$\begin{aligned} f(x) &= 2x^2 - 2 \\ g(x) &= x^3 - 2x^2 + x - 2 \\ h(x) &= \frac{2}{3}x^4 - 2 \end{aligned}$$

There are two types of turning points: a minimal turning point and a maximal turning point. A minimal turning point is a point on the graph where the graph stops decreasing in value and starts increasing in value and a maximal turning point is a point on the graph where the graph stops increasing in value and starts decreasing. These are shown in Figure 10.4.

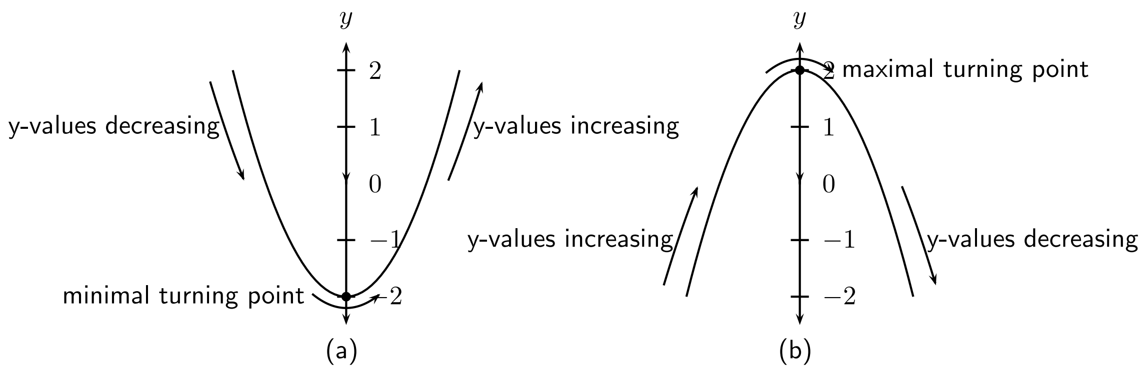


Figure 10.4: (a) Maximal turning point. (b) Minimal turning point.

In Figure 10.3(a), E is a maximal turning point and D is a minimal turning point.

10.11.5 Asymptotes

An asymptote is a straight or curved line, which the graph of a function will approach, but never touch.

In Figure 10.3(b), the y-axis and line h are both asymptotes as the graph approaches both these lines, but never touches them.

10.11.6 Lines of Symmetry

Graphs look the same on either side of lines of symmetry. These lines include the x- and y-axes. For example, in Figure 10.5 is symmetric about the y-axis. This is described as the axis of symmetry.

10.11.7 Intervals on which the Function Increases/Decreases

In the discussion of turning points, we saw that the graph of a function can start or stop increasing or decreasing at a turning point. If the graph in Figure 10.3(a) is examined, we find that the values of the graph increase and decrease over different intervals. We see that the graph increases (i.e. that the y-values increase) from -∞ to point E, then it decreases (i.e. the y-values decrease) from point E to point D and then it increases from point D to $+\infty$.

10.11.8 Discrete or Continuous Nature of the Graph

A graph is said to be continuous if there are no breaks in the graph. For example, the graph in Figure 10.3(a) can be described as a continuous graph, while the graph in Figure 10.3(b) has a

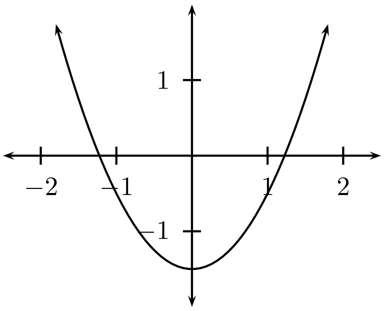
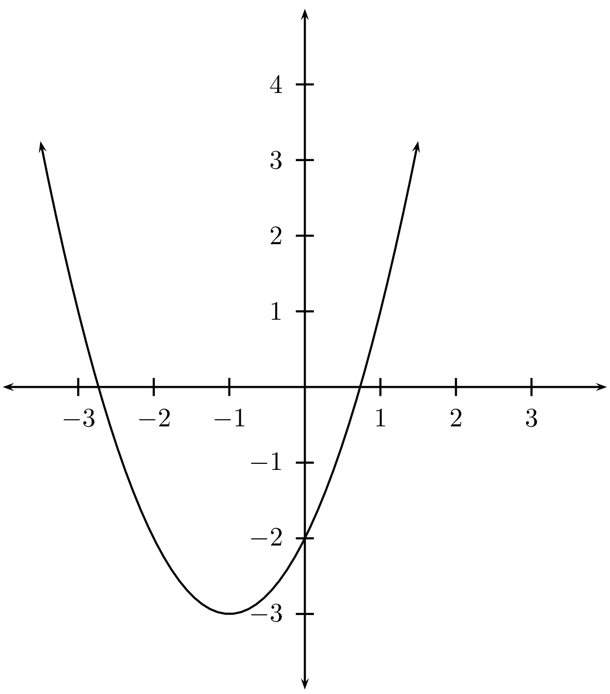


Figure 10.5: Demonstration of axis of symmetry. The y-axis is an axis of symmetry, because the graph looks the same on both sides of the y-axis.

break around the asymptotes. In Figure 10.3(b), it is clear that the graph does have a break in it around the asymptote.

Exercise: Domain and Range

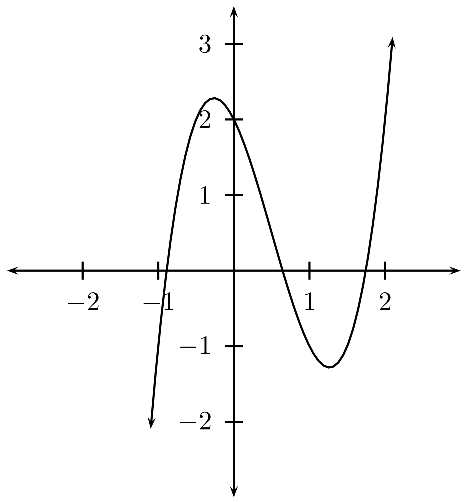
1. The domain of the function $f(x) = 2x+5$ is -3; -3; -3; 0. Determine the range of f.
2. If $g(x) = -x^2 + 5$ and x is between - 3 and 3, determine:
   (a) the domain of $g(x)$
   (b) the range of $g(x)$
3. Label, on the following graph:
   (a) the x-intercept(s)
   (b) the y-intercept(s)
   (c) regions where the graph is increasing
   (d) regions where the graph is decreasing



4. Label, on the following graph:
   (a) the x-intercept(s)
   (b) the y-intercept(s)

(c) regions where the graph is increasing

(d) regions where the graph is decreasing



10.12 Graphs of Functions

10.12.1 Functions of the form $y = ax + q$

Functions with a general form of $y = ax + q$ are called *straight line* functions. In the equation, $y = ax + q$, a and q are constants and have different effects on the graph of the function. The general shape of the graph of functions of this form is shown in Figure 10.6 for the function $f(x) = 2x + 3$.

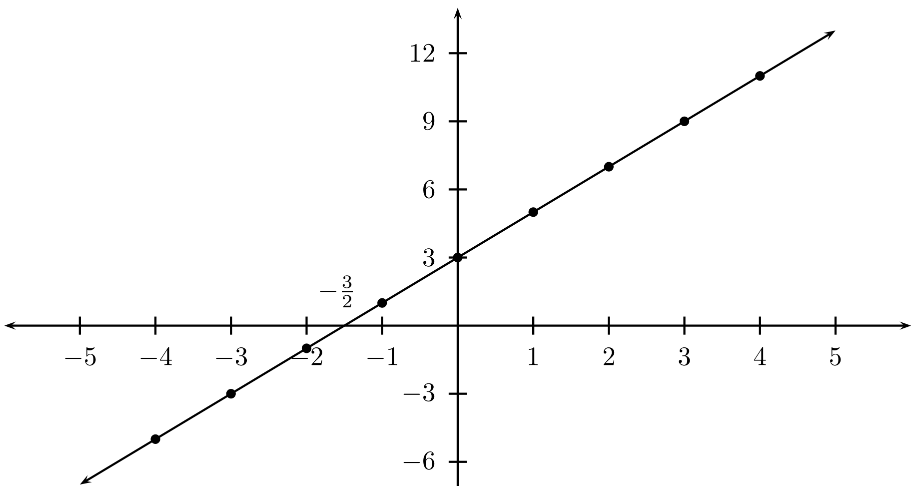


Figure 10.6: Graph of $f(x) = 2x + 3$

Activity :: Investigation : Functions of the Form $y = ax + q$

1. On the same set of axes, plot the following graphs:

   (a) $a(x) = x - 2$

   (b) $b(x) = x - 1$
   (c) $c(x) = x$
   (d) $d(x) = x + 1$
   (e) $e(x) = x + 2$

   Use your results to deduce the effect of q.

2. On the same set of axes, plot the following graphs:
   (a) $f(x) = -2 \cdot x$
   (b) $g(x) = -1 \cdot x$
   (c) $h(x) = 0 \cdot x$
   (d) $j(x) = 1 \cdot x$
   (e) $k(x) = 2 \cdot x$

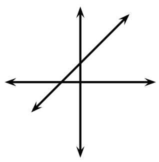
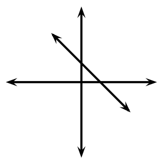
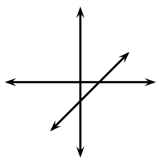
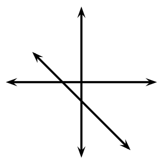
   Use your results to deduce the effect of a.

You should have found that the value of a affects the slope of the graph. As a increases, the slope of the graph increases. If $a > 0$ then the graph increases from left to right (slopes upwards). If $a < 0$ then the graph increases from right to left (slopes downwards). For this reason, a is referred to as the *slope* or *gradient* of a straight-line function.

You should have also found that the value of q affects where the graph passes through the y-axis. For this reason, q is known as the *y-intercept*.

These different properties are summarised in Table 10.1.

Table 10.1: Table summarising general shapes and positions of graphs of functions of the form $y = ax + q$.

	$a > 0$	$a < 0$
$q > 0$		
$q < 0$		

Domain and Range

For $f(x) = ax + q$, the domain is $\{x : x \in \mathbb{R}\}$ because there is no value of $x \in \mathbb{R}$ for which $f(x)$ is undefined.

The range of $f(x) = ax + q$ is also $\{f(x) : f(x) \in \mathbb{R}\}$ because there is no value of $f(x) \in \mathbb{R}$ for which $f(x)$ is undefined.

For example, the domain of $g(x) = x - 1$ is $\{x : x \in \mathbb{R}\}$ because there is no value of $x \in \mathbb{R}$ for which $g(x)$ is undefined. The range of $g(x)$ is $\{g(x) : g(x) \in \mathbb{R}\}$.

Intercepts

For functions of the form, $y = ax + q$, the details of calculating the intercepts with the x and y axis is given.

The y-intercept is calculated as follows:

$$\begin{aligned} y &= ax + q && (10.8)\\ y_{int} &= a(0) + q && (10.9)\\ &= q && (10.10) \end{aligned}$$

For example, the y-intercept of $g(x) = x - 1$ is given by setting $x = 0$ to get:

$$\begin{aligned} g(x) &= x - 1\\ y_{int} &= 0 - 1\\ &= -1 \end{aligned}$$

The x-intercepts are calculated as follows:

$$\begin{aligned} y &= ax + q && (10.11)\\ 0 &= a \cdot x_{int} + q && (10.12)\\ a \cdot x_{int} &= -q && (10.13)\\ x_{int} &= -\frac{q}{a} && (10.14) \end{aligned}$$

For example, the x-intercepts of $g(x) = x - 1$ is given by setting $y = 0$ to get:

$$\begin{aligned} g(x) &= x - 1\\ 0 &= x_{int} - 1\\ x_{int} &= 1 \end{aligned}$$

Turning Points

The graphs of straight line functions do not have any turning points.

Axes of Symmetry

The graphs of straight-line functions do not, generally, have any axes of symmetry.

Sketching Graphs of the Form $f(x) = ax + q$

In order to sketch graphs of the form, $f(x) = ax + q$, we need to determine three characteristics:

1. sign of a
2. y-intercept
3. x-intercept

Only two points are needed to plot a straight line graph. The easiest points to use are the x-*intercept* (where the line cuts the x-axis) and the y-intercept.

For example, sketch the graph of $g(x) = x - 1$. Mark the intercepts.

Firstly, we determine that $a > 0$. This means that the graph will have an upward slope.

The y-intercept is obtained by setting $x = 0$ and was calculated earlier to be $y_{int} = -1$. The x-intercept is obtained by setting $y = 0$ and was calculated earlier to be $x_{int} = 1$.

Worked Example 48: Drawing a straight line graph

Question: Draw the graph of $y = 2x + 2$

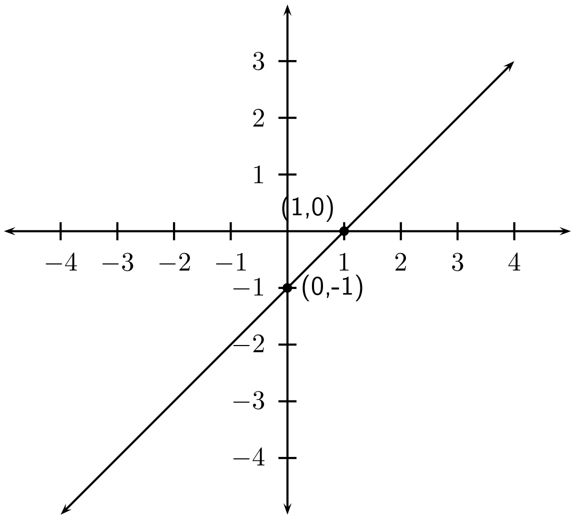


Figure 10.7: Graph of the function $g(x) = x - 1$

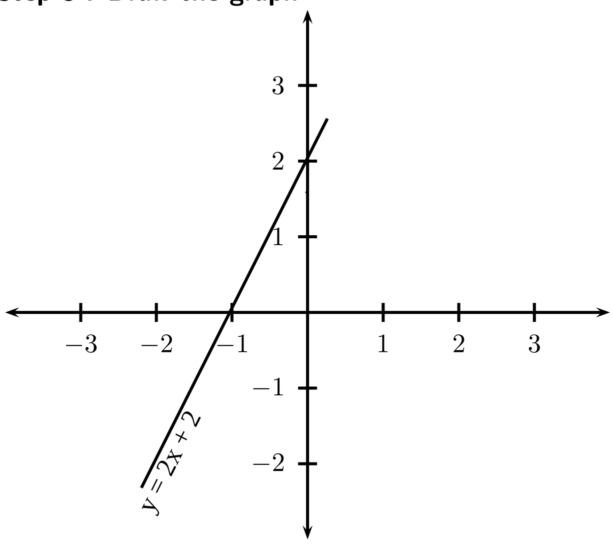
Answer
Step 1 : Find the y-intercept
For the intercept on the y-axis, let $x = 0$

$$\begin{aligned} y &= 2(0) + 2 \\ &= 2 \end{aligned}$$

Step 2 : Find the x-intercept
For the intercept on the x-axis, let $y = 0$

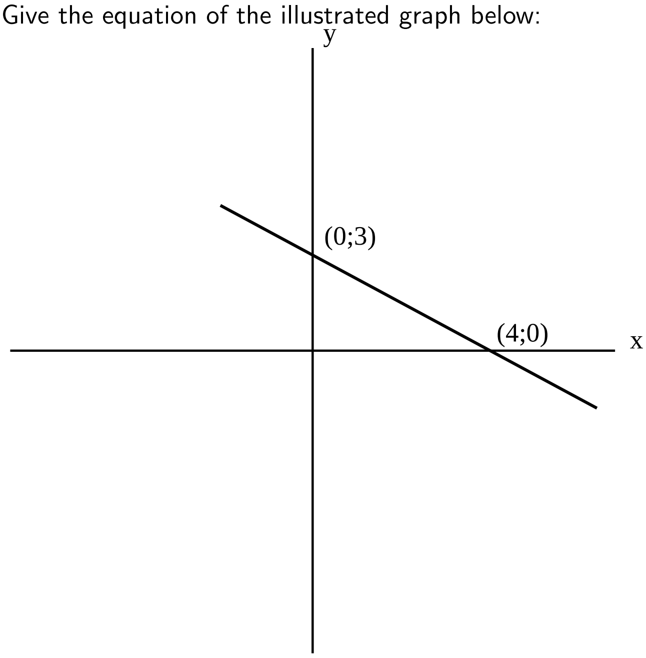
$$\begin{aligned} 0 &= 2x + 2 \\ 2x &= -2 \\ x &= -1 \end{aligned}$$

Step 3 : Draw the graph



Exercise: Intercepts

1. List the y-intercepts for the following straight-line graphs:
   (a) $y = x$
   (b) $y = x - 1$
   (c) $y = 2x - 1$
   (d) $y + 1 = 2x$
2. Give the equation of the illustrated graph below:



3. Sketch the following relations on the same set of axes, clearly indicating the intercepts with the axes as well as the co-ordinates of the point of interception on the graph: $x + 2y - 5 = 0$ and $3x - y - 1 = 0$

10.12.2 Functions of the Form $y = ax^2 + q$

The general shape and position of the graph of the function of the form $f(x) = ax^2 + q$ is shown in Figure 10.8.

Activity :: Investigation : Functions of the Form $y = ax^2 + q$

1. On the same set of axes, plot the following graphs:
   (a) $a(x) = -2 \cdot x^2 + 1$
   (b) $b(x) = -1 \cdot x^2 + 1$
   (c) $c(x) = 0 \cdot x^2 + 1$
   (d) $d(x) = 1 \cdot x^2 + 1$
   (e) $e(x) = 2 \cdot x^2 + 1$

   Use your results to deduce the effect of a.
2. On the same set of axes, plot the following graphs:
   (a) $f(x) = x^2 - 2$
   (b) $g(x) = x^2 - 1$

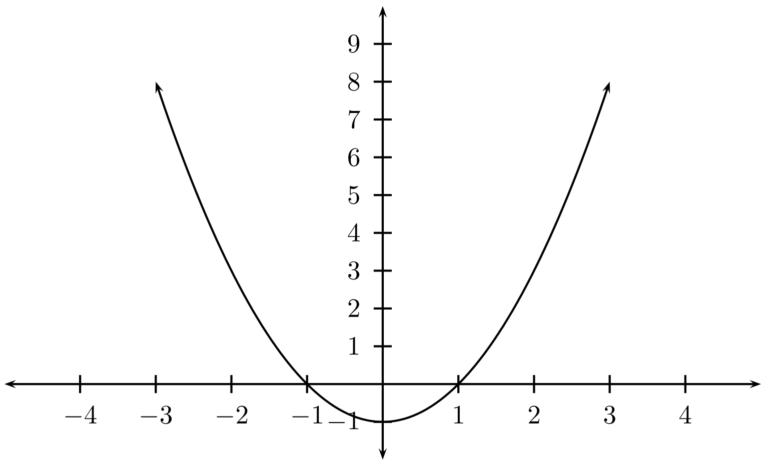


Figure 10.8: Graph of the $f(x) = x^2 - 1$.

(c) $h(x) = x^2 + 0$
(d) $j(x) = x^2 + 1$
(e) $k(x) = x^2 + 2$

Use your results to deduce the effect of q.

Complete the following table of values for the functions a to k to help with drawing the required graphs in this activity:

x	-2	-1	0	1	2
$a(x)$					
$b(x)$					
$c(x)$					
$d(x)$					
$e(x)$					
$f(x)$					
$g(x)$					
$h(x)$					
$j(x)$					
$k(x)$					

From your graphs, you should have found that a affects whether the graph makes a smile or a frown. If $a < 0$, the graph makes a frown and if $a > 0$ then the graph makes a smile. This is shown in Figure 10.9.

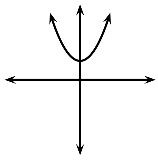
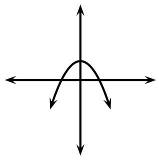
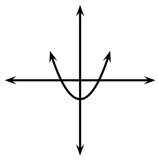
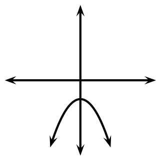
$a > 0$ (a positive smile) $a < 0$ (a negative frown)

Figure 10.9: Distinctive shape of graphs of a parabola if $a > 0$ and $a < 0$.

You should have also found that the value of q affects whether the turning point is to the left of the y-axis ($q > 0$) or to the right of the y-axis ($q < 0$).

These different properties are summarised in Table **??**.

Table 10.2: Table summarising general shapes and positions of functions of the form $y = ax^2+q$.

	$a > 0$	$a < 0$
$q > 0$		
$q < 0$		

Domain and Range

For $f(x) = ax^2 + q$, the domain is $\{x : x \in \mathbb{R}\}$ because there is no value of $x \in \mathbb{R}$ for which $f(x)$ is undefined.

The range of $f(x) = ax^2 + q$ depends on whether the value for a is positive or negative. We will consider these two cases separately.

If $a > 0$ then we have:

$$\begin{aligned} x^2 &\geq 0 \quad \text{(The square of an expression is always positive)} \\ ax^2 &\geq 0 \quad \text{(Multiplication by a positive number maintains the nature of the inequality)} \\ ax^2+q &\geq q \\ f(x) &\geq q \end{aligned}$$

This tells us that for all values of x, $f(x)$ is always greater than q. Therefore if $a > 0$, the range of $f(x) = ax^2 + q$ is $\{f(x) : f(x) \in [q,\infty)\}$.

Similarly, it can be shown that if $a < 0$ that the range of $f(x) = ax^2 + q$ is $\{f(x) : f(x) \in (-\infty,q]\}$. This is left as an exercise.

For example, the domain of $g(x) = x^2 + 2$ is $\{x : x \in \mathbb{R}\}$ because there is no value of $x \in \mathbb{R}$ for which $g(x)$ is undefined. The range of $g(x)$ can be calculated as follows:

$$\begin{aligned} x^2 &\geq 0 \\ x^2+2 &\geq 2 \\ g(x) &\geq 2 \end{aligned}$$

Therefore the range is $\{g(x) : g(x) \in [2,\infty)\}$.

Intercepts

For functions of the form, $y = ax^2 + q$, the details of calculating the intercepts with the x and y axis is given.

The y-intercept is calculated as follows:

$$\begin{aligned} y &= ax^2+q && (10.15) \\ y_{int} &= a(0)^2+q && (10.16) \\ &= q && (10.17) \end{aligned}$$

For example, the y-intercept of $g(x) = x^2 + 2$ is given by setting $x = 0$ to get:

$$\begin{aligned} g(x) &= x^2 + 2 \\ y_{int} &= 0^2 + 2 \\ &= 2 \end{aligned}$$

The x-intercepts are calculated as follows:

$$y = ax^2 + q \quad (10.18)$$

$$0 = ax_{int}^2 + q \quad (10.19)$$

$$ax_{int}^2 = -q \quad (10.20)$$

$$x_{int} = \pm\sqrt{-\frac{q}{a}} \quad (10.21)$$

However, (10.21) is only valid if $-\frac{q}{a} > 0$ which means that either $q < 0$ or $a < 0$. This is consistent with what we expect, since if $q > 0$ and $a > 0$ then $-\frac{q}{a}$ is negative and in this case the graph lies above the x-axis and therefore does not intersect the x-axis. If however, $q > 0$ and $a < 0$, then $-\frac{q}{a}$ is positive and the graph is hat shaped and should have two x-intercepts. Similarly, if $q < 0$ and $a > 0$ then $-\frac{q}{a}$ is also positive, and the graph should intersect with the x-axis.

For example, the x-intercepts of $g(x) = x^2 + 2$ is given by setting $y = 0$ to get:

$$\begin{aligned} g(x) &= x^2 + 2 \\ 0 &= x_{int}^2 + 2 \\ -2 &= x_{int}^2 \end{aligned}$$

which is not real. Therefore, the graph of $g(x) = x^2 + 2$ does not have any x-intercepts.

Turning Points

The turning point of the function of the form $f(x) = ax^2 + q$ is given by examining the range of the function. We know that if $a > 0$ then the range of $f(x) = ax^2 + q$ is $\{f(x) : f(x) \in [q,\infty)\}$ and if $a < 0$ then the range of $f(x) = ax^2 + q$ is $\{f(x) : f(x) \in (-\infty,q]\}$.

So, if $a > 0$, then the lowest value that $f(x)$ can take on is q. Solving for the value of x at which $f(x) = q$ gives:

$$\begin{aligned} q &= ax_{tp}^2 + q \\ 0 &= ax_{tp}^2 \\ 0 &= x_{tp}^2 \\ x_{tp} &= 0 \end{aligned}$$

$\therefore x = 0$ at $f(x) = q$. The co-ordinates of the (minimal) turning point is therefore $(0; q)$.

Similarly, if $a < 0$, then the highest value that $f(x)$ can take on is q and the co-ordinates of the (maximal) turning point is $(0; q)$.

Axes of Symmetry

There is one axis of symmetry for the function of the form $f(x) = ax^2 + q$ that passes through the turning point. Since the turning point lies on the y-axis, the axis of symmetry is the y-axis.

Sketching Graphs of the Form $f(x) = ax^2 + q$

In order to sketch graphs of the form, $f(x) = ax^2 + q$, we need to calculate determine four characteristics:

1. sign of a

2. domain and range
3. turning point
4. y-intercept
5. x-intercept

For example, sketch the graph of $g(x) = -\frac{1}{2}x^2 - 3$. Mark the intercepts, turning point and axis of symmetry.

Firstly, we determine that $a < 0$. This means that the graph will have a maximal turning point.

The domain of the graph is $\{x : x \in \mathbb{R}\}$ because $f(x)$ is defined for all $x \in \mathbb{R}$. The range of the graph is determined as follows:

$$\begin{aligned} x^2 &\geq 0 \\ -\frac{1}{2}x^2 &\leq 0 \\ -\frac{1}{2}x^2 - 3 &\leq -3 \\ \therefore f(x) &\leq -3 \end{aligned}$$

Therefore the range of the graph is $\{f(x) : f(x) \in (-\infty, -3]\}$.

Using the fact that the maximum value that $f(x)$ achieves is -3, then the y-coordinate of the turning point is -3. The x-coordinate is determined as follows:

$$\begin{aligned} -\frac{1}{2}x^2 - 3 &= -3 \\ -\frac{1}{2}x^2 - 3 + 3 &= 0 \\ -\frac{1}{2}x^2 &= 0 \\ \text{Divide both sides by } -\tfrac{1}{2}\text{:}\quad x^2 &= 0 \\ \text{Take square root of both sides:}\quad x &= 0 \\ \therefore\quad x &= 0 \end{aligned}$$

The coordinates of the turning point are: $(0, -3)$.

The y-intercept is obtained by setting $x = 0$. This gives:

$$\begin{aligned} y_{int} &= -\frac{1}{2}(0)^2 - 3 \\ &= -\frac{1}{2}(0) - 3 \\ &= -3 \end{aligned}$$

The x-intercept is obtained by setting $y = 0$. This gives:

$$\begin{aligned} 0 &= -\frac{1}{2}x_{int}^2 - 3 \\ 3 &= -\frac{1}{2}x_{int}^2 \\ -3 \cdot 2 &= x_{int}^2 \\ -6 &= x_{int}^2 \end{aligned}$$

which is not real. Therefore, there are no x-intercepts.

We also know that the axis of symmetry is the y-axis.

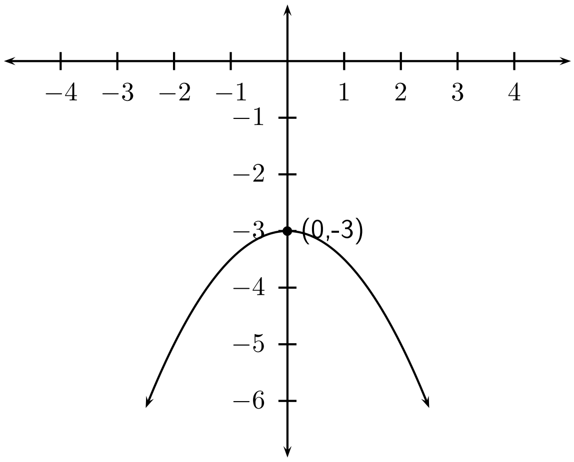
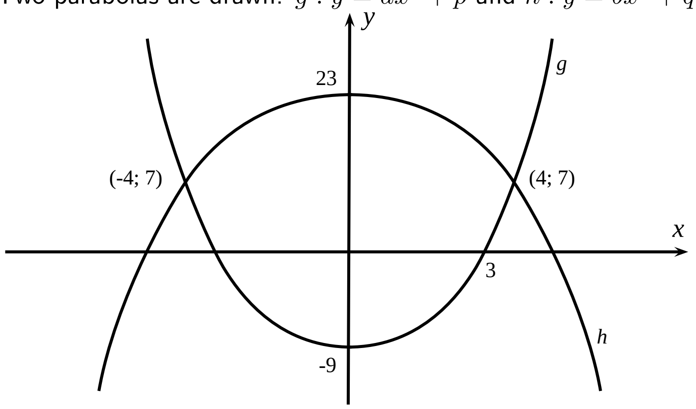


Figure 10.10: Graph of the function $f(x) = -\frac{1}{2}x^2 - 3$

?

Exercise: Parabolas

1. Show that if $a < 0$ that the range of $f(x) = ax^2+q$ is $\{f(x) : f(x) \in (-\infty,q]\}$.
2. Draw the graph of the function $y = -x^2 + 4$ showing all intercepts with the axes.
3. Two parabolas are drawn: $g : y = ax^2 + p$ and $h : y = bx^2 + q$.



   (a) Find the values of a and p.
   (b) Find the values of b and q.
   (c) Find the values of x for which $ax^2 + p \geq bx^2 + q$.
   (d) For what values of x is g increasing ?

10.12.3 Functions of the Form $y = \frac{a}{x} + q$

Functions of the form $y = \frac{a}{x} + q$ are known as *hyperbolic* functions. The general form of the graph of this function is shown in Figure 10.11.

Activity :: Investigation : Functions of the Form $y = \frac{a}{x} + q$

1. On the same set of axes, plot the following graphs:
   (a) $a(x) = \frac{-2}{x} + 1$
   (b) $b(x) = \frac{-1}{x} + 1$

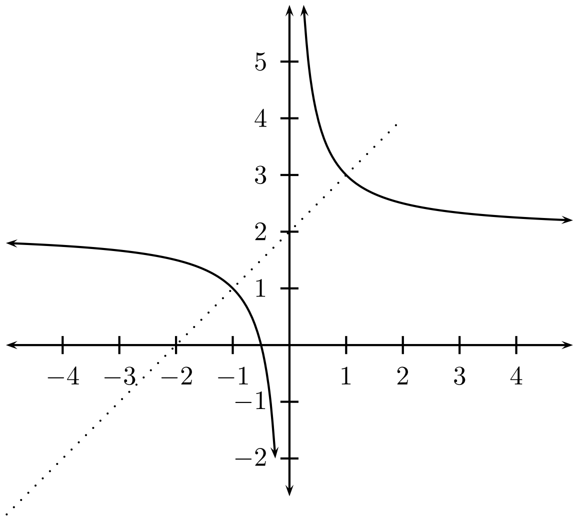


Figure 10.11: General shape and position of the graph of a function of the form $f(x) = \frac{a}{x} + q$.

(c) $c(x) = \frac{0}{x} + 1$
(d) $d(x) = \frac{+1}{x} + 1$
(e) $e(x) = \frac{+2}{x} + 1$

Use your results to deduce the effect of a.

2. On the same set of axes, plot the following graphs:
   (a) $f(x) = \frac{1}{x} - 2$
   (b) $g(x) = \frac{1}{x} - 1$
   (c) $h(x) = \frac{1}{x} + 0$
   (d) $j(x) = \frac{1}{x} + 1$
   (e) $k(x) = \frac{1}{x} + 2$

   Use your results to deduce the effect of q.

You should have found that the value of a affects whether the graph is located in the first and third quadrants of Cartesian plane.

You should have also found that the value of q affects whether the graph lies above the x-axis ($q > 0$) or below the x-axis ($q < 0$).

These different properties are summarised in Table 10.3. The axes of symmetry for each graph are shown as a dashed line.

Domain and Range

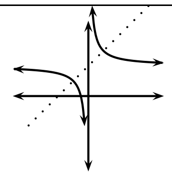
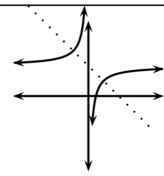
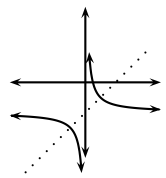
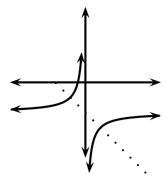
For $y = \frac{a}{x} + q$, the function is undefined for $x = 0$. The domain is therefore $\{x : x \in \mathbb{R}, x \neq 0\}$.

We see that $y = \frac{a}{x} + q$ can be re-written as:

$$\begin{aligned} y &= \frac{a}{x} + q \\ y - q &= \frac{a}{x} \\ \text{If } x \neq 0 \text{ then: } \quad (y - q)(x) &= a \\ x &= \frac{a}{y - q} \end{aligned}$$

This shows that the function is undefined at $y = q$. Therefore the range of $f(x) = \frac{a}{x} + q$ is $\{f(x) : f(x) \in (-\infty, q) \cup (q, \infty)\}$.

Table 10.3: Table summarising general shapes and positions of functions of the form $y = \frac{a}{x} + q$. The axes of symmetry are shown as dashed lines.

	$a > 0$	$a < 0$
$q > 0$		
$q < 0$		

For example, the domain of $g(x) = \frac{2}{x} + 2$ is $\{x : x \in \mathbb{R}, x \neq 0\}$ because $g(x)$ is undefined at $x = 0$.

$$\begin{aligned} y &= \frac{2}{x} + 2 \\ (y - 2) &= \frac{2}{x} \\ \text{If } x \neq 0 \text{ then:} \quad x(y - 2) &= 2 \\ x &= \frac{2}{y - 2} \end{aligned}$$

We see that $g(x)$ is undefined at $y = 2$. Therefore the range is $\{g(x) : g(x) \in (-\infty,2) \cup (2,\infty)\}$.

Intercepts

For functions of the form, $y = \frac{a}{x} + q$, the intercepts with the x and y axis is calculated by setting $x = 0$ for the y-intercept and by setting $y = 0$ for the x-intercept.

The y-intercept is calculated as follows:

$$y = \frac{a}{x} + q \quad (10.22)$$

$$y_{int} = \frac{a}{0} + q \quad (10.23)$$

which is undefined. Therefore there is no y-intercept.

For example, the y-intercept of $g(x) = \frac{2}{x} + 2$ is given by setting $x = 0$ to get:

$$\begin{aligned} y &= \frac{2}{x} + 2 \\ y_{int} &= \frac{2}{0} + 2 \end{aligned}$$

which is undefined.

The x-intercepts are calculated by setting $y = 0$ as follows:

$$\begin{aligned} y &= \frac{a}{x} + q && (10.24) \\ 0 &= \frac{a}{x_{int}} + q && (10.25) \\ \frac{a}{x_{int}} &= -q && (10.26) \\ a &= -q(x_{int}) && (10.27) \\ x_{int} &= \frac{a}{-q} && (10.28) \\ & && (10.29) \end{aligned}$$

For example, the x-intercept of $g(x) = \frac{2}{x} + 2$ is given by setting $x = 0$ to get:

$$\begin{aligned} y &= \frac{2}{x} + 2 \\ 0 &= \frac{2}{x_{int}} + 2 \\ -2 &= \frac{2}{x_{int}} \\ -2(x_{int}) &= 2 \\ x_{int} &= \frac{2}{-2} \\ x_{int} &= -1 \end{aligned}$$

Asymptotes

There are two asymptotes for functions of the form $y = \frac{a}{x} + q$. They are determined by examining the domain and range.

We saw that the function was undefined at $x = 0$ and for $y = q$. Therefore the asymptotes are $x = 0$ and $y = q$.

For example, the domain of $g(x) = \frac{2}{x} + 2$ is $\{x : x \in \mathbb{R}, x \neq 0\}$ because $g(x)$ is undefined at $x = 0$. We also see that $g(x)$ is undefined at $y = 2$. Therefore the range is $\{g(x) : g(x) \in (-\infty,2) \cup (2,\infty)\}$.

From this we deduce that the asymptotes are at $x = 0$ and $y = 2$.

Sketching Graphs of the Form $f(x) = \frac{a}{x} + q$

In order to sketch graphs of functions of the form, $f(x) = \frac{a}{x} + q$, we need to calculate determine four characteristics:

1. domain and range
2. asymptotes
3. y-intercept
4. x-intercept

For example, sketch the graph of $g(x) = \frac{2}{x} + 2$. Mark the intercepts and asymptotes.

We have determined the domain to be $\{x : x \in \mathbb{R}, x \neq 0\}$ and the range to be $\{g(x) : g(x) \in (-\infty,2) \cup (2,\infty)\}$. Therefore the asymptotes are at $x = 0$ and $y = 2$.

There is no y-intercept and the x-intercept is $x_{int} = -1$.

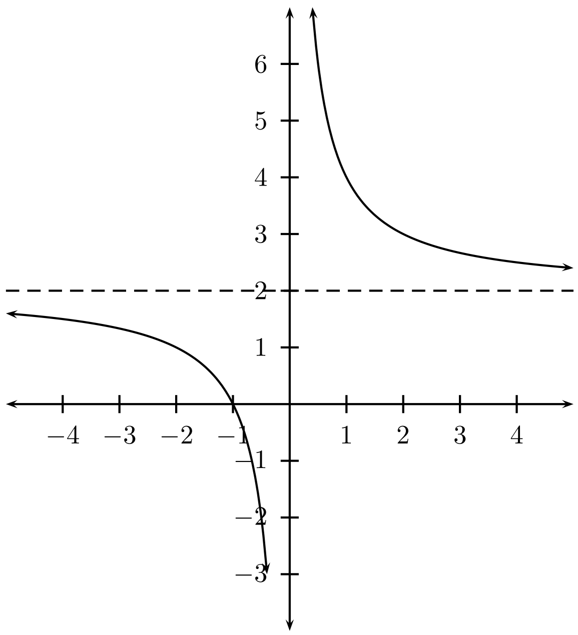


Figure 10.12: Graph of $g(x) = \frac{2}{x} + 2$.

Exercise: Graphs

1. Using grid paper, draw the graph of $xy = -6$.
   (a) Does the point (-2; 3) lie on the graph ? Give a reason for your answer.
   (b) Why is the point (-2; -3) not on the graph ?
   (c) If the x-value of a point on the drawn graph is $0{,}25$, what is the corresponding y-value ?
   (d) What happens to the y-values as the x-values become very large ?
   (e) With the line $y = -x$ as line of symmetry, what is the point symmetrical to (-2; 3) ?
2. Draw the graph of $xy = 8$.
   (a) How would the graph $y = \frac{8}{3} + 3$ compare with that of $xy = 8$? Explain your answer fully.
   (b) Draw the graph of $y = \frac{8}{3} + 3$ on the same set of axes.

10.12.4 Functions of the Form $y = ab^{(x)} + q$

Functions of the form $y = ab^{(x)} + q$ are known as *exponential* functions. The general shape of a graph of a function of this form is shown in Figure 10.13.

Activity :: Investigation : Functions of the Form $y = ab^{(x)} + q$

1. On the same set of axes, plot the following graphs:
   (a) $a(x) = -2 \cdot b^{(x)} + 1$
   (b) $b(x) = -1 \cdot b^{(x)} + 1$
   (c) $c(x) = -0 \cdot b^{(x)} + 1$

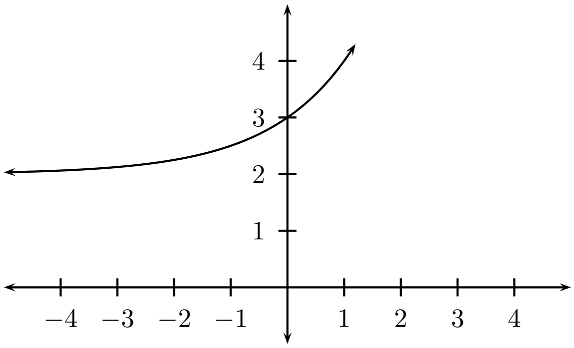


Figure 10.13: General shape and position of the graph of a function of the form $f(x) = ab^{(x)} + q$.

(d) $d(x) = -1 \cdot b^{(x)} + 1$

(e) $e(x) = -2 \cdot b^{(x)} + 1$

Use your results to deduce the effect of a.

2. On the same set of axes, plot the following graphs:

(a) $f(x) = 1 \cdot b^{(x)} - 2$

(b) $g(x) = 1 \cdot b^{(x)} - 1$

(c) $h(x) = 1 \cdot b^{(x)} + 0$

(d) $j(x) = 1 \cdot b^{(x)} + 1$

(e) $k(x) = 1 \cdot b^{(x)} + 2$

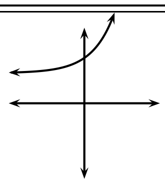
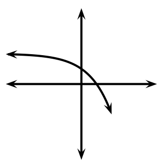
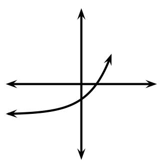
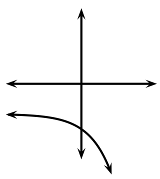
Use your results to deduce the effect of q.

You should have found that the value of a affects whether the graph curves upwards ($a > 0$) or curves downwards ($a < 0$).

You should have also found that the value of q affects the position of the y-intercept.

These different properties are summarised in Table 10.4.

Table 10.4: Table summarising general shapes and positions of functions of the form $y = ab^{(x)} + q$.

	$a > 0$	$a < 0$
$q > 0$		
$q < 0$		

Domain and Range

For $y = ab^{(x)} + q$, the function is defined for all real values of x. Therefore, the domain is $\{x : x \in \mathbb{R}\}$.

The range of $y = ab^{(x)} + q$ is dependent on the sign of a.

If $a > 0$ then:

$$\begin{aligned} b^{(x)} &\geq 0 \\ a \cdot b^{(x)} &\geq 0 \\ a \cdot b^{(x)} + q &\geq q \\ f(x) &\geq q \end{aligned}$$

Therefore, if $a > 0$, then the range is $\{f(x) : f(x) \in [q,\infty)\}$.

If $a < 0$ then:

$$\begin{aligned} b^{(x)} &\leq 0 \\ a \cdot b^{(x)} &\leq 0 \\ a \cdot b^{(x)} + q &\leq q \\ f(x) &\leq q \end{aligned}$$

Therefore, if $a < 0$, then the range is $\{f(x) : f(x) \in (-\infty,q]\}$.

For example, the domain of $g(x) = 3 \cdot 2^x + 2$ is $\{x : x \in \mathbb{R}\}$. For the range,

$$\begin{aligned} 2^x &\geq 0 \\ 3 \cdot 2^x &\geq 0 \\ 3 \cdot 2^x + 2 &\geq 2 \end{aligned}$$

Therefore the range is $\{g(x) : g(x) \in [2,\infty)\}$.

Intercepts

For functions of the form, $y = ab^{(x)} + q$, the intercepts with the x and y axis is calculated by setting $x = 0$ for the y-intercept and by setting $y = 0$ for the x-intercept.

The y-intercept is calculated as follows:

$$\begin{aligned} y &= ab^{(x)} + q && (10.30) \\ y_{int} &= ab^{(0)} + q && (10.31) \\ &= a(1) + q && (10.32) \\ &= a + q && (10.33) \end{aligned}$$

For example, the y-intercept of $g(x) = 3 \cdot 2^x + 2$ is given by setting $x = 0$ to get:

$$\begin{aligned} y &= 3 \cdot 2^x + 2 \\ y_{int} &= 3 \cdot 2^0 + 2 \\ &= 3 + 2 \\ &= 5 \end{aligned}$$

The x-intercepts are calculated by setting $y = 0$ as follows:

$$\begin{aligned} y &= ab^{(x)} + q && (10.34) \\ 0 &= ab^{(x_{int})} + q && (10.35) \\ ab^{(x_{int})} &= -q && (10.36) \\ b^{(x_{int})} &= -\frac{q}{a} && (10.37) \end{aligned}$$

Which only has a real solution if either $a < 0$ or $q < 0$. Otherwise, the graph of the function of form $y = ab^{(x)} + q$ does not have any x-intercepts.

For example, the x-intercept of $g(x) = 3 \cdot 2^x + 2$ is given by setting $y = 0$ to get:

$$\begin{aligned} y &= 3 \cdot 2^x + 2 \\ 0 &= 3 \cdot 2^{x_{int}} + 2 \\ -2 &= 3 \cdot 2^{x_{int}} \\ 2^{x_{int}} &= \frac{-2}{3} \end{aligned}$$

which has no real solution. Therefore, the graph of $g(x) = 3 \cdot 2^x + 2$ does not have any x-intercepts.

Asymptotes

There are two asymptotes for functions of the form $y = ab^{(x)} + q$. They are determined by examining the domain and range.

We saw that the function was undefined at $x = 0$ and for $y = q$. Therefore the asymptotes are $x = 0$ and $y = q$.

For example, the domain of $g(x) = 3 \cdot 2^x + 2$ is $\{x : x \in \mathbb{R}, x \neq 0\}$ because $g(x)$ is undefined at $x = 0$. We also see that $g(x)$ is undefined at $y = 2$. Therefore the range is $\{g(x) : g(x) \in (-\infty{,}2) \cup (2{,}\infty)\}$.

From this we deduce that the asymptotes are at $x = 0$ and $y = 2$.

Sketching Graphs of the Form $f(x) = ab^{(x)} + q$

In order to sketch graphs of functions of the form, $f(x) = ab^{(x)} + q$, we need to calculate determine four characteristics:

1. domain and range
2. y-intercept
3. x-intercept

For example, sketch the graph of $g(x) = 3 \cdot 2^x + 2$. Mark the intercepts.

We have determined the domain to be $\{x : x \in \mathbb{R}\}$ and the range to be $\{g(x) : g(x) \in [2{,}\infty)\}$.

The y-intercept is $y_{int} = 5$ and there are no x-intercepts.

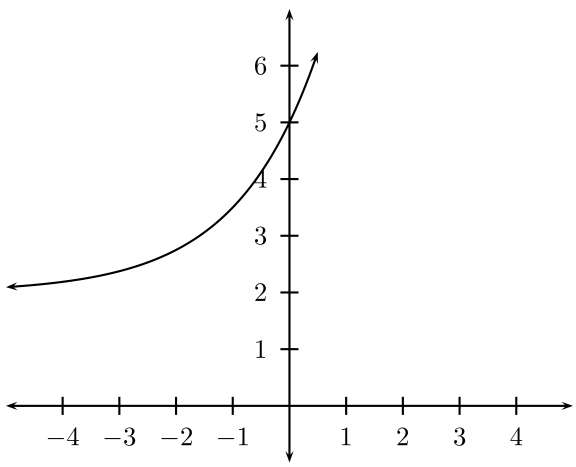
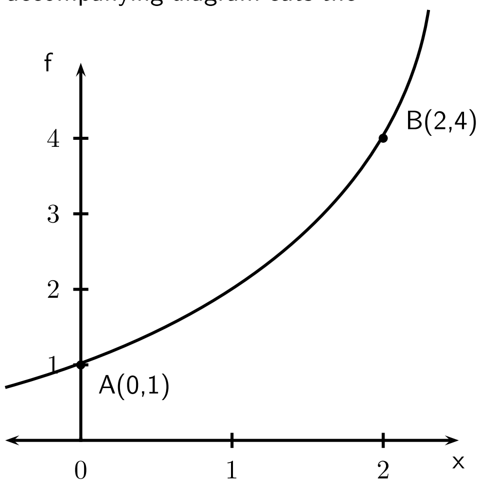


Figure 10.14: Graph of $g(x) = 3 \cdot 2^x + 2$.

Exercise: Exponential Functions and Graphs

1. Draw the graphs of $y = 2^x$ and $y = (\frac{1}{2})^x$ on the same set of axes.
   (a) Is the x-axis and asymptote or and axis of symmetry to both graphs ? Explain your answer.
   (b) Which graph is represented by the equation $y = 2^{-x}$? Explain your answer.
   (c) Solve the equation $2^x = (\frac{1}{2})^x$ graphically and check that your answer is correct by using substitution.
   (d) Predict how the graph $y = 2.2^x$ will compare to $y = 2^x$ and then draw the graph of $y = 2.2^x$ on the same set of axes.
2. The curve of the exponential function f in the accompanying diagram cuts the y-axis at the point A(0; 1) and B(2; 4) is on f.



   (a) Determine the equation of the function f.
   (b) Determine the equation of h, the function of which the curve is the reflection of the curve of f in the x-axis.
   (c) Determine the range of h.

10.13 End of Chapter Exercises

1. Given the functions $f(x) = -2x^2 - 18$ and $g(x) = -2x + 6$
   (a) Draw f and g on the same set of axes.
   (b) Calculate the points of intersection of f and g.
   (c) Hence use your graphs and the points of intersection to solve for x when:
      i. $f(x) > 0$
      ii. $\frac{f(x)}{g(x)} \leq 0$
   (d) Give the equation of the reflection of f in the x-axis.
2. After a ball is dropped, the rebound height of each bounce decreases. The equation $y = 5(0.8)^x$ shows the relationship between x, the number of bounces, and y, the height of the bounce, for a certain ball. What is the approximate height of the fifth bounce of this ball to the nearest tenth of a unit ?
3. Marc had 15 coins in five rand and two rand pieces. He had 3 more R2-coins than R5-coins. He wrote a system of equations to represent this situation, letting x represent the number of five rand coins and y represent the number of two rand coins. Then he solved the system by graphing.

(a) Write down the system of equations.
(b) Draw their graphs on the same set of axes.
(c) What is the solution?

Chapter 11

Average Gradient - Grade 10 Extension

11.1 Introduction

In chapter 10.7.4, we saw that the gradient of a straight line graph is calculated as:

$$\frac{y_2 - y_1}{x_2 - x_1} \tag{11.1}$$

for two points (x_1,y_1) and (x_2,y_2) on the graph.

We can now define the *average gradient* between any two points, (x_1,y_1) and (x_2,y_2) as:

$$\frac{y_2 - y_1}{x_2 - x_1}. \tag{11.2}$$

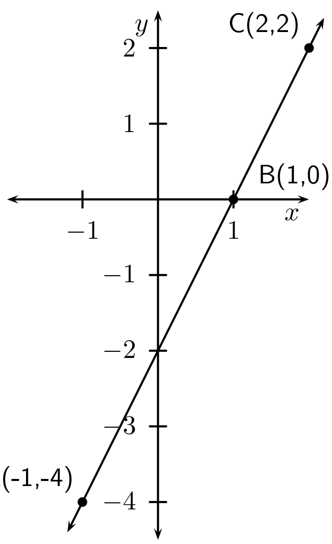
This is the same as (11.1).

11.2 Straight-Line Functions

Activity :: Investigation : Average Gradient - Straight Line Function

Fill in the table by calculating the average gradient over the indicated intervals for the function $f(x) = 2x - 2$:

	x_1	x_2	y_1	y_2	$\frac{y_2-y_1}{x_2-x_1}$
A-B					
A-C					
B-C					



What do you notice about the gradients over each interval?

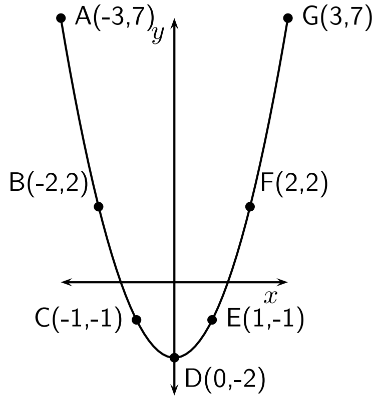
The average gradient of a straight-line function is the same over any two intervals on the function.

11.3 Parabolic Functions

Activity :: Investigation : Average Gradient - Parabolic Function

Fill in the table by calculating the average gradient over the indicated intervals for the function $f(x) = 2x - 2$:

	x_1	x_2	y_1	y_2	$\frac{y_2-y_1}{x_2-x_1}$
A-B					
B-C					
C-D					
D-E					
E-F					
F-G					



What do you notice about the average gradient over each interval? What can you say about the average gradients between A and D compared to the average gradients between D and G?

The average gradient of a parabolic function depends on the interval and is the gradient of a straight line that passes through the points on the interval.

For example, in Figure 11.1 the various points have been joined by straight-lines. The average gradients between the joined points are then the gradients of the straight lines that pass through the points.

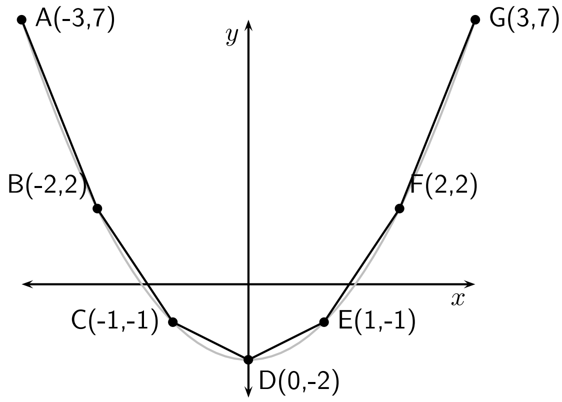


Figure 11.1: The average gradient between two points on a curve is the gradient of the straight line that passes through the points.

Method: Average Gradient

Given the equation of a curve and two points (x_1, x_2):

1. Write the equation of the curve in the form $y = \ldots$.
2. Calculate y_1 by substituting x_1 into the equation for the curve.
3. Calculate y_2 by substituting x_2 into the equation for the curve.
4. Calculate the average gradient using:

$$\frac{y_2 - y_1}{x_2 - x_1}$$

Worked Example 49: Average Gradient

Question: Find the average gradient of the curve $y = 5x^2 - 4$ between the points $x = -3$ and $x = 3$

Answer

Step 1 : Label points

Label the points as follows:

$$x_1 = -3$$

$$x_2 = 3$$

to make it easier to calculate the gradient.

Step 2 : Calculate the y coordinates

We use the equation for the curve to calculate the y-value at x_1 and x_2.

$$\begin{aligned} y_1 &= 5x_1^2 - 4 \\ &= 5(-3)^2 - 4 \\ &= 5(9) - 4 \\ &= 41 \end{aligned}$$

$$\begin{aligned} y_2 &= 5x_2^2 - 4 \\ &= 5(3)^2 - 4 \\ &= 5(9) - 4 \\ &= 41 \end{aligned}$$

Step 3 : Calculate the average gradient

$$\begin{aligned} \frac{y_2 - y_1}{x_2 - x_1} &= \frac{41 - 41}{3 - (-3)} \\ &= \frac{0}{3 + 3} \\ &= \frac{0}{6} \\ &= 0 \end{aligned}$$

Step 4 : Write the final answer

The average gradient between $x = -3$ and $x = 3$ on the curve $y = 5x^2 - 4$ is 0.

11.4 End of Chapter Exercises

1. An object moves according to the function $d = 2t^2 + 1$, where d is the distance in metres and t the time in seconds. Calculate the average speed of the object between 2 and 3 seconds.
2. Given: $f(x) = x^3 - 6x$.
   Determine the average gradient between the points where $x = 1$ and $x = 4$.

Chapter 12

Geometry Basics

12.1 Introduction

The purpose of this chapter is to recap some of the ideas that you learned in geometry and trigonometry in earlier grades. You should feel comfortable with the work covered in this chapter before attempting to move onto the Grade 10 Geometry Chapter (Chapter 13) or the Grade 10 Trigonometry Chapter (Chapter 14). This chapter revises:

1. Terminology: quadrilaterals, vertices, sides, angles, parallel lines, perpendicular lines, diagonals, bisectors, transversals
2. Similarities and differences between quadrilaterals
3. Properties of triangles and quadrilaterals
4. Congruence
5. Classification of angles into acute, right, obtuse, straight, reflex or revolution
6. Theorem of Pythagoras which is used to calculate the lengths of sides of a right-angled triangle

12.2 Points and Lines

The two simplest objects in geometry are *points* and *lines*.

A point is something that is not very wide or high and is usually used in geometry as a marker of a position. Points are usually labelled with a capital letter. Some examples of points are shown in Figure 12.1.

A line is formed when many points are placed next to each other. Lines can be straight or curved, but are always continuous. This means that there are never any breaks in the lines. The endpoints of lines are labelled with capital letters. Examples of two lines are shown in Figure 12.1.

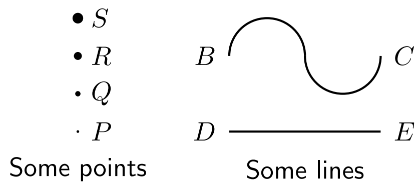


Figure 12.1: Examples of some points (labelled P, Q, R and S) and some lines (labelled BC and DE).

Lines are labelled according to the start point and end point. We call the line that starts at a point A and ends at a point B, AB. Since the line from point B to point A is the same as the line from point A to point B, we have that AB=BA.

The length of the line between points A and B is AB. So if we say $AB = CD$ we mean that the length of the line between A and B is equal to the length of the line between C and D.

In science, we sometimes talk about a *vector* and this is just a fancy way of saying the we are referring to the line that starts at one point and moves in the direction of the other point. We label a vector in a similar manner to a line, with $\vec{AB}$ referring to the vector from the point A with length AB and in the direction from point A to point B. Similarly, $\vec{BA}$ is the line segment with the same length but direction from point B to point A. Usually, vectors are only equal if they have the same length *and* same direction. So, usually, $\vec{AB} \neq \vec{BA}$.

A line is measured in *units of length*. Some common units of length are listed in Table 12.1.

Table 12.1: Some common units of length and their abbreviations.

Unit of Length	Abbreviation
kilometre	km
metre	m
centimetre	cm
millimetre	mm

12.3 Angles

An *angle* is formed when two straight lines meet at a point. The point at which two lines meet is known as a *vertex*. Angles are labelled with a ˆ on a letter, for example, in Figure 12.3, the angle is at $\hat{B}$. Angles can also be labelled according to the line segments that make up the angle. For example, in Figure 12.3, the angle is made up when line segments CB and BA meet. So, the angle can be referred to as $\angle CBA$ or $\angle ABC$. The $\angle$ symbol is a short method of writing angle in geometry.

Angles are measured in *degrees* which is denoted by $^\circ$.

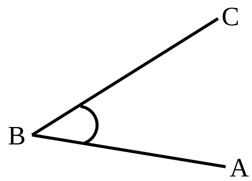


Figure 12.2: Angle labelled as $\hat{B}$, $\angle CBA$ or $\angle ABC$

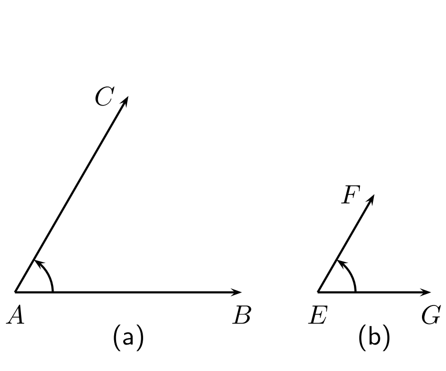


Figure 12.3: Examples of angles. $\hat{A} = \hat{E}$, even though the lines making up the angles are of different lengths.

12.3.1 Measuring angles

The size of an angle does not depend on the length of the lines that are joined to make up the angle, but depends only on how both the lines are placed as can be seen in Figure 12.3. This means that the idea of length cannot be used to measure angles. An angle is a rotation around the vertex.

Using a Protractor

A protractor is a simple tool that is used to measure angles. A picture of a protractor is shown in Figure 12.4.

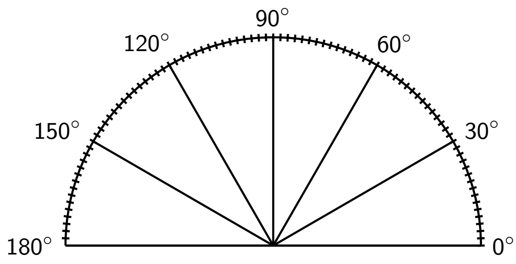


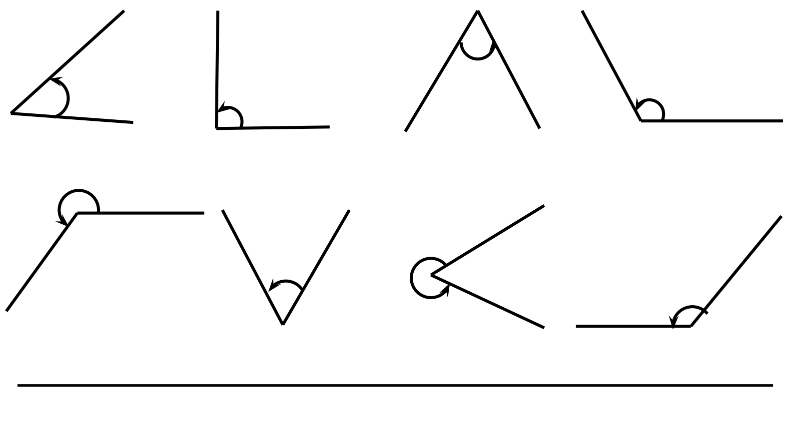
Figure 12.4: Diagram of a protractor.

Method:

Using a protractor

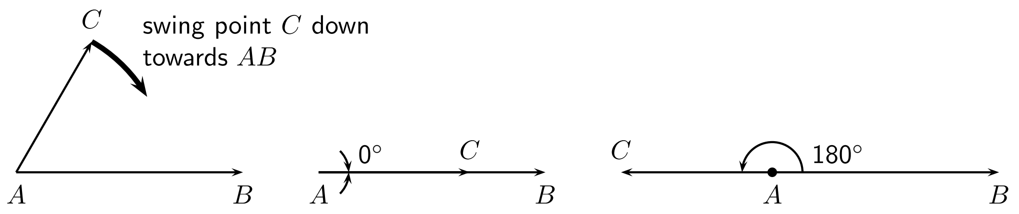
1. Place the bottom line of the protractor along one line of the angle.
2. Move the protractor along the line so that the centre point on the protractor is at the vertex of the two lines that make up the angle.
3. Follow the second line until it meets the marking on the protractor and read off the angle. Make sure you start measuring at 0°.

Activity :: Measuring Angles : Use a protractor to measure the following angles:



12.3.2 Special Angles

What is the smallest angle that can be drawn? The figure below shows two lines (CA and AB) making an angle at a common vertex A. If line CA is rotated around the common vertex A, down towards line AB, then the smallest angle that can be drawn occurs when the two lines are pointing in the same direction. This gives an angle of 0°.



If line CA is now swung upwards, any other angle can be obtained. If line CA and line AB point in opposite directions (the third case in the figure) then this forms an angle of 180°.

Important: If three points A, B and C lie on a straight line, then the angle between them is 180°. Conversely, if the angle between three points is 180°, then the points lie on a straight line.

An angle of 90° is called a *right angle*. A right angle is half the size of the angle made by a straight line (180°). We say CA is *perpendicular* to AB or $CA \perp AB$. An angle twice the size of a straight line is 360°. An angle measuring 360° looks identical to an angle of 0°, except for the labelling. We call this a *revolution*.

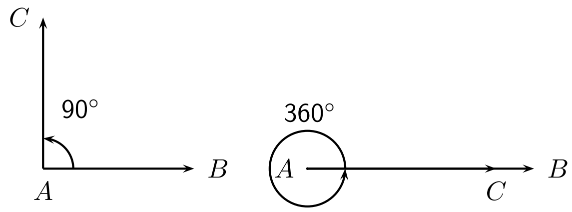


Figure 12.5: An angle of 90° is known as a *right angle*.

Extension: Angles larger than 360°

All angles larger than 360° also look like we have seen them before. If you are given an angle that is larger than 360°, continue subtracting 360° from the angle, until you get an answer that is between 0°and 360°. Angles that measure more than 360° are largely for mathematical convenience.

Important:

- *Acute angle*: An angle $\geq 0°$ and $< 90°$.
- *Right angle*: An angle measuring $90°$.
- *Obtuse angle*: An angle $> 90°$ and $< 180°$.
- *Straight angle*: An angle measuring 180°.
- *Reflex angle*: An angle $> 180°$ and $< 360°$.
- *Revolution*: An angle measuring $360°$.

These are simply labels for angles in particular ranges, shown in Figure 12.6.

Once angles can be measured, they can then be compared. For example, all right angles are 90°, therefore all right angles are equal, and an obtuse angle will always be larger than an acute angle.

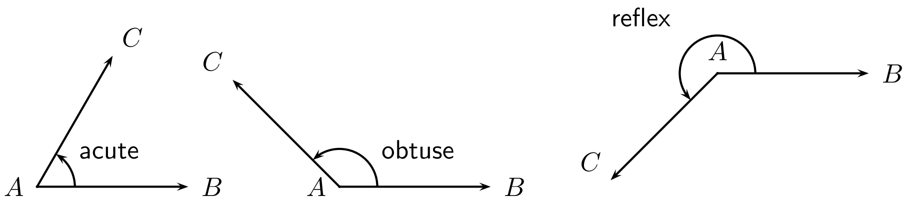


Figure 12.6: Three types of angles defined according to their ranges.

12.3.3 Special Angle Pairs

In Figure 12.7, straight lines AB and CD intersect at point X, forming four angles: $\hat{X}_1$, $\hat{X}_2$, $\hat{X}_3$ and $\hat{X}_4$.

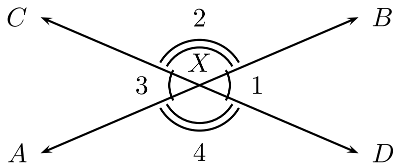


Figure 12.7: Two intersecting straight lines with vertical angles $\hat{X}_1$,$\hat{X}_3$ and $\hat{X}_2$,$\hat{X}_4$.

The table summarises the special angle pairs that result.

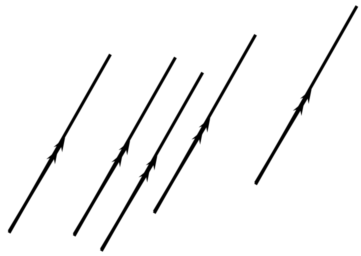
Special Angle	Property	Example
adjacent angles	share a common vertex and a common side	($\hat{X}_1$,$\hat{X}_2$), ($\hat{X}_2$,$\hat{X}_3$), ($\hat{X}_3$,$\hat{X}_4$), ($\hat{X}_4$,$\hat{X}_1$)
linear pair (adjacent angles on a straight line)	adjacent angles formed by two intersecting straight lines that by definition add to 180°	$\hat{X}_1 + \hat{X}_2 = 180°$ $\hat{X}_2 + \hat{X}_3 = 180°$ $\hat{X}_3 + \hat{X}_4 = 180°$ $\hat{X}_4 + \hat{X}_1 = 180°$
vertically opposite angles	angles formed by two intersecting straight lines that share a vertex but do not share any sides	$\hat{X}_1 = \hat{X}_3$ $\hat{X}_2 = \hat{X}_4$
supplementary angles	two angles whose sum is 180°	
complementary angles	two angles whose sum is 90°	

Important: The vertically opposite angles formed by the intersection of two straight lines are equal. Adjacent angles on a straight line are supplementary.

12.3.4 Parallel Lines intersected by Transversal Lines

Two lines intersect if they cross each other at a point. For example, at a traffic intersection, two or more streets intersect; the middle of the intersection is the common point between the streets.

Parallel lines are lines that never intersect. For example the tracks of a railway line are parallel. We wouldn't want the tracks to intersect as that would be catastrophic for the train!



All these lines are parallel to each other. Notice the arrow symbol for parallel.

Interesting Fact

A section of the Australian National Railways Trans-Australian line is perhaps one of the longest pairs of man-made parallel lines.

> **Longest Railroad Straight** (Source: www.guinnessworldrecords.com)
> The Australian National Railways Trans-Australian line over the Nullarbor Plain, is 478 km (297 miles) dead straight, from Mile 496, between Nurina and Loongana, Western Australia, to Mile 793, between Ooldea and Watson, South Australia.

A *transversal* of two or more lines is a line that intersects these lines. For example in Figure 12.8, AB and CD are two parallel lines and EF is a transversal. We say $AB \parallel CD$. The properties of the angles formed by these intersecting lines are summarised in the table below.

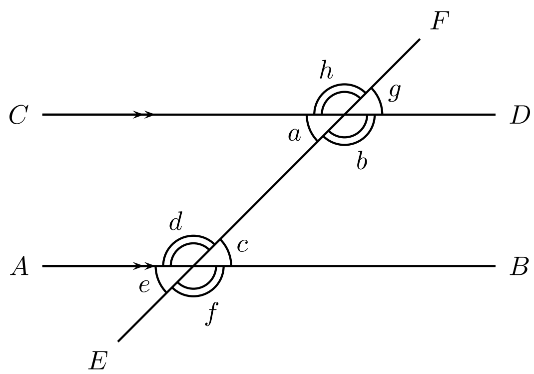
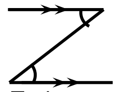
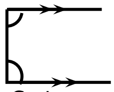
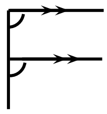


Figure 12.8: Parallel lines intersected by a transversal

Extension: Euclid's Parallel Line Postulate

If a straight line falling on two straight lines makes the two interior angles on the same side less than two right angles (180°), the two straight lines, if produced indefinitely, will meet on that side. This postulate can be used to prove many identities about the angles formed when two parallel lines are cut by a transversal.

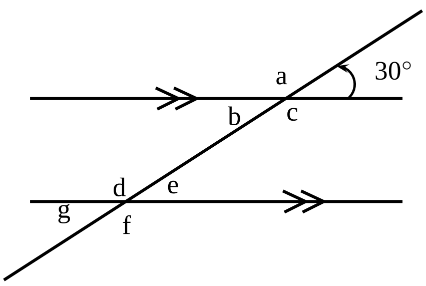
Name of angle	Definition	Examples	Notes
interior angles	the angles that lie inside the parallel lines	a, b, c and d are interior angles	the word *interior* means inside
exterior angles	the angles that lie outside the parallel lines	e, f, g and h are exterior angles	the word *exterior* means outside
alternate interior angles	the interior angles that lie on opposite sides of the transversal	(a,c) and (b,d) are pairs of alternate interior angles, $a = c$, $b = d$	 Z shape
co-interior angles on the same side	co-interior angles that lie on the same side of the transversal	(a,d) and (b,c) are interior angles on the same side. $a + d = 180°$, $b + c = 180°$	 C shape
corresponding angles	the angles on the same side of the transversal and the same side of the parallel lines	(a,e), (b,f), (c,g) and (d,h) are pairs of corresponding angles. $a = e$, $b = f$, $c = g$, $d = h$	 F shape

Important:

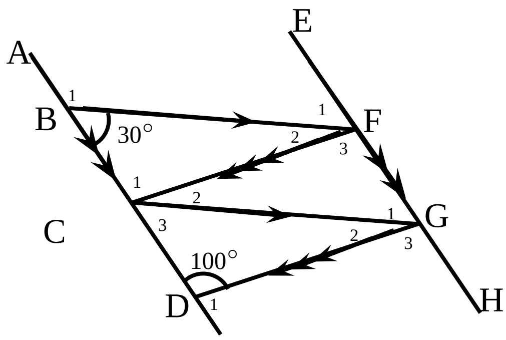
1. If two parallel lines are intersected by a transversal, the sum of the co-interior angles on the same side of the transversal is 180°.
2. If two parallel lines are intersected by a transversal, the alternate interior angles are equal.
3. If two parallel lines are intersected by a transversal, the corresponding angles are equal.
4. If two lines are intersected by a transversal such that any pair of co-interior angles on the same side is supplementary, then the two lines are parallel.
5. If two lines are intersected by a transversal such that a pair of alternate interior angles are equal, then the lines are parallel.
6. If two lines are intersected by a transversal such that a pair of alternate corresponding angles are equal, then the lines are parallel.

Exercise: Angles

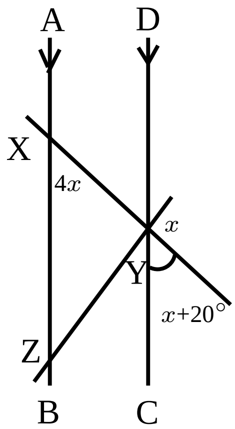
1. Use adjacent, corresponding, co-interior and alternate angles to fill in all the angles labeled with letters in the diagram alongside:



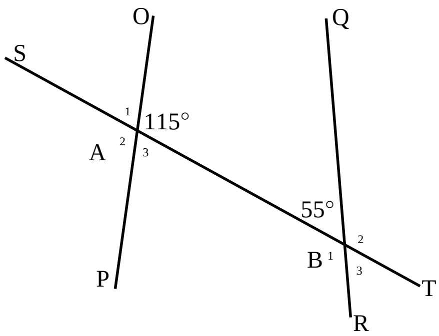
2. Find all the unknown angles in the figure alongside:



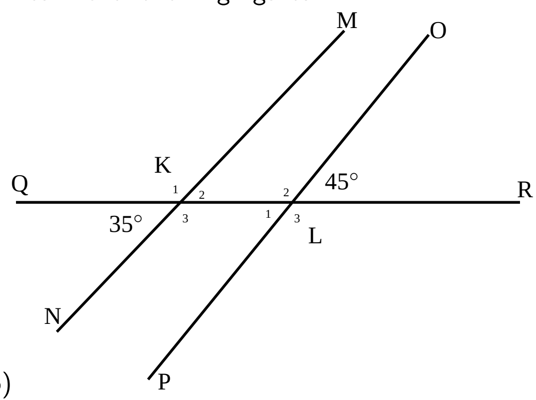
3. Find the value of x in the figure alongside:



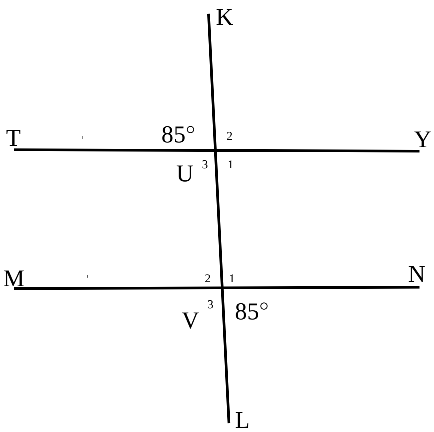
4. Determine whether there are pairs of parallel lines in the following figures.



a)

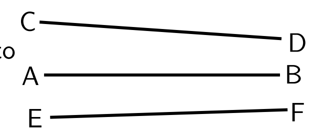


b)



c)

5. If AB is parallel to CD and AB is parallel to EF, prove that CD is parallel to EF:



12.4 Polygons

If you take some lines and join them such that the end point of the first line meets the starting point of the last line, you will get a *polygon*. Each line that makes up the polygon is known as a *side*. A polygon has interior angles. These are the angles that are inside the polygon. The number of sides of a polygon equals the number of interior angles. If a polygon has equal length sides and equal interior angles then the polygon is called a *regular polygon*. Some examples of polygons are shown in Figure 12.9.

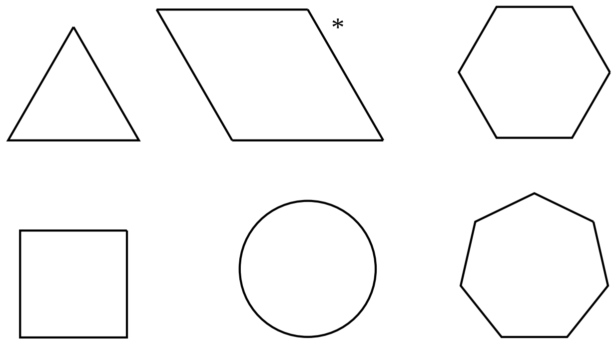


Figure 12.9: Examples of polygons. They are all regular, except for the one marked *

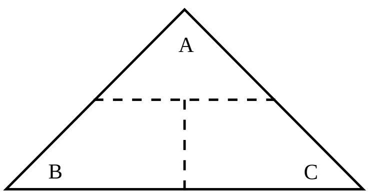
12.4.1 Triangles

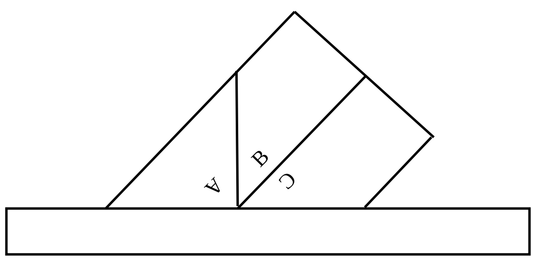
A triangle is a three-sided polygon. There are four types of triangles: equilateral, isosceles, right-angled and scalene. The properties of these triangles are summarised in Table 12.2.

Properties of Triangles

Activity :: Investigation : Sum of the angles in a triangle

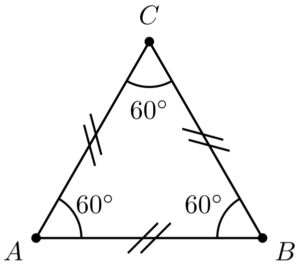
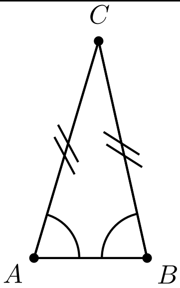
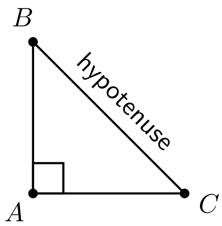
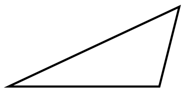
1. Draw on a piece of paper a triangle of any size and shape
2. Cut it out and label the angles $\hat{A}$, $\hat{B}$ and $\hat{C}$ on both sides of the paper
3. Draw dotted lines as shown and cut along these lines to get three pieces of paper
4. Place them along your ruler as shown to see that $\hat{A} + \hat{B} + \hat{C} = 180°$





Important: The sum of the angles in a triangle is 180°.

Table 12.2: Types of Triangles

Name	Diagram	Properties
equilateral		All three sides are equal in length and all three angles are equal.
isosceles		Two sides are equal in length. The angles opposite the equal sides are equal.
right-angled		This triangle has one right angle. The side opposite this angle is called the *hypotenuse*.
scalene (non-syllabus)		All sides and angles are different.

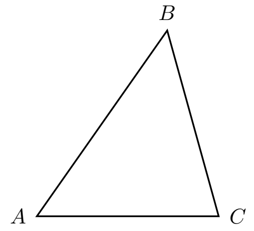


Figure 12.10: In any triangle, $\angle A + \angle B + \angle C = 180°$

Important: Any exterior angle of a triangle is equal to the sum of the two opposite interior angles. An exterior angle is formed by extending any one of the sides.

$B\hat{A}C + B\hat{C}A = C\hat{B}D$ $\quad$ $B\hat{A}C + C\hat{B}A = B\hat{C}D$ $\quad$ $A\hat{B}C + B\hat{A}C = A\hat{C}D$

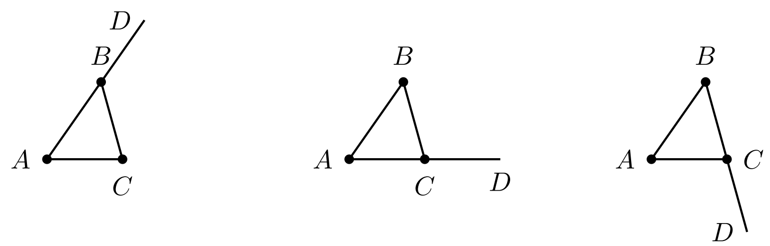
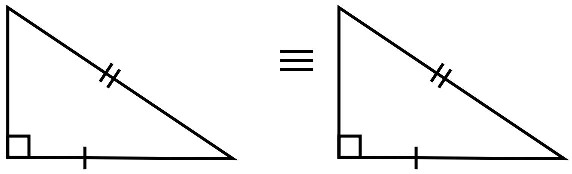
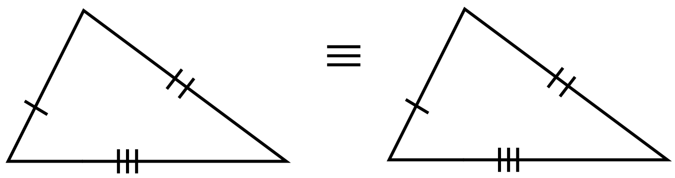
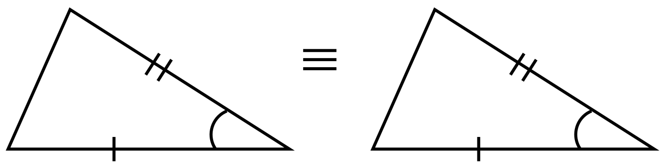
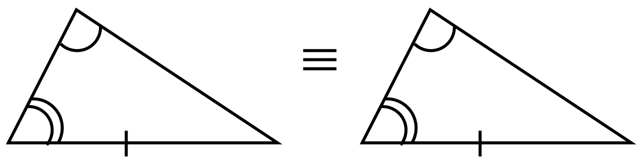
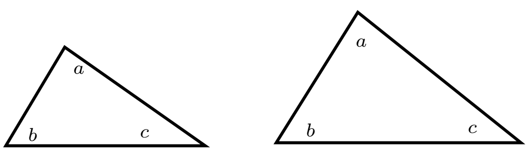
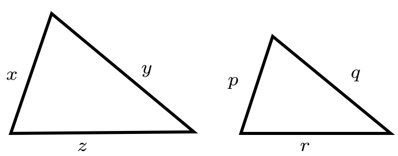


Figure 12.11: In any triangle, any exterior angle is equal to the sum of the two opposite interior angles.

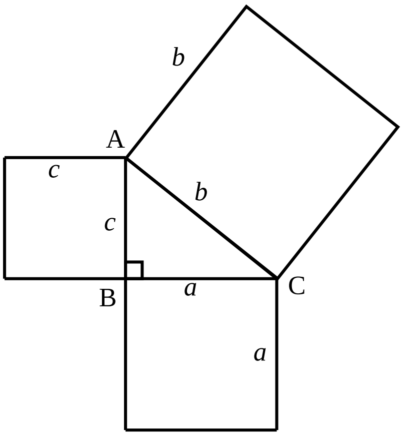
Congruent Triangles

Label	Description	Diagram
RHS	If the hypotenuse and one side of a right-angled triangle are equal to the hypotenuse and the respective side of another triangle then the triangles are congruent.	
SSS	If three sides of a triangle are equal in length to the same sides of another triangle then the two triangles are congruent	
SAS	If two sides and the included angle of one triangle are equal to the same two sides and included angle of another triangle, then the two triangles are congruent.	
AAS	If one side and two angles of one triangle are equal to the same one side and two angles of another triangle, then the two triangles are congruent.	

Similar Triangles

Description	Diagram
If all three pairs of corresponding angles of two triangles are equal, then the triangles are similar.	
If all pairs of corresponding sides of two triangles are in proportion, then the triangles are similar.	 $\frac{x}{p} = \frac{y}{q} = \frac{z}{r}$

The theorem of Pythagoras

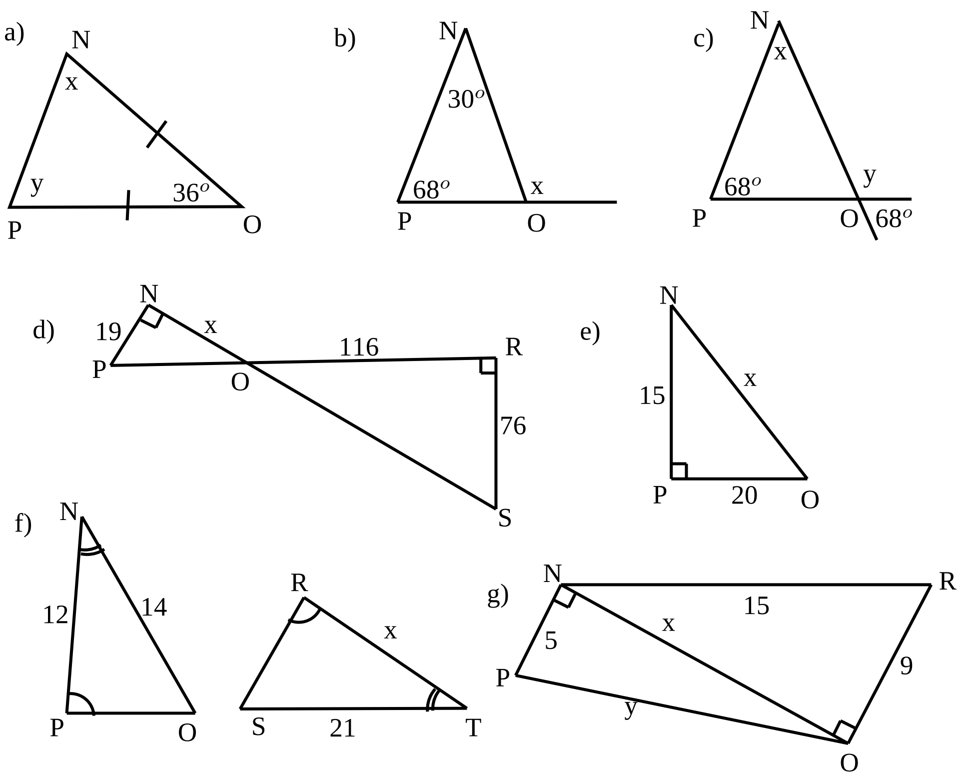


If $\triangle$ABC is right-angled ($\hat{B} = 90°$) then
$b^2 = a^2 + c^2$

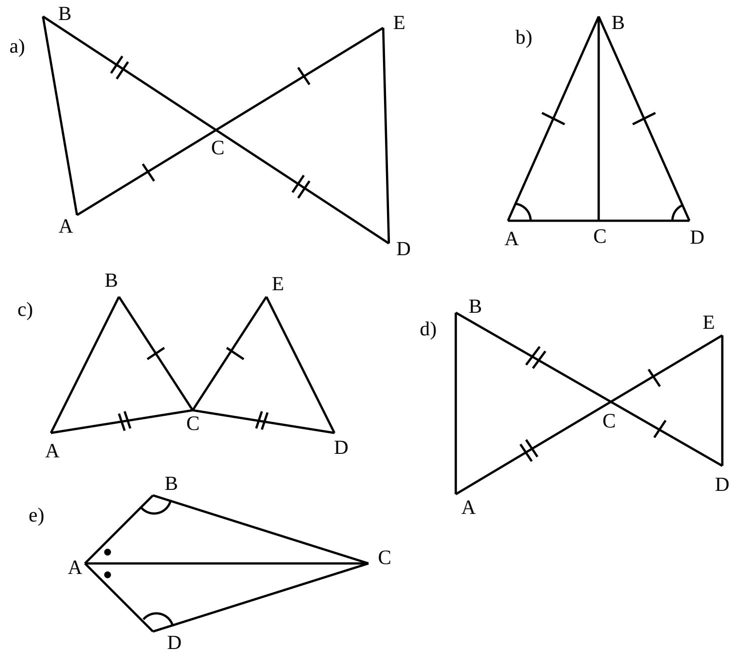
Converse:
If $b^2 = a^2 + c^2$, then
$\triangle$ABC is right-angled ($\hat{B} = 90°$).

Exercise: Triangles

1. Calculate the unknown variables in each of the following figures. All lengths are in mm.



2. State whether or not the following pairs of triangles are congruent or not. Give reasons for your answers. If there is not enough information to make a descision, say why.



12.4.2 Quadrilaterals

A *quadrilateral* is any polygon with four sides. The basic quadrilaterals are the trapezium, parallelogram, rectangle, rhombus, square and kite.

Name of quadrilateral	Figure
trapezium	Figure 12.12
parallelogram	Figure 12.13
rectangle	Figure 12.14
rhombus	Figure 12.15
square	Figure 12.16
kite	Figure 12.17

Table 12.3: Examples of quadrilaterals.

Trapezium

A trapezium is a quadrilateral with one pair of parallel opposite sides. It may also be called a *trapezoid*. A special type of trapezium is the *isosceles trapezium*, where one pair of opposite sides is parallel, the other pair of sides is equal in length and the angles at the ends of each parallel side are equal. An isosceles trapezium has one line of symmetry and its diagonals are equal in length.

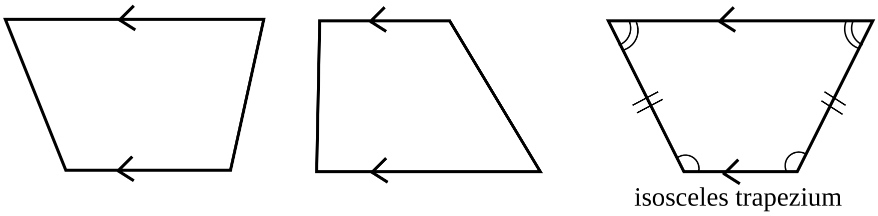


Figure 12.12: Examples of trapeziums.

Parallelogram

A trapezium with both sets of opposite sides parallel is called a *parallelogram*. A summary of the properties of a parallelogram is:

- Both pairs of opposite sides are parallel.
- Both pairs of opposite sides are equal in length.
- Both pairs of opposite angles are equal.
- Both diagonals bisect each other (i.e. they cut each other in half).

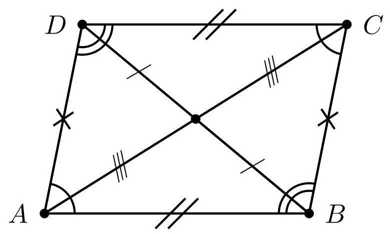


Figure 12.13: An example of a parallelogram.

Rectangle

A *rectangle* is a parallelogram that has all four angles equal to $90°$. A summary of the properties of a rectangle is:

- Both pairs of opposite sides are parallel.
- Both pairs of opposite sides are of equal length equal.
- Both diagonals bisect each other.
- Diagonals are equal in length.
- All angles are right angles.

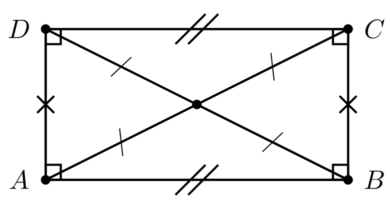


Figure 12.14: Example of a rectangle.

Rhombus

A *rhombus* is a parallelogram that has all four side of equal length. A summary of the properties of a rhombus is:

- Both pairs of opposite sides are parallel.
- All sides are equal in length.

- Both pairs of opposite angles equal.
- Both diagonals bisect each other at $90°$.
- Diagonals of a rhombus bisect both pairs of opposite angles.

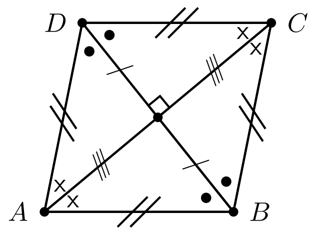


Figure 12.15: An example of a rhombus. A rhombus is a parallelogram with all sides equal.

Square

A *square* is a rhombus that has all four angles equal to 90°.

A summary of the properties of a rhombus is:

- Both pairs of opposite sides are parallel.
- All sides are equal in length.
- All angles are equal to $90°$.
- Both pairs of opposite angles equal.
- Both diagonals bisect each other at $90°$.
- Diagonals are equal in length.
- Diagonals bisect both pairs of opposite angles (ie. all $45°$).

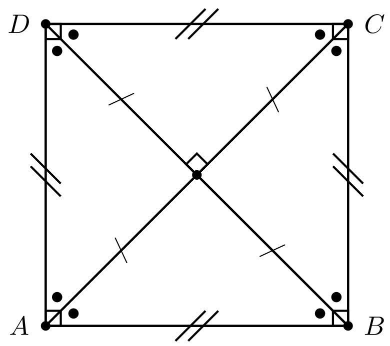


Figure 12.16: An example of a square. A square is a rhombus with all angles equal to 90°.

Kite

A *kite* is a quadrilateral with two pairs of adjacent sides equal.

A summary of the properties of a kite is:

- Two pairs of adjacent sides are equal in length.

- One pair of opposite angles are equal where the angles must be between unequal sides.
- One diagonal bisects the other diagonal and one diagonal bisects one pair of opposite angles.
- Diagonals intersect at right-angles.

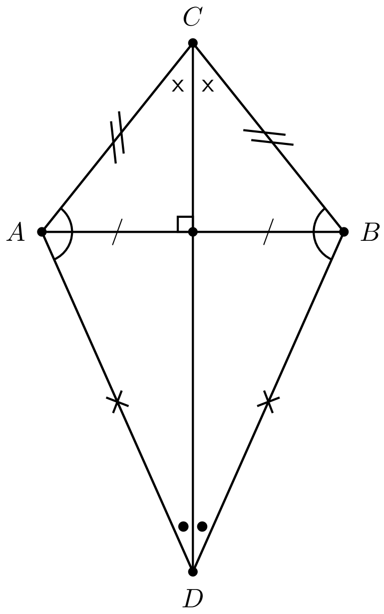


Figure 12.17: An example of a kite.

12.4.3 Other polygons

There are many other polygons, some of which are given in the table below.

Sides	Name
5	pentagon
6	hexagon
7	heptagon
8	octagon
10	decagon
15	pentadecagon

Table 12.4: Table of some polygons and their number of sides.

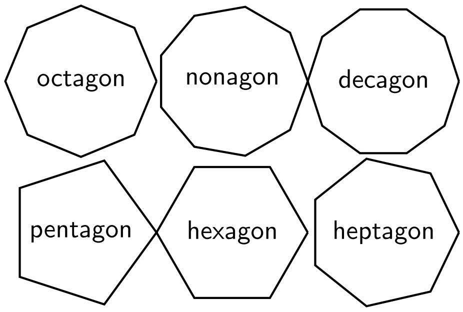


Figure 12.18: Examples of other polygons.

12.4.4 Extra

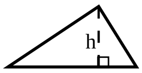
Angles of regular polygons

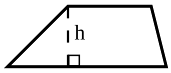
You can calculate the size of the interior angle of a regular polygon by using:

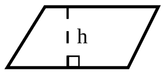
$$\hat{A} = \frac{n-2}{n} \times 180^\circ \qquad (12.1)$$

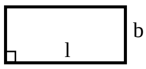
where n is the number of sides and $\hat{A}$ is any angle.

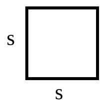
Areas of Polygons

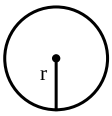
1. Area of triangle: $\frac{1}{2}\times$ base $\times$ perpendicular height 

2. Area of trapezium: $\frac{1}{2}\times$ (sum of $\parallel$ sides) $\times$ perpendicular height 

3. Area of parallelogram and rhombus: base $\times$ perpendicular height 

4. Area of rectangle: length $\times$ breadth 

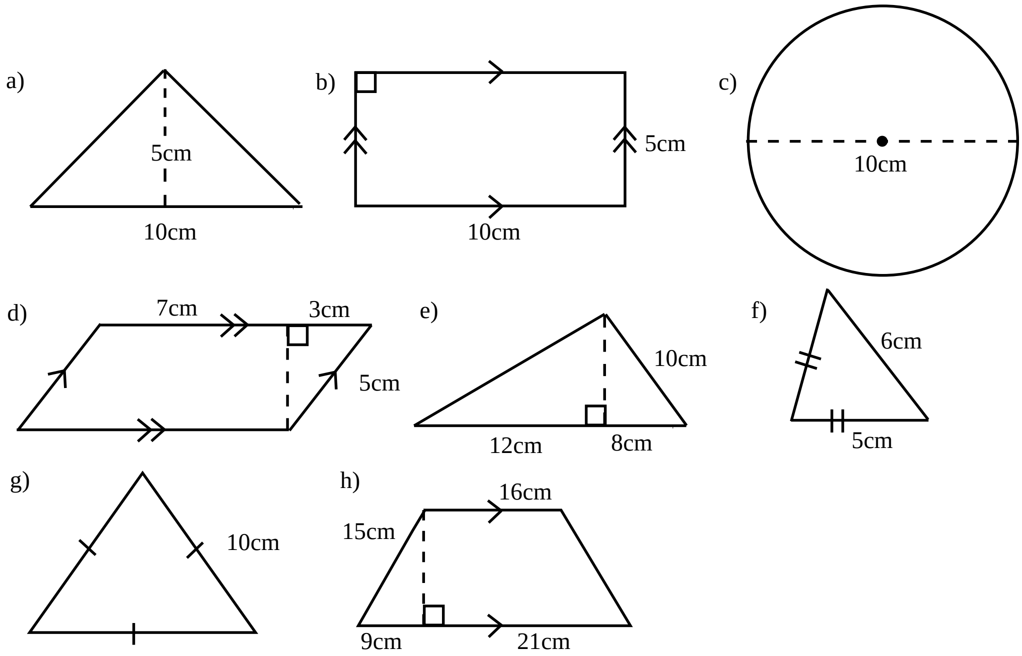
5. Area of square: length of side $\times$ length of side 

6. Area of circle: π x radius^2 

Exercise: Polygons

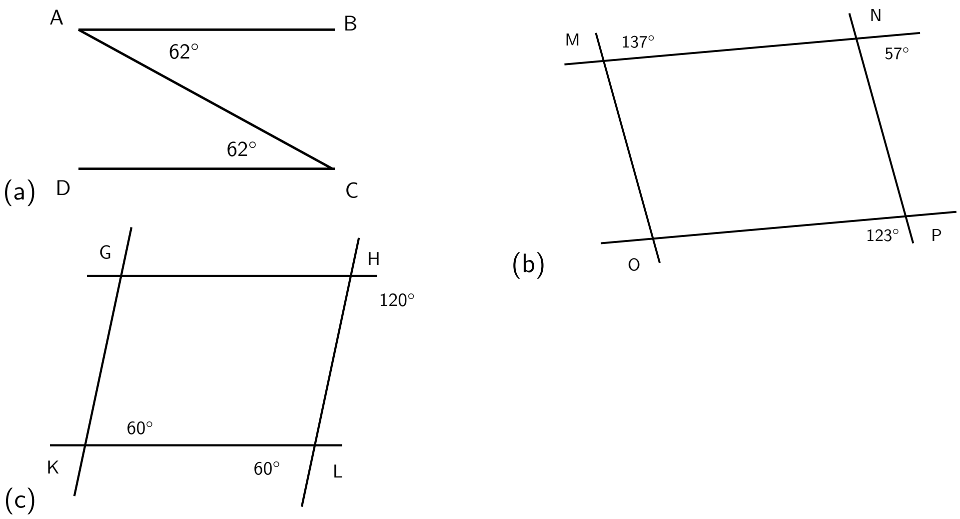
1. For each case below, say whether the statement is true or false. For false statements, give a counter-example to prove it:
   (a) All squares are rectangles
   (b) All rectangles are squares
   (c) All pentagons are similar
   (d) All equilateral triangles are similar
   (e) All pentagons are congruent
   (f) All equilateral triangles are congruent

2. Find the areas of each of the given figures - remember area is measured in square units (cm^2, m^2, mm^2).

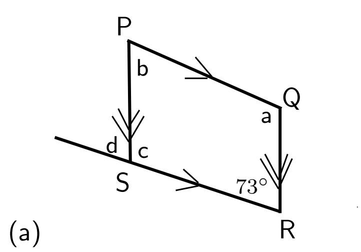


12.5 Exercises

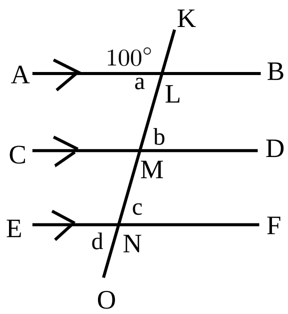
1. Find all the pairs of parallel lines in the following figures, giving reasons in each case.



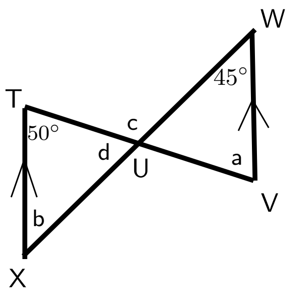
2. Find a, b, c and d in each case, giving reasons.



(b)



(c)



(a) Which of the following claims are true? Give a counter-example for those that are incorrect.

i. All equilateral triangles are similar.

ii. All regular quadrilaterals are similar.

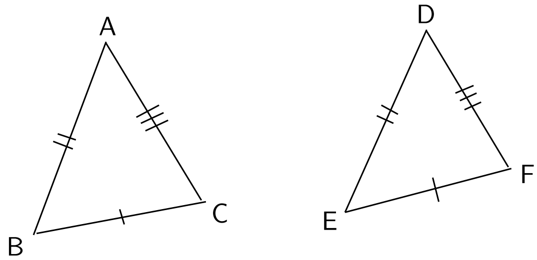
iii. In any $\triangle ABC$ with $\angle ABC = 90°$ we have $AB^3 + BC^3 = CA^3$.

iv. All right-angled isosceles triangles with perimeter 10 cm are congruent.

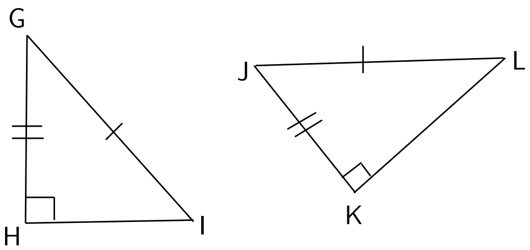
v. All rectangles with the same area are similar.

(b) Say which of the following pairs of triangles are congruent with reasons.

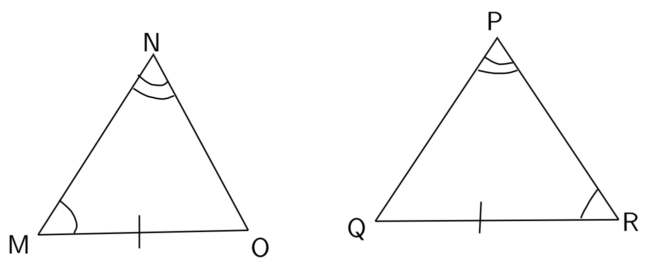
i.



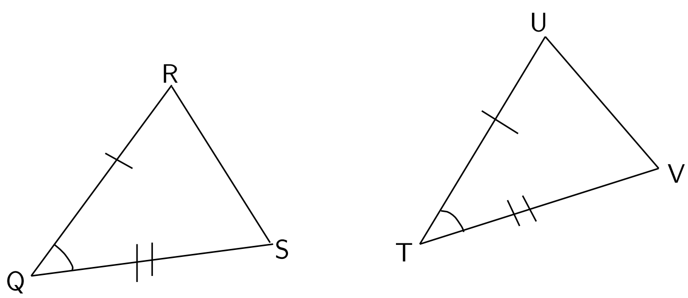
ii.



iii.

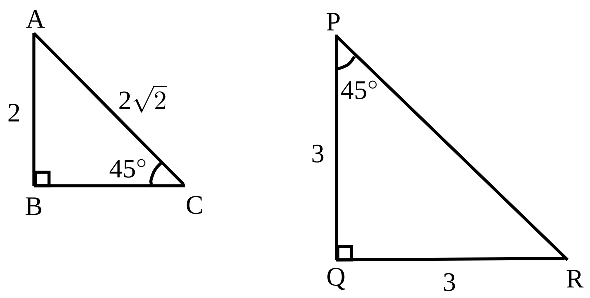


iv.

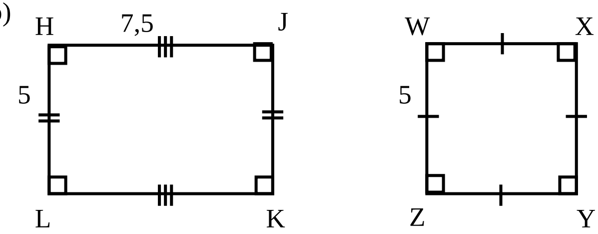


(c) For each pair of figures state whether they are similar or not. Give reasons.

(a)

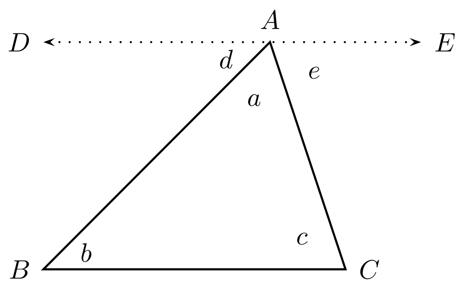


(b)



12.5.1 Challenge Problem

1. Using the figure below, show that the sum of the three angles in a triangle is 180°. Line DE is parallel to BC.



Chapter 13

Geometry - Grade 10

13.1 Introduction

Geometry (Greek: geo = earth, metria = measure) arose as the field of knowledge dealing with spatial relationships. It was one of the two fields of pre-modern mathematics, the other being the study of numbers. In modern times, geometric concepts have become very complex and abstract and are barely recognizable as the descendants of early geometry.

Activity :: Researchproject : History of Geometry

Work in pairs or groups and investigate the history of the foundation of geometry. Describe the various stages of development and how the following cultures used geometry to improve their lives.

1. Ancient Indian geometry (c. 3000 - 500 B.C.)
   (a) Harappan geometry
   (b) Vedic geometry
2. Classical Greek geometry (c. 600 - 300 B.C.)
   (a) Thales and Pythagoras
   (b) Plato
3. Hellenistic geometry (c. 300 B.C - 500 C.E.)
   (a) Euclid
   (b) Archimedes

13.2 Right Prisms and Cylinders

In this section we study how to calculate the surface areas and volumes of right prisms and cylinders. A right prism is a polygon that has been stretched out into a tube so that the height of the tube is perpendicular to the base. A square prism has a base that is a square and a triangular prism has a base that is a triangle.

It is relatively simple to calculate the surface areas and volumes of prisms.

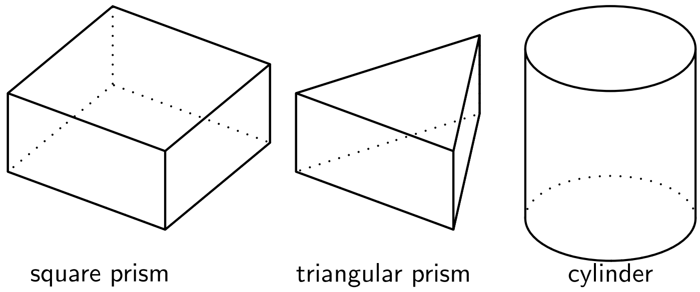


Figure 13.1: Examples of a right square prism, a right triangular prism and a cylinder.

13.2.1 Surface Area

The term *surface area* refers to the total area of the exposed or outside surfaces of a prism. This is easier to understand if you imagine the prism as a solid object.

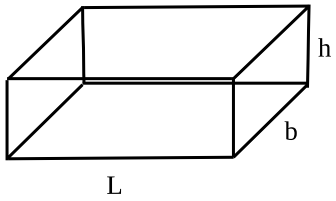
If you examine the prisms in Figure 13.1, you will see that each face of a prism is a simple polygon. For example, the triangular prism has two faces that are triangles and three faces that are rectangles. Therefore, in order to calculate the surface area of a prism you simply have to calculate the area of each face and add it up. In the case of a cylinder the top and bottom faces are circles, while the curved surface flattens into a rectangle.

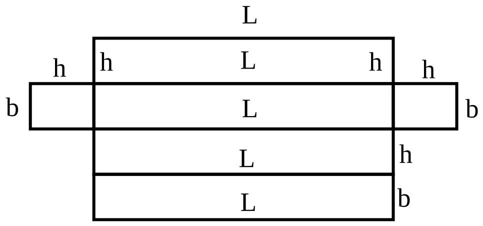
Surface Area of Prisms
Calculate the area of each face and add the areas together to get the surface area.

Activity :: Discussion : surface areas

Study the following prisms, nets and formulae. Explain to your partner, how each relates to the other.

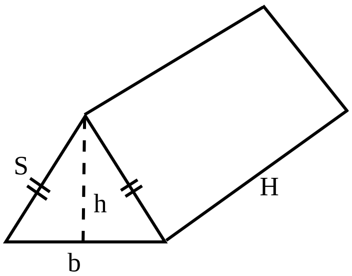
Rectangular Prism

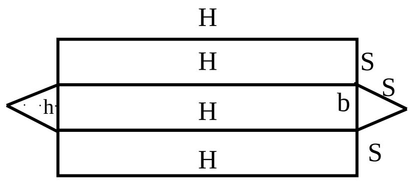




$$S.A. = 2[(L \times b) + (b \times h) + (L \times b)]$$

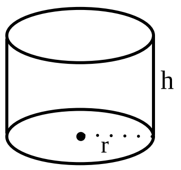
Triangular Prism

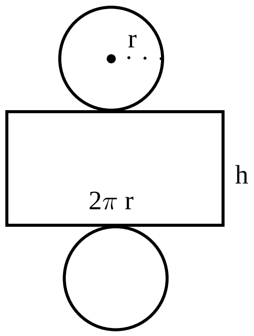




$$S.A. = 2(\frac{1}{2}b \times h) + 2(H \times S) + (H \times b)$$

Cylinder



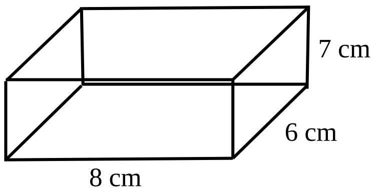


$$S.A. = 2\pi r^2 + 2\pi rh$$

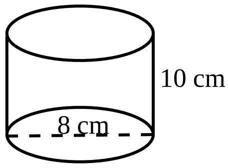
Exercise: Surface areas

1. Calculate the surface area in each of the following:

1.

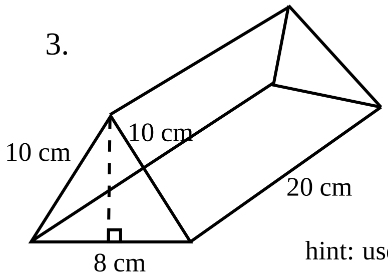


2.



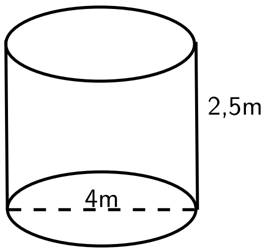
hint: diameter = 2 × radius

3.



hint: use Pythagoras to find height of triangular face.

2. If a litre of paint, paints $2m^2$, how much paint is needed to paint:
   (a) A rectangular swimming pool with dimensions $4m \times 3m \times 2{,}5m$, inside walls and floor only.
   (b) The inside walls and floor of a circular reservoir with diameter $4m$ and height $2{,}5m$



13.2.2 Volume

The volume of a right prism is calculated by multiplying the area of the base by the height. So, for a square prism of side length a and height h the volume is $a \times a \times h = a^2h$.

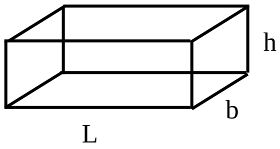
Volume of Prisms
Calculate the area of the base and multiply by the height to get the volume of a prism.

?

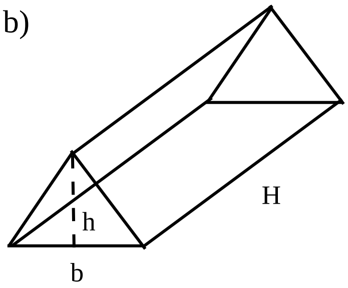
Exercise: Volume

1. Write down the formula for each of the following volumes:

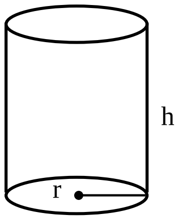
a)



b)

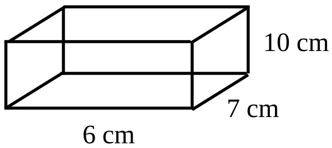


c)

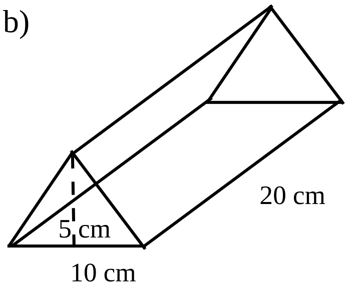


2. Calculate the following volumes:

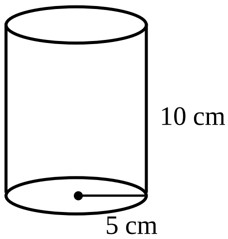
a)



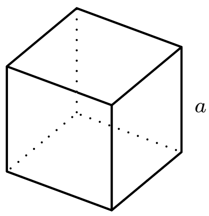
b)



c)

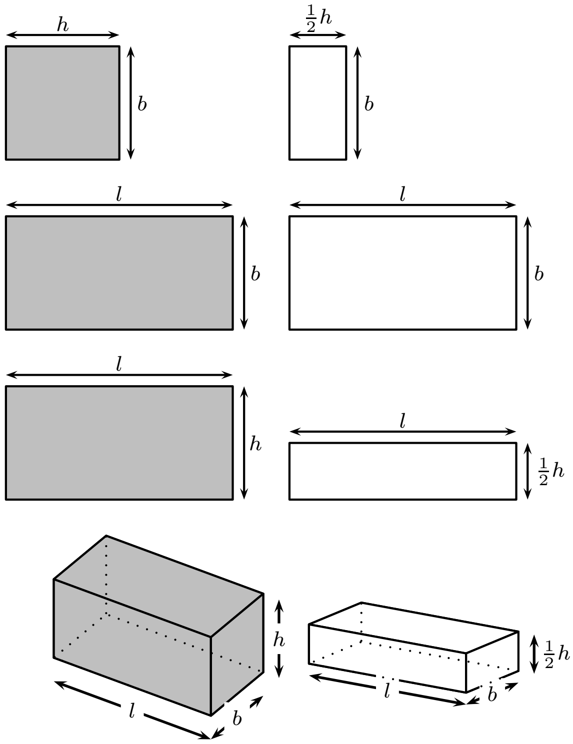


3. A cube is a special prism that has all edges equal. This means that each face is a square. An example of a cube is a die. Show that for a cube with side length a, the surface area is $6a^2$ and the volume is a^3.



Now, what happens to the surface area if one dimension is multiplied by a constant? For example, how does the surface area change when the height of a rectangular prism is divided by 2?

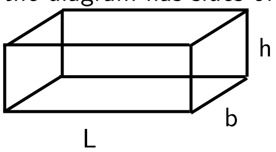
Worked Example 50: Scaling the dimensions of a prism



Surface Area	=	$2(l \times h + l \times b + b \times h)$	Surface Area	=	$2(l \times \frac{1}{2}h + l \times b + b \times \frac{1}{2}h)$
Volume	=	$l \times b \times h$	Volume	=	$l \times b \times \frac{1}{2}h$
				=	$\frac{1}{2}(l \times b \times h)$

Figure 13.2: Rectangular prisms

Question: The size of a prism is specified by the length of its sides. The prism in the diagram has sides of lengths L, b and h.



a) Consider enlarging all sides of the prism by a constant factor x. Where $x > 1$. Calculate the volume and surface area of the enlarged prism as a function of the factor x and the volume of the original volume.

a) In the same way as above now consider the case, where $0 < x < 1$. Now calculate the reduction factor in the volume and the surface area.

Answer

Step 1 : Identify

The volume of a prism is given by:
$V = L \times b \times h$
The surface area of the prism is given by:
$A = 2 \times (L \times b + L \times h + b \times h)$

Step 2 : Rescale

If all the sides of the prism get rescaled, the new sides will be:

$$\begin{aligned} L' &= x \times L \\ b' &= x \times b \\ h' &= x \times h \end{aligned}$$

The new volume will then be given by:

$$\begin{aligned} V' &= L' \times b' \times h' \\ &= x \times L \times x \times b \times x \times h \\ &= x^3 \times L \times b \times h \\ &= x^3 \times V \end{aligned}$$

The new surface area of the prism will be given by:

$$\begin{aligned} A' &= 2 \times (L' \times b' + L' \times h' + b' \times h') \\ &= 2 \times (x \times L \times x \times b + x \times L \times x \times h + x \times b \times x \times h) \\ &= x^2 \times 2 \times (L \times b + L \times h + b \times h) \\ &= x^2 \times A \end{aligned}$$

Step 3 : Interpret

a) We found above that the new volume is given by:
$V' = x^3 \times V$
Since $x > 1$, the volume of the prism will be increased by a factor of x^3.
The surface area of the rescaled prism was given by:
$A' = x^2 \times A$
Again, since $x > 1$, the surface area will be increased by a factor of x^2.

b) The answer here is based on the same ideas as above.
In analogy, since here $0 < x < 1$, the volume will de reduced by a factoe of x^3 and the surface area will be decreased by a factor of x^2

When the length of one of the sides is multiplied by a constant the effect is to multiply the original volume by that constant, as for the example in Figure 13.2.

13.3 Polygons

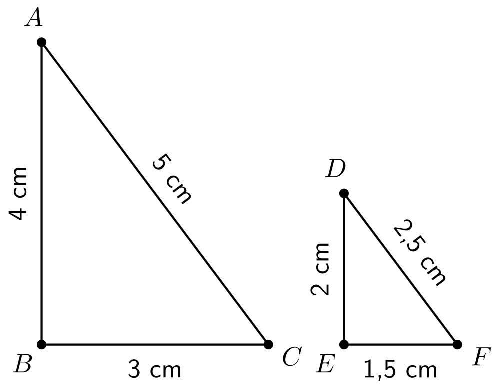
Polygons are all around us. A stop sign is in the shape of an octagon, an eight-sided polygon. The honeycomb of a beehive consists of hexagonal cells.

In this section, you will learn about similar polygons.

13.3.1 Similarity of Polygons

Activity :: Discussion : Similar Triangles
Fill in the table using the diagram and then answer the questions that follow.

$\frac{\text{AB}}{\text{DE}} = \frac{...cm}{...cm} = ...$	$\hat{A}$=...°	$\hat{D}$...°
$\frac{\text{BC}}{\text{EF}} = \frac{...cm}{...cm} = ...$	$\hat{B}$=...°	$\hat{E}$=...°
$\frac{\text{AC}}{\text{DF}} = \frac{...cm}{...cm} = ...$	$\hat{C}$...°	$\hat{F}$=...°



1. What can you say about the numbers you calculated for: $\frac{\mathrm{AB}}{\mathrm{DE}}$, $\frac{\mathrm{BC}}{\mathrm{EF}}$, $\frac{\mathrm{AC}}{\mathrm{DF}}$?
2. What can you say about $\hat{A}$ and $\hat{D}$?
3. What can you say about $\hat{B}$ and $\hat{E}$?
4. What can you say about $\hat{C}$ and $\hat{F}$?

If two polygons are *similar*, one is an enlargement of the other. This means that the two polygons will have the same angles and their sides will be in the same proportion.

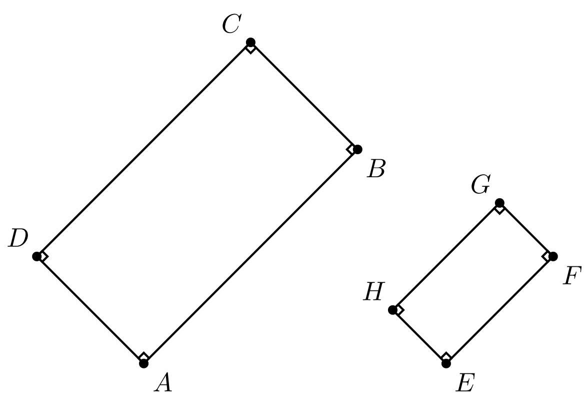
We use the symbol $\equiv$ to mean *is similar to*.

Definition: Similar Polygons
Two polygons are similar if:

1. their corresponding angles are equal, or
2. the ratios of corresponding sides are equal.

Worked Example 51: Similarity of Polygons

Question: Show that the following two polygons are similar.



Answer
Step 1 : Determine what is required
We are required to show that the pair of polygons is similar. We can do this by showing that the ratio of corresponding sides is equal or by showing that corresponding angles are equal.
Step 2 : Decide how to approach the problem

We are not given the lengths of the sides, but we are given the angles. So, we can show that corresponding angles are equal.

Step 3 : Show that corresponding angles are equal

All angles are given to be 90° and

$$\begin{aligned}\hat{A} &= \hat{E}\\ \hat{B} &= \hat{F}\\ \hat{C} &= \hat{G}\\ \hat{D} &= \hat{H}\end{aligned}$$

Step 4 : Final answer

Since corresponding angles are equal, the polygons ABCD and EFGH are similar.

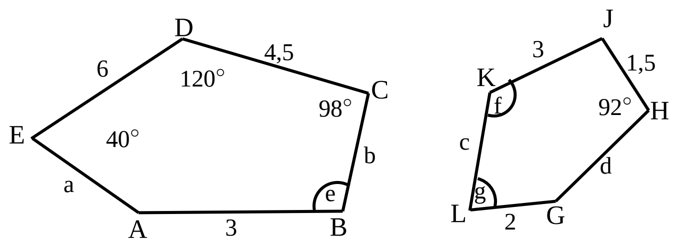
Step 5 : Comment on result

This result shows that all rectangles are similar to each other, because all rectangles will always have corresponding angles equal to each other.

Important: All rectangles and squares are similar.

Worked Example 52: Similarity of Polygons

Question: If two pentagons ABCDE and GHJKL are similar, determine the lengths of the sides and angles labelled with letters:



Answer

Step 1 : Determine what is given

We are given that ABCDE and GHJKL are similar. This means that:

$$\frac{\text{AB}}{\text{GH}} = \frac{\text{BC}}{\text{HJ}} = \frac{\text{CD}}{\text{JK}} = \frac{\text{DE}}{\text{KL}} = \frac{\text{EA}}{\text{LG}}$$

and

$$\begin{aligned}\hat{A} &= \hat{G}\\ \hat{B} &= \hat{H}\\ \hat{C} &= \hat{J}\\ \hat{D} &= \hat{K}\\ \hat{E} &= \hat{L}\end{aligned}$$

Step 2 : Determine what is required

We are required to determine the following lengths:

1. a, b, c and d

and the following angles:

1. e, f and g

Step 3 : Decide how to approach the problem

The corresponding angles are equal, so no calculation is needed. We are given one pair of sides DC and KJ that correspond $\frac{DC}{KJ} = \frac{4,5}{3} = 1,5$ so we know that all sides of $KJHGL$ are 1,5 times smaller than $ABCDE$.

Step 4 : Calculate lengths

$$\begin{aligned}
\frac{a}{2} = 1,5 &\quad \therefore \quad a = 2 \times 1,5 = 3 \\
\frac{b}{1,5} = 1,5 &\quad \therefore \quad b = 1,5 \times 1,5 = 2,25 \\
\frac{6}{c} = 1,5 &\quad \therefore \quad c = 6 \div 1,5 = 4 \\
\frac{3}{d} = 1,5 &\quad \therefore \quad d = 2
\end{aligned}$$

Step 5 : Calculate angles

$$\begin{aligned}
e &= 92^\circ \text{(corresponds to H)} \\
f &= 120^\circ \text{(corresponds to D)} \\
g &= 40^\circ \text{(corresponds to E)}
\end{aligned}$$

Step 6 : Write the final answer

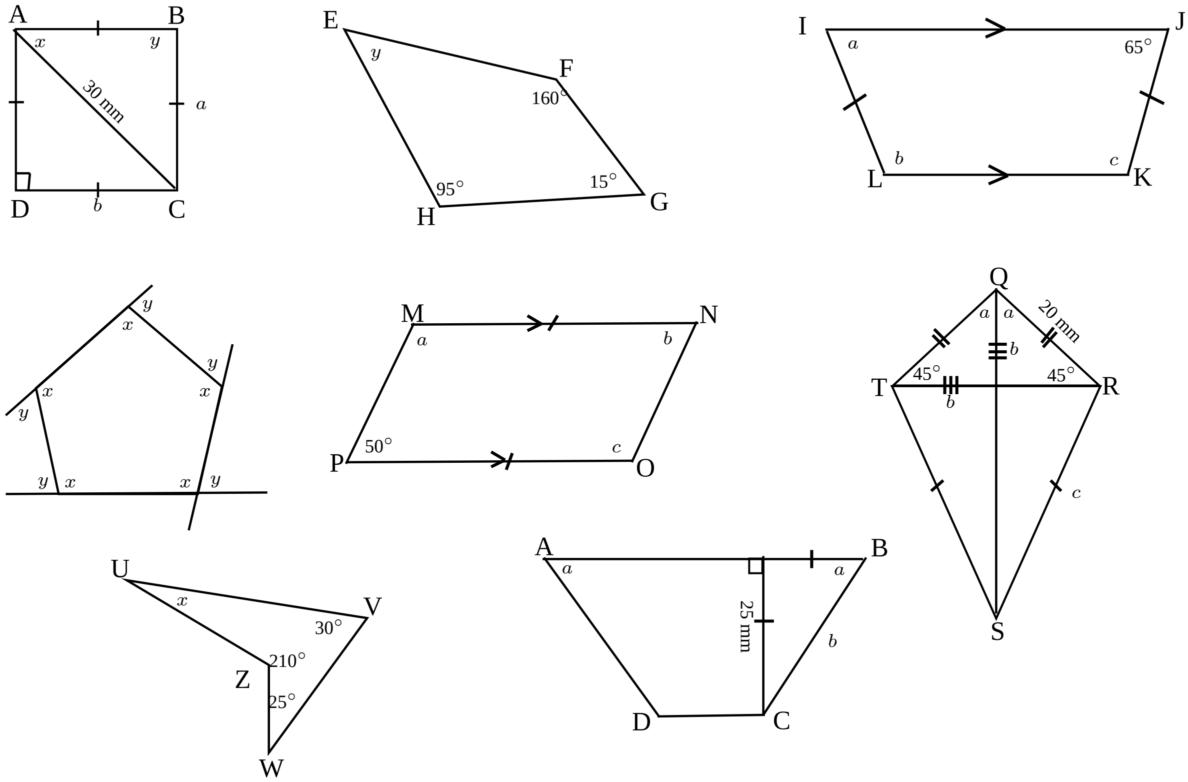
$$\begin{aligned}
a &= 3 \\
b &= 2,25 \\
c &= 4 \\
d &= 2 \\
e &= 92^\circ \\
f &= 120^\circ \\
g &= 40^\circ
\end{aligned}$$

Activity :: Similarity of Equilateral Triangles : Working in pairs, show that all equilateral triangles are similar.

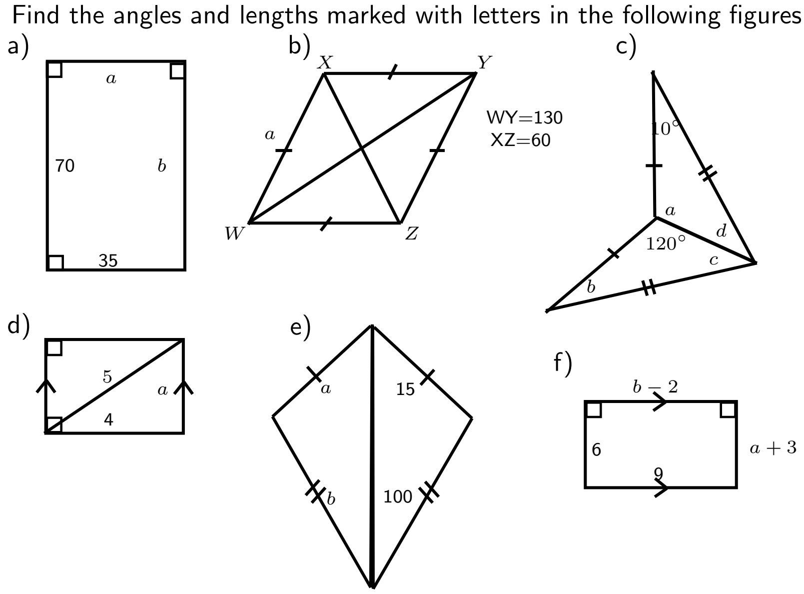
?

Exercise: Polygons-mixed

1. Find the values of the unknowns in each case. Give reasons.



2. Find the angles and lengths marked with letters in the following figures:



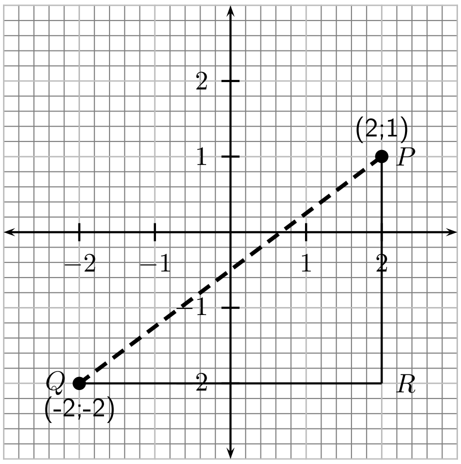
13.4 Co-ordinate Geometry

13.4.1 Introduction

Analytical geometry, also called co-ordinate geometry and earlier referred to as Cartesian geometry, is the study of geometry using the principles of algebra, and the Cartesian co-ordinate system. It is concerned with defining geometrical shapes in a numerical way, and extracting numerical information from that representation. Some consider that the introduction of analytic geometry was the beginning of modern mathematics.

13.4.2 Distance between Two Points

One of the simplest things that can be done with analytical geometry is to calculate the distance between two points. *Distance* is a number that describes how far apart two point are. For example, point P has co-ordinates $(2;1)$ and point Q has co-ordinates $(-2;-2)$. How far apart are points A and B? In the figure, this means how long is the dashed line?

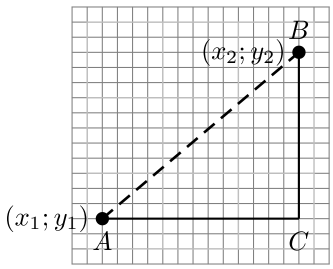


In the figure, it can be seen that the length of the line PR is 3 units and the length of the line QR is four units. However, the $\triangle PQR$, has a right angle at R. Therefore, the length of the side PQ can be obtained by using the Theorem of Pythagoras:

$$\begin{aligned} PQ^2 &= PR^2 + QR^2 \\ \therefore PQ^2 &= 3^2 + 4^2 \\ \therefore PQ &= \sqrt{3^2 + 4^2} = 5 \end{aligned}$$

The length of AB is the distance between the points A and B.

In order to generalise the idea, assume A is any point with co-ordinates $(x_1;y_1)$ and B is any other point with co-ordinates $(x_2;y_2)$.



The formula for calculating the distance between two points is derived as follows. The distance between the points A and B is the length of the line AB. According to the Theorem of Pythagoras, the length of AB is given by:

$$AB = \sqrt{AC^2 + BC^2}$$

However,

$$\begin{aligned} BC &= y_2 - y_1 \\ AC &= x_2 - x_1 \end{aligned}$$

Therefore,

$$\begin{aligned} AB &= \sqrt{AC^2 + BC^2} \\ &= \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \end{aligned}$$

Therefore, for any two points, $(x_1; y_1)$ and $(x_2; y_2)$, the formula is:

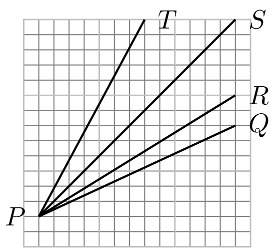
$$\text{Distance} = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

Using the formula, distance between the points P and Q with co-ordinates (2;1) and (-2;-2) is then found as follows. Let the co-ordinates of point P be $(x_1; y_1)$ and the co-ordinates of point Q be $(x_2; y_2)$. Then the distance is:

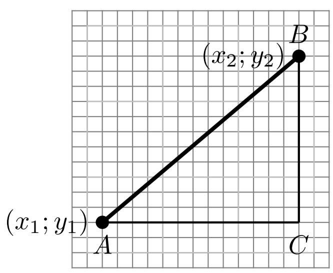
$$\begin{aligned}\text{Distance} &= \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \\ &= \sqrt{(2 - (-2))^2 + (1 - (-2))^2} \\ &= \sqrt{(2 + 2)^2 + (1 + 2)^2} \\ &= \sqrt{16 + 9} \\ &= \sqrt{25} \\ &= 5\end{aligned}$$

13.4.3 Calculation of the Gradient of a Line

The gradient of a line describes how steep the line is. In the figure, line PT is the steepest. Line PS is less steep than PT but is steeper than PR, and line PR is steeper than PQ.



The gradient of a line is defined as the ratio of the vertical distance to the horizontal distance. This can be understood by looking at the line as the hypotenuse of a right-angled triangle. Then the gradient is the ratio of the length of the vertical side of the triangle to the horizontal side of the triangle. Consider a line between a point A with co-ordinates $(x_1; y_1)$ and a point B with co-ordinates $(x_2; y_2)$.



$$\text{Gradient} = \frac{y_2 - y_1}{x_2 - x_1}$$

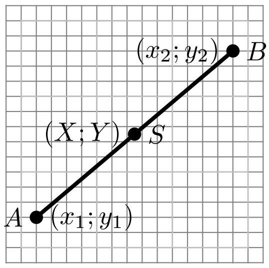
For example the gradient of the line between the points P and Q, with co-ordinates (2;1) and (-2;-2) (Figure 13.4.2) is:

$$\begin{aligned}\text{Gradient} &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{-2-1}{-2-2} \\ &= \frac{-3}{-4} \\ &= \frac{3}{4}\end{aligned}$$

13.4.4 Midpoint of a Line

Sometimes, knowing the co-ordinates of the middle point or *midpoint* of a line is useful. For example, what is the midpoint of the line between point P with co-ordinates $(2;1)$ and point Q with co-ordinates $(-2;-2)$.

The co-ordinates of the midpoint of any line between any two points A and B with co-ordinates $(x_1;y_1)$ and $(x_2;y_2)$, is generally calculated as follows. Let the midpoint of AB be at point S with co-ordinates $(X;Y)$. The aim is to calculate X and Y in terms of $(x_1;y_1)$ and $(x_2;y_2)$.



$$\begin{aligned}X &= \frac{x_1 + x_2}{2} \\ Y &= \frac{y_1 + y_2}{2} \\ \therefore & \quad S\left(\frac{x_1 + x_2}{2}; \frac{y_1 + y_2}{2}\right)\end{aligned}$$

Then the co-ordinates of the midpoint (S) of the line between point P with co-ordinates $(2;1)$ and point Q with co-ordinates $(-2;-2)$ is:

$$\begin{aligned}X &= \frac{x_1 + x_2}{2} \\ &= \frac{-2+2}{2} \\ &= 0 \\ Y &= \frac{y_1 + y_2}{2} \\ &= \frac{-2+1}{2} \\ &= -\frac{1}{2}\end{aligned}$$

$$\therefore S(0; -\frac{1}{2})$$

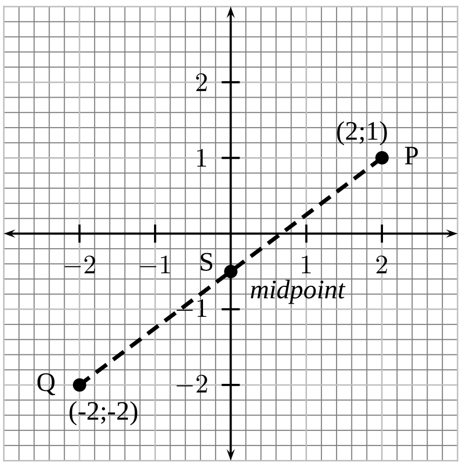
It can be confirmed that the distance from the each end point to the midpoint is equal. The co-ordinate of the midpoint S is $(0;-0{,}5)$.

$$\begin{aligned} PS &= \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \\ &= \sqrt{(0-2)^2 + (-0.5-1)^2} \\ &= \sqrt{(-2)^2 + (-1.5)^2} \\ &= \sqrt{4 + 2.25} \\ &= \sqrt{6.25} \end{aligned}$$

and

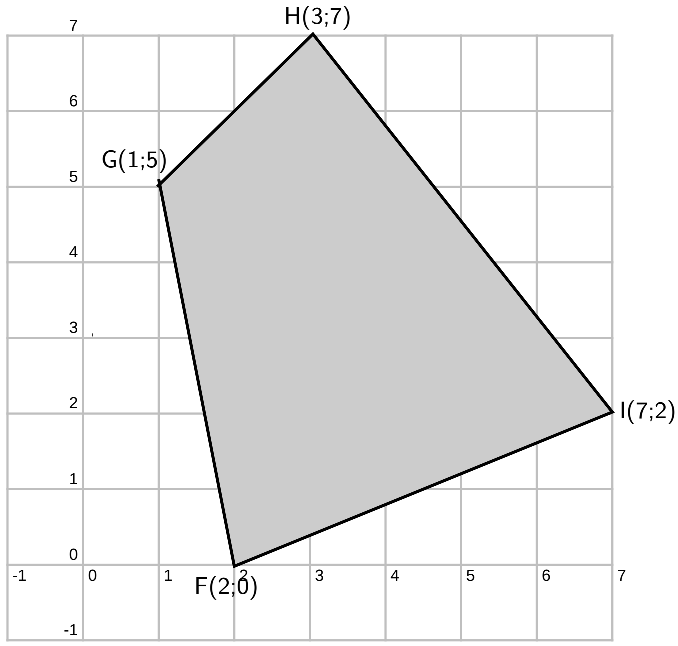
$$\begin{aligned} QS &= \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \\ &= \sqrt{(0-(-2))^2 + (-0.5-(-2))^2} \\ &= \sqrt{(0+2))^2 + (-0.5+2))^2} \\ &= \sqrt{(2))^2 + (-1.5))^2} \\ &= \sqrt{4 + 2.25} \\ &= \sqrt{6.25} \end{aligned}$$

It can be seen that $PS = QS$ as expected.



Exercise: Co-ordinate Geometry

1. In the diagram given the vertices of a quadrilateral are F(2;0), G(1;5), H(3;7)



and I(7;2).

a) What are the lengths of the opposite sides of FGHI?
b) Are the opposite sides of FGHI parallel?
c) Do the diagonals of FGHI bisect each other?
d) Can you state what type of quadrilateral FGHI is? Give reasons for your answer.

2. A quadrialteral ABCD with vertices A(3;2), B(1;7), C(4;5) and D(1;3) is given.
   a) Draw the qaudrilateral.
   b) Find the lengths of the sides of the quadrilateral.
3. S(1;4), T(-1;2), U(0;-1) and V(4;-1) are the vertices of a pentagon.
   a) Are two of the sides of this pentagon parallel? If yes, find them.
   b) Are two of the sides of this pentagon of equal length? If yes, find them.
4. ABCD is a quadrilateral with verticies A(0;3), B(4;3), C(5;-1) and D(-1;-1).
   a) Show that:
      (i) $AD = BC$
      (ii) $AB \parallel DC$
   b) What name would you give to ABCD?
   c) Show that the diagonals AC and BD do not bisect each other.
5. P, Q, R and S are the points (-2;0), (2;3), (5;3), (-3;-3) respectively.
   a) Show that:
      (i) $SR = 2PQ$
      (ii) $SR \parallel PQ$
   b) Calculate:
      (i) PS
      (ii) QR
   c) What kind of a quadrilateral is PQRS? Give reasons for your answers.
6. EFGH is a parallelogram with verticies E(-1;2), F(-2;-1) and G(2;0). Find the co-ordinates of H by using the fact that the diagonals of a parallelogram bisect each other.

13.5 Transformations

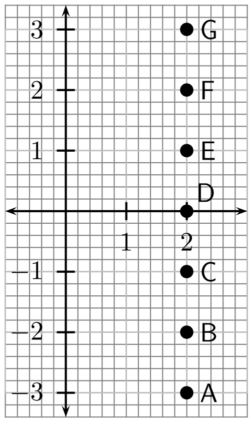
In this section you will learn about how the co-ordinates of a point change when the point is moved horizontally and vertically on the Cartesian plane. You will also learn about what happens to the co-ordinates of a point when it is reflected on the x-axis, y-axis and the line $y = x$.

13.5.1 Translation of a Point

When something is moved in a straight line, we say that it is *translated*. What happens to the co-ordinates of a point that is translated horizontally or vertically?

Activity :: Discussion : Translation of a Point Vertically

Complete the table, by filling in the co-ordinates of the points shown in the figure.



Point	x co-ordinate	y co-ordinate
A		
B		
C		
D		
E		
F		
G		

What do you notice about the x co-ordinates? What do you notice about the y co-ordinates?
What would happen to the co-ordinates of point A, if it was moved to the position of point G?

When a point is moved vertically up or down on the Cartesian plane, the x co-ordinate of the point remains the same, but the y co-ordinate changes by the amount that the point was moved up or down.

For example, in Figure 13.3 Point A is moved 4 units upwards to the position marked by G. The new x co-ordinate of point A is the same (x=1), but the new y co-ordinate is shifted in the positive y direction 4 units and becomes y=-2+**4**=2. The new co-ordinates of point A are therefore G(1;2). Similarly, for point B that is moved downwards by 5 units, the x co-ordinate is the same ($x = -2{,}5$), but the y co-ordinate is shifted in the negative y-direction by 5 units. The new y co-ordinate is therefore y=2,5 -**5**=-2,5.

Important: If a point is shifted upwards, the new y co-ordinate is given by adding the shift to the old y co-ordinate. If a point is shifted downwards, the new y co-ordinate is given by subtracting the shift from the old y co-ordinate.

Activity :: Discussion : Translation of a Point Horizontally

Complete the table, by filling in the co-ordinates of the points shown in the figure.

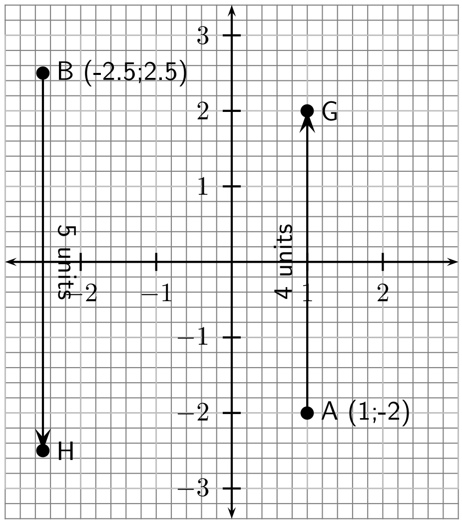
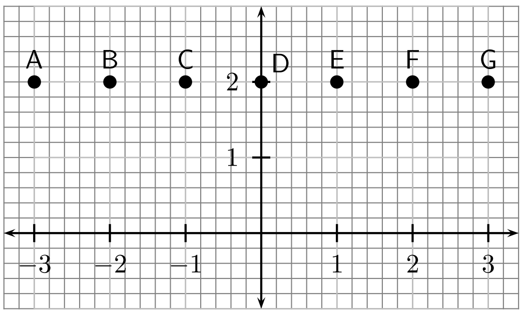


Figure 13.3: Point A is moved 4 units upwards to the position marked by G. Point B is 5 units downwards to the position marked by H.



Point	x co-ordinate	y co-ordinate
A		
B		
C		
D		
E		
F		
G		

What do you notice about the x co-ordinates? What do you notice about the y co-ordinates?

What would happen to the co-ordinates of point A, if it was moved to the position of point G?

When a point is moved horizontally left or right on the Cartesian plane, the y co-ordinate of the point remains the same, but the x co-ordinate changes by the amount that the point was moved left or right.

For example, in Figure 13.4 Point A is moved 4 units right to the position marked by G. The new y co-ordinate of point A is the same (y=1), but the new x co-ordinate is shifted in the positive x direction 4 units and becomes x=-2**+4**=2. The new co-ordinate of point A at G is therefore (2;1). Similarly, for point B that is moved left by 5 units, the y co-ordinate is the same ($y = -2{,}5$), but the x co-ordinate is shifted in the negative x-direction by 5 units. The new x co-ordinate is therefore x=2,5 **-5**=-2,5. The new co-ordinates of point B at H is therefore (-2,5;1).

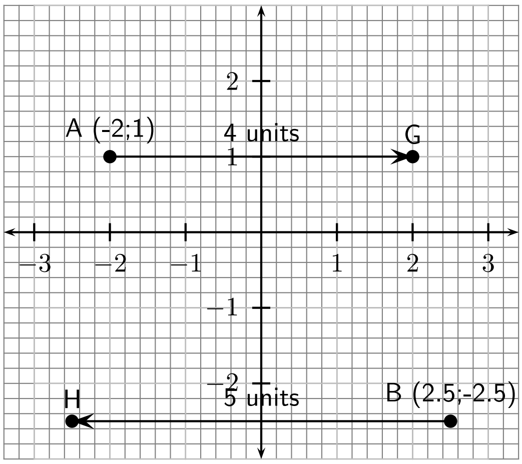
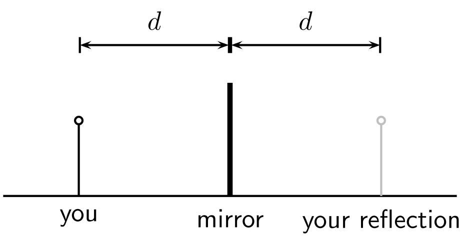


Figure 13.4: Point A is moved 4 units to the right to the position marked by G. Point B is 5 units to the left to the position marked by H.

Important: If a point is shifted to the right, the new x co-ordinate is given by adding the shift to the old x co-ordinate. If a point is shifted to the left, the new x co-ordinate is given by subtracting the shift from the old x co-ordinate.

13.5.2 Reflection of a Point

When you stand in front of a mirror your reflection is located the same distance (d) behind the mirror as you are standing in front of the mirror.



We can apply the same idea to a point that is reflected on the x-axis, the y-axis and the line $y = x$.

Reflection on the x-axis

If a point is reflected on the x-axis, then the reflection must be the same distance below the x-axis as the point is above the x-axis and vice-versa.

Important: When a point is reflected about the x-axis, only the y co-ordinate of the point changes.

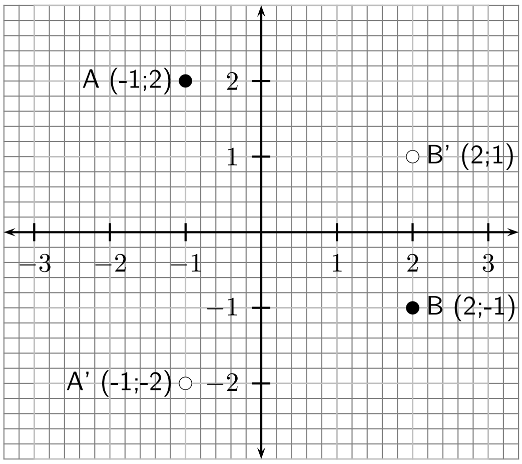


Figure 13.5: Points A and B are reflected on the x-axis. The original points are shown with • and the reflected points are shown with ∘.

Worked Example 53: Reflection on the x-axis

Question: Find the co-ordinates of the reflection of the point P, if P is reflected on the x-axis. The co-ordinates of P are (5;10).

Answer

Step 1 : Determine what is given and what is required

We are given the point P with co-ordinates (5;10) and need to find the co-ordinates of the point if it is reflected on the x-axis.

Step 2 : Determine how to approach the problem

The point P is above the x-axis, therefore its reflection will be the same distance below the x-axis as the point P is above the x-axis. Therefore, y=-10.

For a reflection on the x-axis, the x co-ordinate remains unchanged. Therefore, x=5.

Step 3 : Write the final answer

The co-ordinates of the reflected point are (5;-10).

Reflection on the y-axis

If a point is reflected on the y-axis, then the reflection must be the same distance to the left of the y-axis as the point is to the right of the y-axis and vice-versa.

Important: When a point is reflected on the y-axis, only the x co-ordinate of the point changes. The y co-ordinate remains unchanged.

Worked Example 54: Reflection on the y-axis

Question: Find the co-ordinates of the reflection of the point Q, if Q is reflected on the y-axis. The co-ordinates of Q are (15;5).

Answer

Step 1 : Determine what is given and what is required

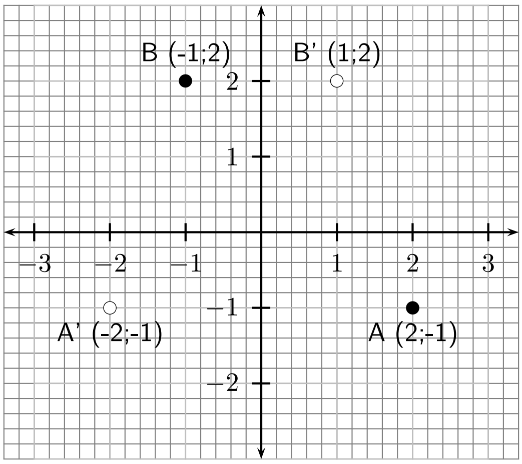


Figure 13.6: Points A and B are reflected on the y-axis. The original points are shown with • and the reflected points are shown with ∘.

We are given the point Q with co-ordinates (15;5) and need to find the co-ordinates of the point if it is reflected on the y-axis.

Step 2 : Determine how to approach the problem

The point Q is to the right of the y-axis, therefore its reflection will be the same distance to the left of the y-axis as the point Q is to the right of the y-axis. Therefore, x=-15.

For a reflection on the y-axis, the y co-ordinate remains unchanged. Therefore, y=5.

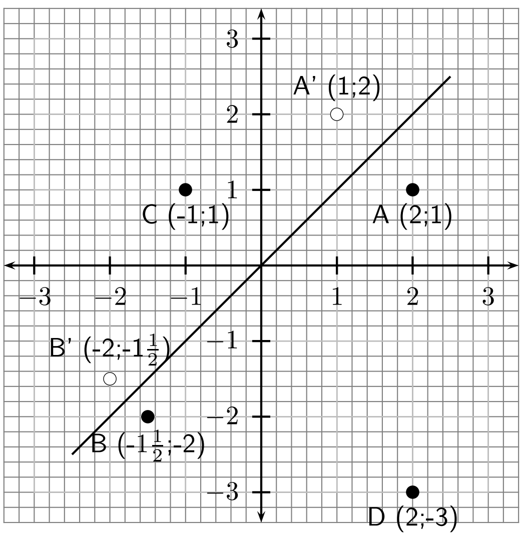
Step 3 : Write the final answer

The co-ordinates of the reflected point are (-15;5).

Reflection on the line $y = x$

The final type of reflection you will learn about is the reflection of a point on the line $y = x$.

Activity :: Casestudy : Reflection of a point on the line $y = x$



Study the information given and complete the following table:

	Point	Reflection
A	(2;1)	(1;2)
B	(-1$\frac{1}{2}$;-2)	(-2;-1$\frac{1}{2}$)
C	(-1;1)	
D	(2;-3)	

What can you deduce about the co-ordinates of points that are reflected about the line $y = x$?

The x and y co-ordinates of points that are reflected on the line $y = x$ are swapped around, or interchanged. This means that the x co-ordinate of the original point becomes the y co-ordinate of the reflected point and the y co-ordinate of the original point becomes the x co-ordinate of the reflected point.

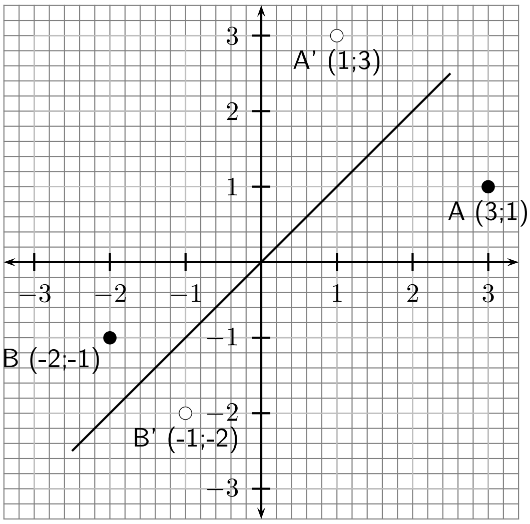


Figure 13.7: Points A and B are reflected on the line $y = x$. The original points are shown with • and the reflected points are shown with ◦.

Important: The x and y co-ordinates of points that are reflected on the line $y = x$ are interchanged.

Worked Example 55: Reflection on the line $y = x$

Question: Find the co-ordinates of the reflection of the point R, if R is reflected on the line $y = x$. The co-ordinates of R are (-5;5).

Answer

Step 1 : Determine what is given and what is required

We are given the point R with co-ordinates (-5;5) and need to find the co-ordinates of the point if it is reflected on the line $y = x$.

Step 2 : Determine how to approach the problem

The x co-ordinate of the reflected point is the y co-ordinate of the original point. Therefore, x=5.

The y co-ordinate of the reflected point is the x co-ordinate of the original point. Therefore, y=-5.

Step 3 : Write the final answer

The co-ordinates of the reflected point are (5;-5).

Rules of Translation

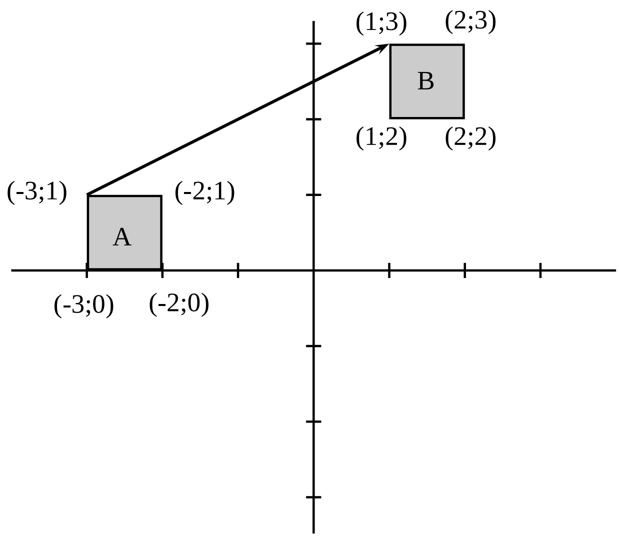
A quick way to write a translation is to use a 'rule of translation'. For example $(x; y) \rightarrow (x + a; y + b)$ means translate point (x;y) by moving a units horizontally and b units vertically. So if we translate (1;2) by the rule $(x; y) \rightarrow (x + 3; y - 1)$ it becomes (4;1). We have moved 3 units right and 1 unit down.

Translating a Region

To translate a region, we translate each point in the region.

Example

Region A has been translated to region B by the rule: $(x; y) \rightarrow (x + 4; y + 2)$



Activity :: Discussion : Rules of Transformations

Work with a friend and decide which item from column 1 matches each description in column 2.

Column 1	Column 2
$(x; y) \rightarrow (x; y - 3)$	a reflection on x-y line
$(x; y) \rightarrow (x - 3; y)$	a reflection on the x axis
$(x; y) \rightarrow (x; -y)$	a shift of 3 units left
$(x; y) \rightarrow (-x; y)$	a shift of 3 units down
$(x; y) \rightarrow (y; x)$	a reflection on the y-axis

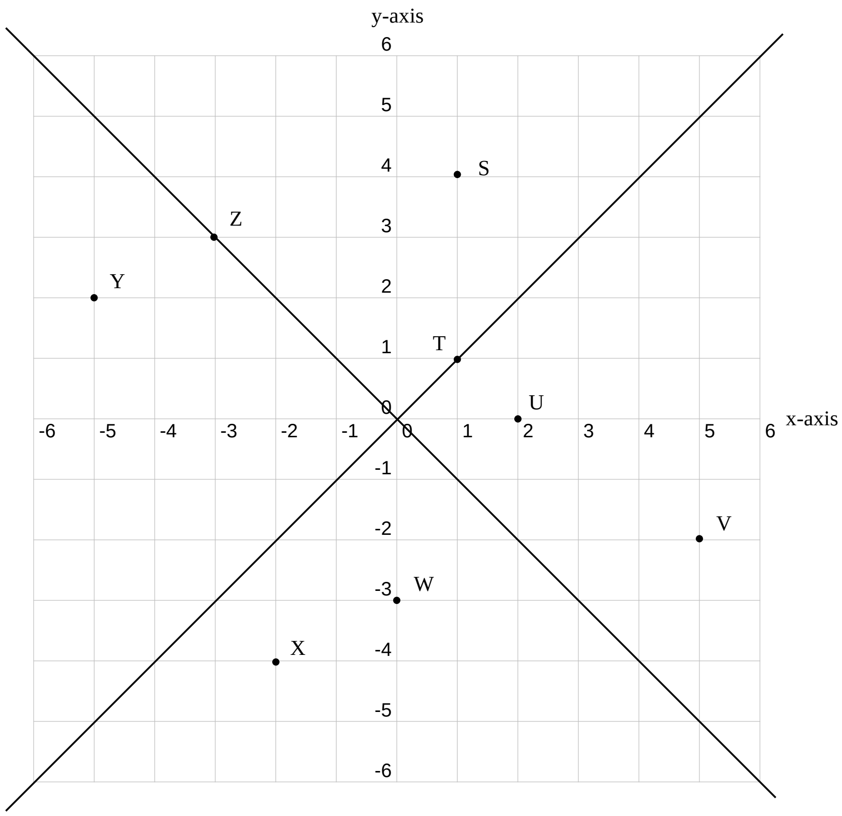
?

Exercise: Transformations

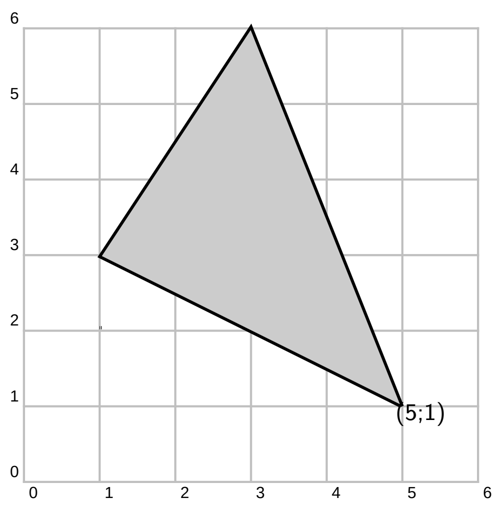
1. Find the co-ordinates of each of the points (S - Z) if they are reflected about the given lines:

   a) y-axis (x=0)

b) x-axis (y=0)
c) y=-x
d) y=x



2. Write down the rule used for each of the following reflections:
   a) Z(7;3), Z'(3;7)
   b) Y(-1;-8), Y'(1;-8)
   c) X(5;9), X'(-5;9)
   d) W(4;6), W'(4;6)
   e) V($\frac{-3}{7}$;$\frac{5}{3}$), V'($\frac{5}{3}$;$\frac{-3}{7}$)
3. a) Reflect the given points using the rules that are given.
   b) Identify the line of reflection in each case (some may not exist):
      (i) H(-4;3); (x;y)→ (-x;y)
      (ii) H(-4;3); (x;y) → (-y;-x)
      (iii) H(-4;3); (x;y) → (y;x)
      (iv) H(-4;3); (x;y) → (-x;-y)
      (v) H(-4;3); (x;y) → (x;-y)



4. Using squared paper, copy the diagram given. Let -10≤x≤10, -10≤y≤10.

i) Identify the transformation.

ii) Draw the image of the figure according the rules given.

a) $(x; y) \to (-x; y)$

b) $(x; y) \to (y; x)$

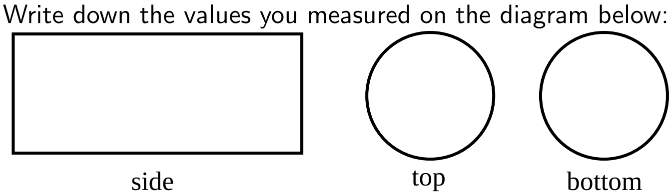
c) $(x; y) \to (x; y - 3)$

d) $(x; y) \to (x + 5; y)$

e) $(x; y) \to (x; -y)$

Activity :: Investigation : Calculation of Volume, Surface Area and scale factors of objects

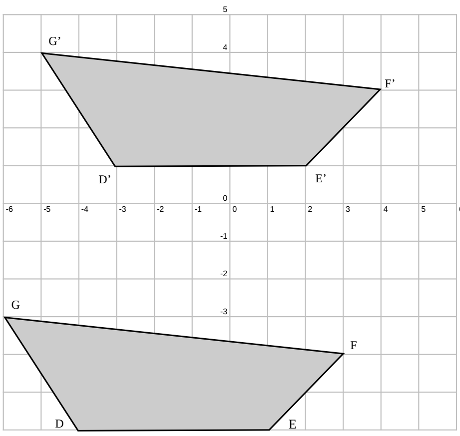
1. Look around the house or school and find a can or a tin of any kind (e.g. beans, soup, cooldrink, paint etc.)
2. Measure the height of the tin and the diameter of its top or bottom.
3. Write down the values you measured on the diagram below:

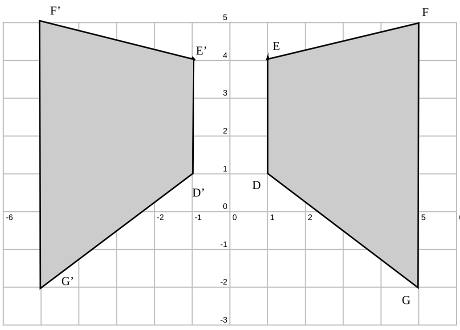


4. Using your measurements, calculate the following (in cm^2, rounded off to 2 decimal places):
   (a) the area of the side of the tin (i.e. the rectangle)
   (b) the area of the top and bottom of the tin (i.e. the circles)
   (c) the total surface area of the tin
5. If the tin metal costs 0,17 cents/cm^2, how much does it cost to make the tin?
6. Find the volume of your tin (in cm^3, rounded off to 2 decimal places).
7. What is the volume of the tin given on its label?
8. Compare the volume you calculated with the value given on the label. How much air is contained in the tin when it contains the product (i.e. cooldrink, soup etc.)
9. Why do you think space is left for air in the tin?
10. If you wanted to double the volume of the tin, but keep the radius the same, by how much would you need to increase the height?
11. If the height of the tin is kept the same, but now the radius is doubled, by what scale factor will the:
    (a) area of the side surface of the tin increase?
    (b) area of the bottom/top of the tin increase?

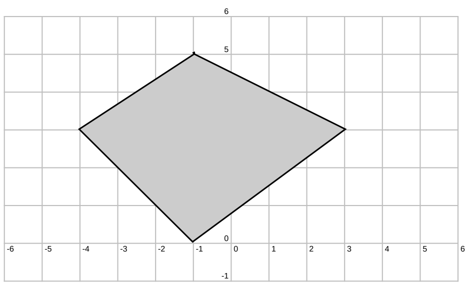
13.6 End of Chapter Exercises

1. Write a rule that will give the following transformations of DEFG to D'E'F'G in each case.

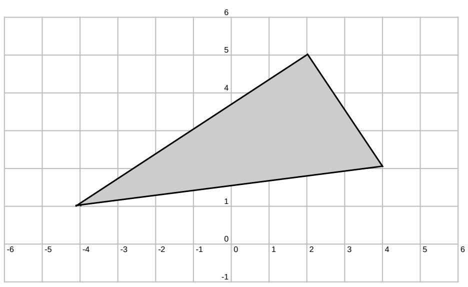
a) 

b) 

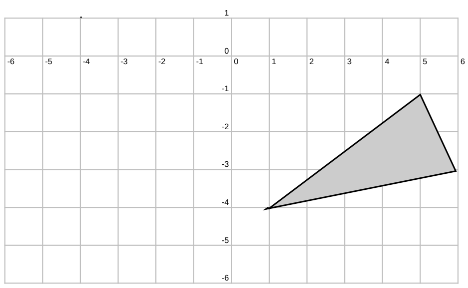
2. Using the rules given, identify the type of transformation and draw the image of the shapes.



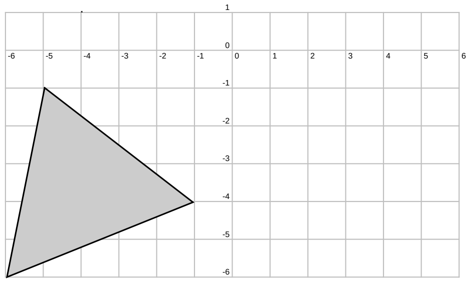
a) (x;y)→(x+3;y-3)



b) (x;y)→(x-4;y)



c) (x;y)→(y;x)

d) (x;y)→(-x;-y) 

Chapter 14

Trigonometry - Grade 10

14.1 Introduction

In geometry we learn about how the sides of polygons relate to the angles in the polygons, but we have not learned how to calculate an angle if we only know the lengths of the sides. Trigonometry (pronounced: trig-oh-nom-eh-tree) deals with the relationship between the angles and the sides of a right-angled triangle. We will learn about trigonometric functions, which form the basis of trigonometry.

Activity :: Investigation : History of Trigonometry

Work in pairs or groups and investigate the history of the foundation of trigonometry. Describe the various stages of development and how the following cultures used trigonometry to improve their lives.

The works of the following people or cultures can be investigated:

1. Cultures
   (a) Ancient Egyptians
   (b) Mesopotamians
   (c) Ancient Indians of the Indus Valley
2. People
   (a) Lagadha (circa 1350-1200 BC)
   (b) Hipparchus (circa 150 BC)
   (c) Ptolemy (circa 100)
   (d) Aryabhata (circa 499)
   (e) Omar Khayyam (1048-1131)
   (f) Bhaskara (circa 1150)
   (g) Nasir al-Din (13th century)
   (h) al-Kashi and Ulugh Beg (14th century)
   (i) Bartholemaeus Pitiscus (1595)

You should be familiar with the idea of measuring angles from geometry but have you ever stopped to think why there are 360 degrees in a circle? The reason is purely historical. There are 360 degrees in a circle because the ancient

Babylonians had a number system with base 60. A base is the number you count up to before you get an extra digit. The number system that we use everyday is called the decimal system (the base is 10), but computers use the binary system (the base is 2). $360 = 6 \times 60$ so for them it made sense to have 360 degrees in a circle.

14.2 Where Trigonometry is Used

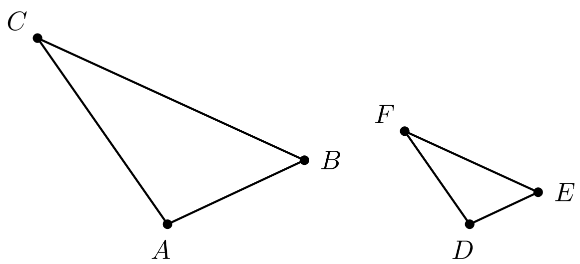
There are many applications of trigonometry. Of particular value is the technique of triangulation, which is used in astronomy to measure the distance to nearby stars, in geography to measure distances between landmarks, and in satellite navigation systems. GPSs (global positioning systems) would not be possible without trigonometry. Other fields which make use of trigonometry include astronomy (and hence navigation, on the oceans, in aircraft, and in space), music theory, acoustics, optics, analysis of financial markets, electronics, probability theory, statistics, biology, medical imaging (CAT scans and ultrasound), pharmacy, chemistry, number theory (and hence cryptology), seismology, meteorology, oceanography, many physical sciences, land surveying and geodesy, architecture, phonetics, economics, electrical engineering, mechanical engineering, civil engineering, computer graphics, cartography, crystallography and game development.

Activity :: Discussion : Uses of Trigonometry
Select one of the uses of trigonometry from the list given and write a 1-page report describing *how* trigonometry is used in your chosen field.

14.3 Similarity of Triangles

If $\triangle ABC$ is similar to $\triangle DEF$, then this is written as:

$$\triangle ABC \sim \triangle DEF$$



Then, it is possible to deduce proportionalities between corresponding sides of the two triangles, such as the following:

AB$_{\overline{BC=\frac{DE}{EF}}}$

AB$_{\overline{AC=\frac{DE}{DF}}}$

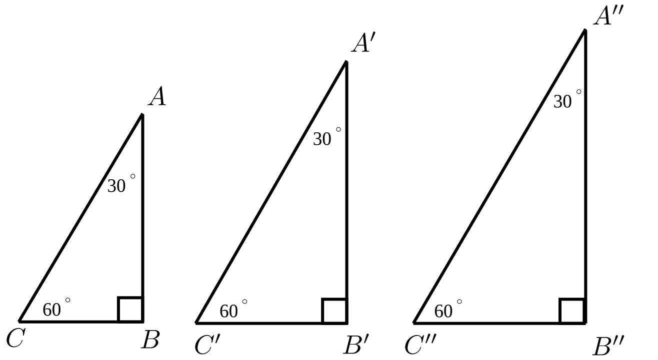
AC$\overline{BC=\frac{DF}{EF}}$

AB$\overline{DE=\frac{BC}{EF}=\frac{AC}{DF}}$The most important fact about similar triangles ABC and DEF is that the angle at vertex A is equal to the angle at vertex D, the angle at B is equal to the angle at E, and the angle at C is equal to the angle at F.

$$\begin{aligned} \angle A &= \angle D \\ \angle B &= \angle E \\ \angle C &= \angle F \end{aligned}$$

Activity :: Investigation : Ratios of Similar Triangles

In your exercise book, draw three similar triangles of different sizes, but each with $\hat{A} = 30°$; $\hat{B} = 90°$ and $\hat{C} = 60°$. Measure angles and lengths very accurately in order to fill in the table below (round answers to one decimal place).



Dividing lengths of sides (Ratios)		
$\frac{AB}{BC} =$	$\frac{AB}{AC} =$	$\frac{CB}{AC} =$
$\frac{A'B'}{B'C'} =$	$\frac{A'B'}{A'C'} =$	$\frac{C'B'}{A'C'} =$
$\frac{A''B''}{B''C''} =$	$\frac{A''B''}{A''C''} =$	$\frac{C''B''}{A''C''} =$

What observations can you make about the ratios of the sides?
These equal ratios are used to define the trigonometric functions.

Note: In algebra, we often use the letter x for our unknown variable (although we can use any other letter too, such as a, b, k, etc). In trigonometry, we often use the Greek symbol θ for an unknown angle (we also use α , β , γ etc).

14.4 Definition of the Trigonometric Functions

We are familiar with a function of the form $f(x)$ where f is the function and x is the argument. Examples are:

$$\begin{aligned} f(x) &= 2^x && \text{(exponential function)} \\ g(x) &= x + 2 && \text{(linear function)} \\ h(x) &= 2x^2 && \text{(parabolic function)} \end{aligned}$$

The basis of trigonometry are the *trigonometric functions*. There are three basic trigonometric functions:

1. sine

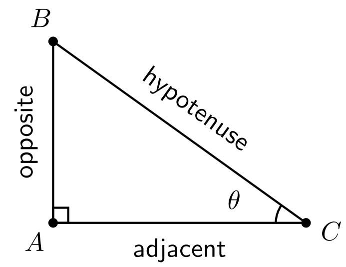
2. cosine
3. tangent

These are abbreviated to:

1. sin
2. cos
3. tan

These functions are defined from a **right-angled triangle**.

Consider a right-angled triangle.



In the right-angled triangle, we refer to the lengths of the three sides according to how they are placed in relation to the angle θ. The side opposite to θ is labelled *opposite*, the side next to θ is labelled *adjacent* and the side opposite the right-angle is labelled the *hypotenuse*.

We define:

$$\begin{aligned}\sin\theta &= \frac{opposite}{hypotenuse}\\ \cos\theta &= \frac{adjacent}{hypotenuse}\\ \tan\theta &= \frac{opposite}{adjacent}\end{aligned}$$

These functions relate the lengths of the sides of a triangle to its interior angles.

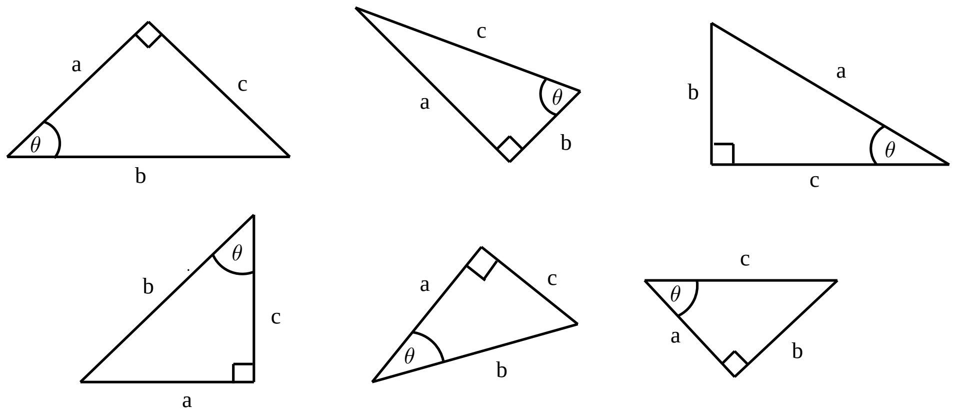
One way of remembering the definitions is to use the following mnemonic that is perhaps easier to remember:

Silly **O**ld **H**ens	$\text{Sin} = \frac{\textbf{O}\text{pposite}}{\textbf{H}\text{ypotenuse}}$
Cackle **A**nd **H**owl	$\text{Cos} = \frac{\textbf{A}\text{djacent}}{\textbf{H}\text{ypotenuse}}$
Till **O**ld **A**ge	$\text{Tan} = \frac{\textbf{O}\text{pposite}}{\textbf{A}\text{djacent}}$

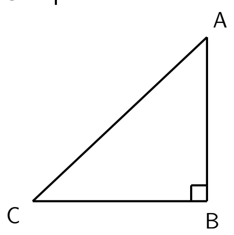
Important: The definitions of opposite, adjacent and hypotenuse only make sense when you are working with right-angled triangles! Always check to make sure your triangle has a right-angle before you use them, otherwise you will get the wrong answer. We will find ways of using our knowledge of right-angled triangles to deal with the trigonometry of non right-angled triangles in Grade 11.

Activity :: Investigation : Definitions of Trigonometric Functions

1. In each of the following triangles, state whether a, b and c are the hypotenuse, opposite or adjacent sides of the triangle.

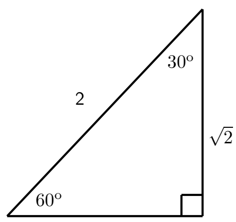


2. Complete each of the following, the first has been done for you

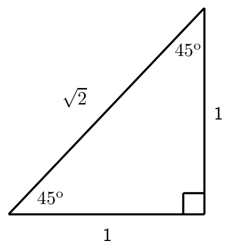


$$a)\quad \sin\hat{A} = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{CB}{AC} \qquad d)\quad \sin\hat{C} =$$
$$b)\quad \cos\hat{A} = \qquad e)\quad \cos\hat{C} =$$
$$c)\quad \tan\hat{A} = \qquad f)\quad \tan\hat{C} =$$

3. Complete each of the following:



$$\sin 60 = $$
$$\cos 30 = $$
$$\tan 60 = $$



$$\sin 45 = $$
$$\cos 45 = $$
$$\tan 45 = $$

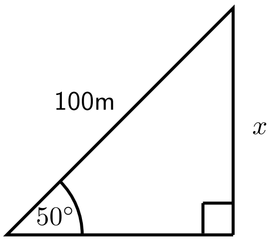
For most angles θ, it is very difficult to calculate the values of $\sin\theta$, $\cos\theta$ and $\tan\theta$. One usually needs to use a calculator to do so. However, we saw in the above Activity that we could work these values out for some special angles. Some of these angles are listed in the table below, along with the values of the trigonometric functions at these angles.

	$0°$	$30°$	$45°$	$60°$	$90°$	$180°$
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	-1
$\sin\theta$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	0
$\tan\theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	$-$	0

These values are useful when asked to solve a problem involving trig functions *without* using a calculator.

Worked Example 56: Finding Lengths

Question: Find the length of x in the following triangle.



Answer

Step 1 : Identify the trig identity that you need

In this case you have an angle ($50°$), the opposite side and the hypotenuse.
So you should use $\sin$

$$\sin 50° = \frac{x}{100}$$

Step 2 : Rearrange the question to solve for x

$$\Rightarrow x = 100 \times \sin 50°$$

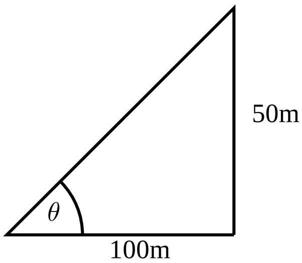
Step 3 : Use your calculator to find the answer

Use the sin button on your calculator

$$\Rightarrow x = 76.6\text{m}$$

Worked Example 57: Finding Angles

Question: Find the value of θ in the following triangle.



Answer

Step 1 : Identify the trig identity that you need

In this case you have the opposite side and the hypotenuse to the angle θ.
So you should use $\tan$

$$\tan\theta = \frac{50}{100}$$

Step 2 : Calculate the fraction as a decimal number

$$\Rightarrow \tan\theta = 0.5$$

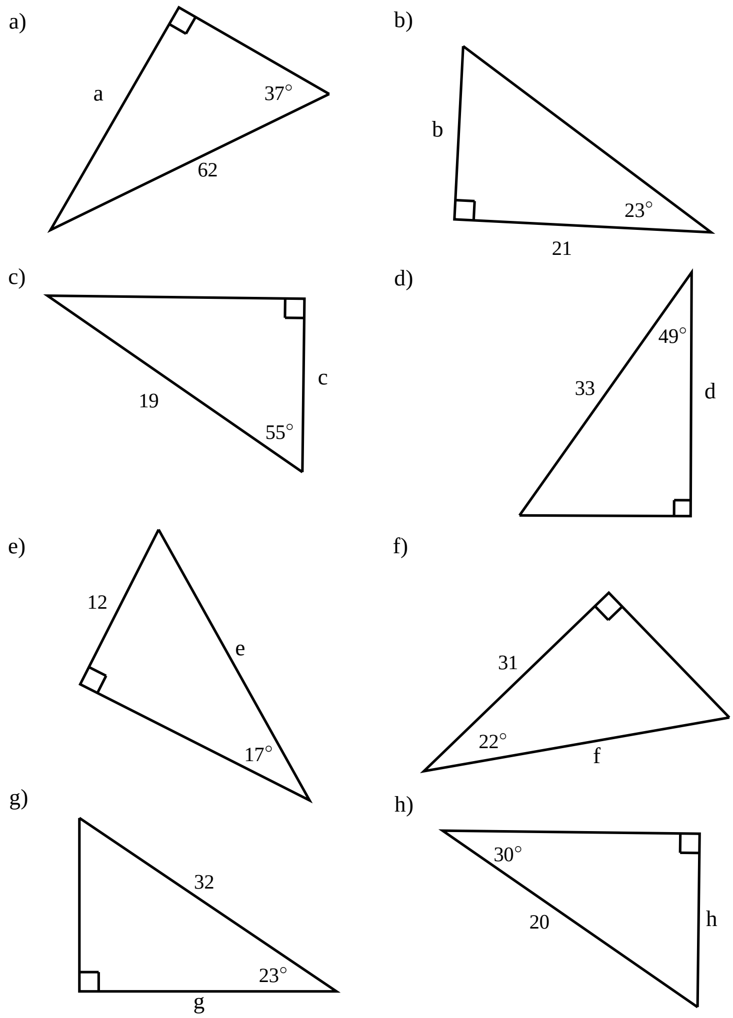
Step 3 : Use your calculator to find the angle

Since you are finding the *angle*,
use $\tan^{-1}$ on your calculator
Don't forget to set your calculator to 'deg' mode!

$$\Rightarrow \theta = 26.6°$$

Exercise: Finding Lengths

Find the length of the sides marked with letters. Give answers correct to 2 decimal places.



14.5 Simple Applications of Trigonometric Functions

Trigonometry was probably invented in ancient civilisations to solve practical problems involving constructing buildings and navigating their ships by the stars. In this section we will show how trigonometry can be used to solve some other practical problems.

14.5.1 Height and Depth

One simple task is to find the height of a building by using trigonometry. We could just use a tape measure lowered from the roof but this is impractical (and dangerous) for tall buildings. It is much more sensible to measure a distance along the ground and use trigonometry to find the height of the building.

Figure 14.1 shows a building whose height we do not know. We have walked 100 m away from the building and measured the angle up to the top. This angle is found to be $38{,}7°$. We call

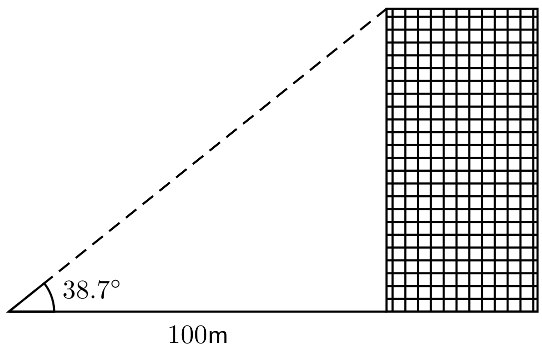


Figure 14.1: Determining the height of a building using trigonometry.

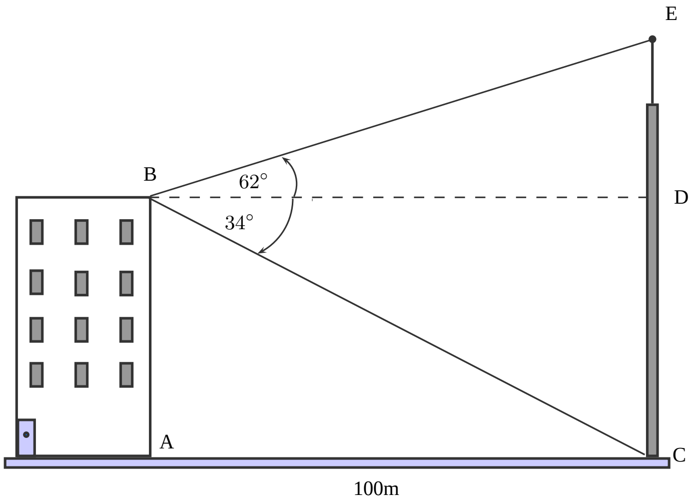
this angle the *angle of elevation*. As you can see from Figure 14.1, we now have a right-angled triangle. As we know the length of one side and an angle, we can calculate the height of the triangle, which is the height of the building we are trying to find.

If we examine the figure, we see that we have the *opposite* and the *adjacent* of the angle of elevation and we can write:

$$\begin{aligned} \tan 38{,}7^\circ &= \frac{\text{opposite}}{\text{adjacent}} \\ &= \frac{\text{height}}{100\,\text{m}} \\ \Rightarrow \text{height} &= 100\,\text{m} \times \tan 38{,}7^\circ \\ &= 80\,\text{m} \end{aligned}$$

Worked Example 58: Height of a tower

Question: A block of flats is 100m away from a cellphone tower. Someone at A measures the angle of elevation to the top of the tower E to be $62°$, and the angle of depression to the bottom of the tower C to be $34°$. What is the height of the cellphone tower correct to 1 decimal place?



Answer
Step 1 : Identify a strategy
To find the height of the tower, all we have to do is find the length of CD and DE. We see that $\triangle ACD$ and $\triangle AED$ are both right-angled. For each of the triangles, we have an angle and we have the length AD. Thus we can calculate the sides of the triangles.
Step 2 : Calculate CD

$$\begin{aligned}\tan(C\hat{A}D) &= \frac{CD}{AD}\\ \Longrightarrow \quad CD &= AD \times \tan(C\hat{A}D)\\ &= 100 \times \tan 34^\circ\end{aligned}$$

Use your calculator to find that $\tan 34^\circ = 0{,}6745$. Using this, we find that $CD = 67{,}45$m

Step 3 : Calculate DE

$$\begin{aligned}\tan(D\hat{A}E) &= \frac{DE}{AD}\\ \Longrightarrow \quad DE &= AD \times \tan(D\hat{A}E)\\ &= 100 \times \tan 62^\circ\\ &= 188{,}07\,\text{m}\end{aligned}$$

Step 4 : Combine the previous answers

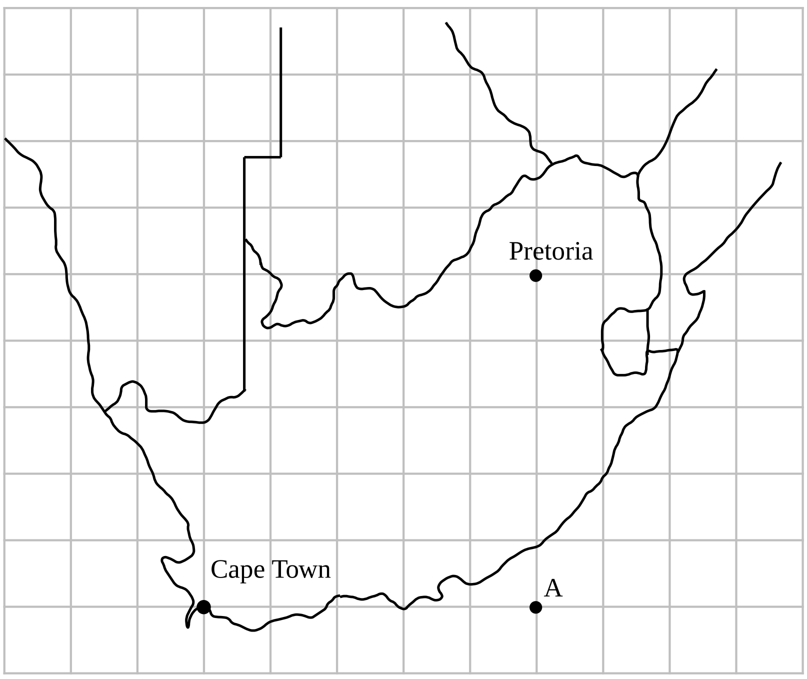
We have that the height of the tower $CE = CD + DE = 67{,}45\,\text{m} + 188{,}07\,\text{m} = 255.5\,\text{m}$.

14.5.2 Maps and Plans

Maps and plans are usually scale drawings. This means that they are an exact copy of the real thing, but are usually smaller. So, only lengths are changed, but all angles are the same. We can use this idea to make use of maps and plans by adding information from the real world.

Worked Example 59: Scale Drawing

Question: A ship approaching Cape Town Harbour reaches point A on the map, due South of Pretoria and due East of Cape Town. If the distance from Cape Town to Pretoria is 1000km, use trigonometry to find out how far East the ship is to Cape Town, and hence find the scale of the map.



Answer

Step 1 : Identify what happens in the question

We already know the distance between Cape Town and A in blocks from the given map (it is 5 blocks). Thus if we work out how many kilometers this same distance is, we can calculate how many kilometers each block represents, and thus we have the scale of the map.

Step 2 : Identify given information

Let us denote Cape Town with C and Pretoria with P. We can see that triangle APC is a right-angled triangle. Furthermore, we see that the distance AC and distance AP are both 5 blocks. Thus it is an isoceles triangle, and so $A\hat{C}P = A\hat{P}C = 45°$.

Step 3 : Carry out the calculation

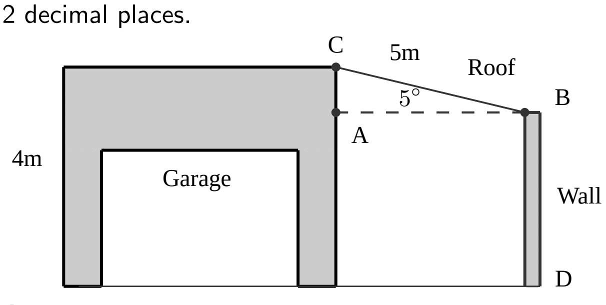
$$
\begin{aligned}
CA = & \quad CP \times \cos(A\hat{C}P) \\
= & \quad 1000 \times \cos(45°) \\
= & \quad \frac{1000}{\sqrt{2}}\ \mathrm{km}
\end{aligned}
$$

To work out the scale, we see that

$$
\begin{aligned}
& 5 \text{ blocks} = \frac{1000}{\sqrt{2}} \text{ km} \\
\implies & 1 \text{ block} = \frac{200}{\sqrt{2}} \text{ km}
\end{aligned}
$$

Worked Example 60: Building plan

Question: Mr Nkosi has a garage at his house, and he decides that he wants to add a corrugated iron roof to the side of the garage. The garage is 4m high, and his sheet for the roof is 5m long. If he wants the roof to be at an angle of $5°$, how high must he build the wall BD, which is holding up the roof? Give the answer to 2 decimal places.



Answer

Step 1 : Set out strategy

We see that the triangle ABC is a right-angled triangle. As we have one side and an angle of this triangle, we can calculate AC. The height of the wall is then the height of the garage minus AC.

Step 2 : Execute strategy

If BC=5m, and angle $A\hat{B}C = 5°$, then

$$
\begin{aligned}
AC &= BC \times \sin(A\hat{B}C) \\
&= 5 \times \sin 5° \\
&= 5 \times 0{,}0871 \\
&= 0.4358\,\mathrm{m}
\end{aligned}
$$

Thus we have that the height of the wall $BD = 5\,\mathrm{m} - 0.4358\,\mathrm{m} = 4.56\,\mathrm{m}$.

Exercise: Applications of Trigonometric Functions

1. A boy flying a kite is standing 30 m from a point directly under the kite. If the string to the kite is 50 m. long, find the angle of elevation of the kite.
2. What is the angle of elevation of the sun when a tree 7,15 m tall casts a shadow 10,1 m long?

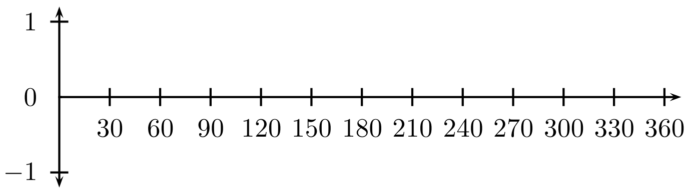
14.6 Graphs of Trigonometric Functions

This section describes the graphs of trigonometric functions.

14.6.1 Graph of $\sin\theta$

Activity :: Graph of $\sin\theta$: Complete the following table, using your calculator to calculate the values. Then plot the values with $\sin\theta$ on the y-axis and θ on the x-axis. Round answers to 1 decimal place.

θ	**0°**	**30°**	**60°**	**90°**	**120°**	**150°**	
$\sin\theta$							
θ	**180°**	**210°**	**240°**	**270°**	**300°**	**330°**	**360°**
$\sin\theta$							



Let us look back at our values for $\sin\theta$

θ	$0°$	$30°$	$45°$	$60°$	$90°$	$180°$
$\sin\theta$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	0

As you can see, the function $\sin\theta$ has a value of 0 at $\theta = 0°$. Its value then smoothly increases until $\theta = 90°$ when its value is 1. We then know that it later decreases to 0 when $\theta = 180°$. Putting all this together we can start to picture the full extent of the sine graph. The sine graph is shown in Figure 14.2. Notice the wave shape, with each wave having a length of $360°$. We say the graph has a *period* of $360°$. The height of the wave above (or below) the x-axis is called the waves' *amplitude*. Thus the maximum amplitude of the sine-wave is 1, and its minimum amplitude is -1.

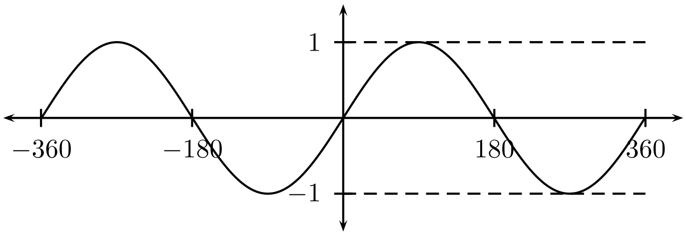


Figure 14.2: The graph of $\sin\theta$.

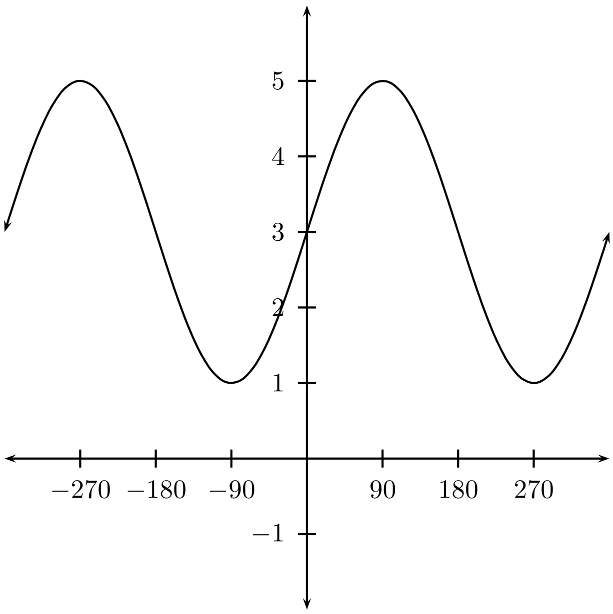


Figure 14.3: Graph of $f(\theta) = 2\sin\theta + 3$

14.6.2 Functions of the form $y = a\sin(x) + q$

In the equation, $y = a\sin(x) + q$, a and q are constants and have different effects on the graph of the function. The general shape of the graph of functions of this form is shown in Figure 14.3 for the function $f(\theta) = 2\sin\theta + 3$.

Activity :: Functions of the Form $y = a\sin(\theta) + q$:

1. **On the same set of axes, plot the following graphs:**
   (a) $a(\theta) = \sin\theta - 2$
   (b) $b(\theta) = \sin\theta - 1$
   (c) $c(\theta) = \sin\theta$
   (d) $d(\theta) = \sin\theta + 1$
   (e) $e(\theta) = \sin\theta + 2$

   Use your results to deduce the effect of q.
2. **On the same set of axes, plot the following graphs:**
   (a) $f(\theta) = -2 \cdot \sin\theta$
   (b) $g(\theta) = -1 \cdot \sin\theta$
   (c) $h(\theta) = 0 \cdot \sin\theta$

(d) $j(\theta) = 1 \cdot \sin\theta$

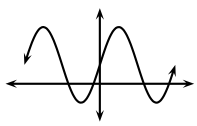
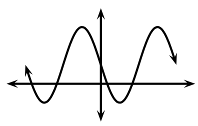
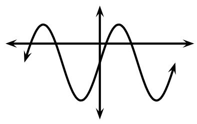
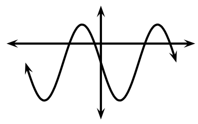
(e) $k(\theta) = 2 \cdot \sin\theta$

Use your results to deduce the effect of a.

You should have found that the value of a affects the height of the peaks of the graph. As the magnitude of a increases, the peaks get higher. As it decreases, the peaks get lower.
q is called the *vertical shift*. If $q = 2$, then the whole sine graph shifts up 2 units. If $q = -1$, the whole sine graph shifts down 1 unit.

These different properties are summarised in Table 14.1.

Table 14.1: Table summarising general shapes and positions of graphs of functions of the form $y = a\sin(x) + q$.

	$a > 0$	$a < 0$
$q > 0$		
$q < 0$		

Domain and Range

For $f(\theta) = a\sin(\theta) + q$, the domain is $\{\theta : \theta \in \mathbb{R}\}$ because there is no value of $\theta \in \mathbb{R}$ for which $f(\theta)$ is undefined.

The range of $f(\theta) = a\sin\theta + q$ depends on whether the value for a is positive or negative. We will consider these two cases separately.

If $a > 0$ we have:

-1 $\leq \sin\theta \leq 1$

-a $\leq a\sin\theta \leq a$ (Multiplication by a positive number maintains the nature of the inequality)

-a+q $\leq a\sin\theta + q \leq a + q$

-a+q $\leq f(\theta) \leq a + q$This tells us that for all values of θ, $f(\theta)$ is always between $-a + q$ and $a + q$. Therefore if $a > 0$, the range of $f(\theta) = a\sin\theta + q$ is $\{f(\theta) : f(\theta) \in [-a + q, a + q]\}$.

Similarly, it can be shown that if $a < 0$, the range of $f(\theta) = a\sin\theta + q$ is $\{f(\theta) : f(\theta) \in [a + q, -a + q]\}$. This is left as an exercise.

Important: The easiest way to find the range is simply to look for the "bottom" and the "top" of the graph.

Intercepts

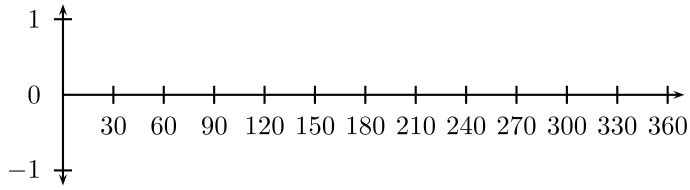
The y-intercept, y_{int}, of $f(\theta) = a\sin(x) + q$ is simply the value of $f(\theta)$ at $\theta = 0°$.

$$\begin{aligned} y_{int} &= f(0°) \\ &= a\sin(0°) + q \\ &= a(0) + q \\ &= q \end{aligned}$$

14.6.3 Graph of $\cos\theta$

Activity :: Graph of $\cos\theta$: Complete the following table, using your calculator to calculate the values correct to 1 decimal place. Then plot the values with $\cos\theta$ on the y-axis and θ on the x-axis.

θ	**0°**	**30°**	**60°**	**90°**	**120°**	**150°**	
$\cos\theta$							
θ	**180°**	**210°**	**240°**	**270°**	**300°**	**330°**	**360°**
$\cos\theta$							



Let us look back at our values for $\cos\theta$

θ	0°	30°	45°	60°	90°	180°
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	-1

If you look carefully, you will notice that the cosine of an angle θ is the same as the sine of the angle $90° - \theta$. Take for example,

$$\cos 60° = \frac{1}{2} = \sin 30° = \sin(90° - 60°)$$

This tells us that in order to create the cosine graph, all we need to do is to shift the sine graph 90° to the left. The graph of $\cos\theta$ is shown in figure 14.6. As the cosine graph is simply a shifted sine graph, it will have the same period and amplitude as the sine graph.

14.6.4 Functions of the form $y = a\cos(x) + q$

In the equation, $y = a\cos(x) + q$, a and q are constants and have different effects on the graph of the function. The general shape of the graph of functions of this form is shown in Figure 14.5 for the function $f(\theta) = 2\cos\theta + 3$.

Activity :: Functions of the Form $y = a\cos(\theta) + q$:

1. **On the same set of axes, plot the following graphs:**

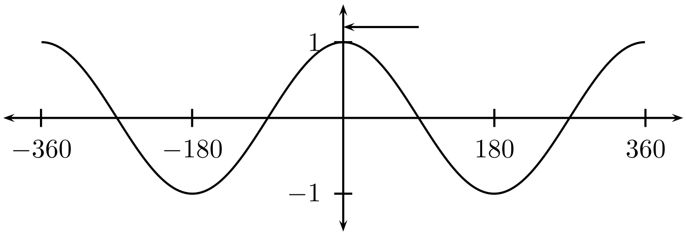


Figure 14.4: The graph of $\cos\theta$.

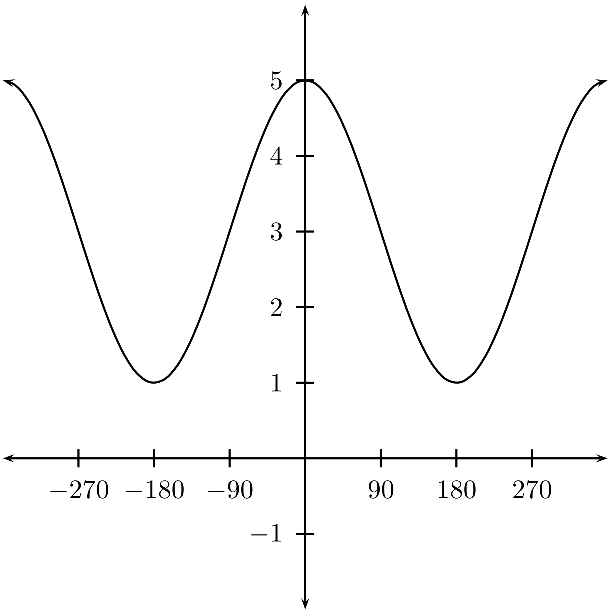


Figure 14.5: Graph of $f(\theta) = 2\cos\theta + 3$

(a) $a(\theta) = \cos\theta - 2$
(b) $b(\theta) = \cos\theta - 1$
(c) $c(\theta) = \cos\theta$
(d) $d(\theta) = \cos\theta + 1$
(e) $e(\theta) = \cos\theta + 2$

Use your results to deduce the effect of q.

2. On the same set of axes, plot the following graphs:

(a) $f(\theta) = -2 \cdot \cos\theta$
(b) $g(\theta) = -1 \cdot \cos\theta$
(c) $h(\theta) = 0 \cdot \cos\theta$
(d) $j(\theta) = 1 \cdot \cos\theta$
(e) $k(\theta) = 2 \cdot \cos\theta$

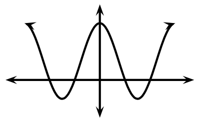
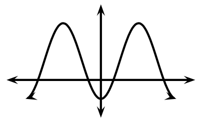
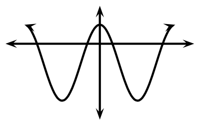
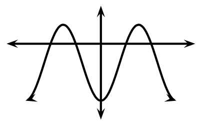
Use your results to deduce the effect of a.

You should have found that the value of a affects the amplitude of the cosine graph in the same way it did for the sine graph.

You should have also found that the value of q shifts the cosine graph in the same way as it did the sine graph.

These different properties are summarised in Table 14.2.

Table 14.2: Table summarising general shapes and positions of graphs of functions of the form $y = a\cos(x) + q$.

	$a > 0$	$a < 0$
$q > 0$		
$q < 0$		

Domain and Range

For $f(\theta) = a\cos(\theta) + q$, the domain is $\{\theta : \theta \in \mathbb{R}\}$ because there is no value of $\theta \in \mathbb{R}$ for which $f(\theta)$ is undefined.

It is easy to see that the range of $f(\theta)$ will be the same as the range of $a\sin(\theta) + q$. This is because the maximum and minimum values of $a\cos(\theta) + q$ will be the same as the maximum and minimum values of $a\sin(\theta) + q$.

Intercepts

The y-intercept of $f(\theta) = a\cos(x) + q$ is calculated in the same way as for sine.

$$\begin{aligned} y_{int} &= f(0^\circ) \\ &= a\cos(0^\circ) + q \\ &= a(1) + q \\ &= a + q \end{aligned}$$

14.6.5 Comparison of Graphs of $\sin\theta$ and $\cos\theta$

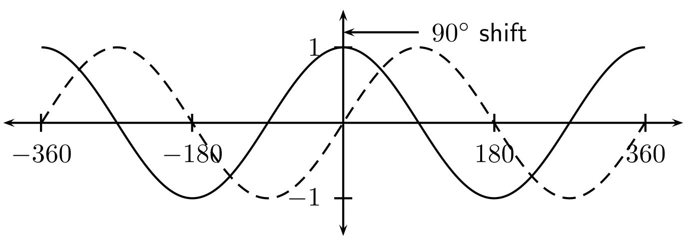
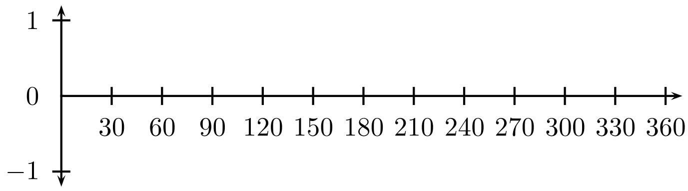


Figure 14.6: The graph of $\cos\theta$ (solid-line) and the sine graph (dashed-line).

14.6.6 Graph of $\tan\theta$

Activity :: Graph of $\tan\theta$: Complete the following table, using your calculator to calculate the values correct to 1 decimal place. Then plot the values with $\tan\theta$ on the y-axis and θ on the x-axis.

θ	**0°**	**30°**	**60°**	**90°**	**120°**	**150°**	
$\tan\theta$							
θ	**180°**	**210°**	**240°**	**270°**	**300°**	**330°**	**360°**
$\tan\theta$							



Let us look back at our values for $\tan\theta$

θ	0°	30°	45°	60°	90°	180°
$\tan\theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞	0

Now that we have graphs for $\sin\theta$ and $\cos\theta$, there is an easy way to visualise the tangent graph. Let us look back at our definitions of $\sin\theta$ and $\cos\theta$ for a right angled triangle.

$$\frac{\sin\theta}{\cos\theta} = \frac{\frac{\text{opposite}}{\text{hypotenuse}}}{\frac{\text{adjacent}}{\text{hypotenuse}}} = \frac{\text{opposite}}{\text{adjacent}} = \tan\theta$$

This is the first of an important set of equations called *trigonometric identities*. An identity is an equation, which holds true for any value, which is put into it. In this case we have shown that

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

for any value of θ.

So we know that for values of θ for which $\sin\theta = 0$, we must also have $\tan\theta = 0$. Also, if $\cos\theta = 0$ our value of $\tan\theta$ is undefined as we cannot divide by 0. The graph is shown in Figure 14.7. The dashed vertical lines are at the values of θ where $\tan\theta$ is not defined.

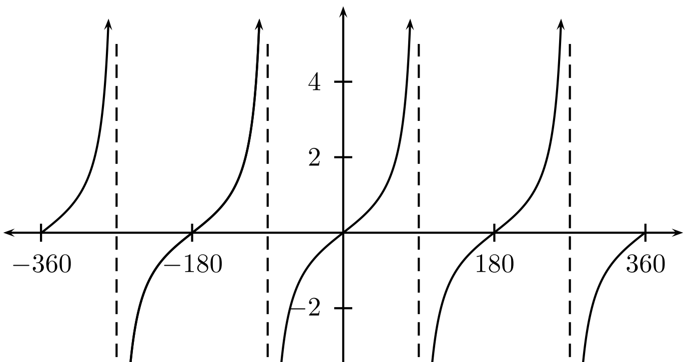


Figure 14.7: The graph of $\tan\theta$.

14.6.7 Functions of the form $y = a\tan(x) + q$

In the figure below is an example of a function of the form $y = a\tan(x) + q$.

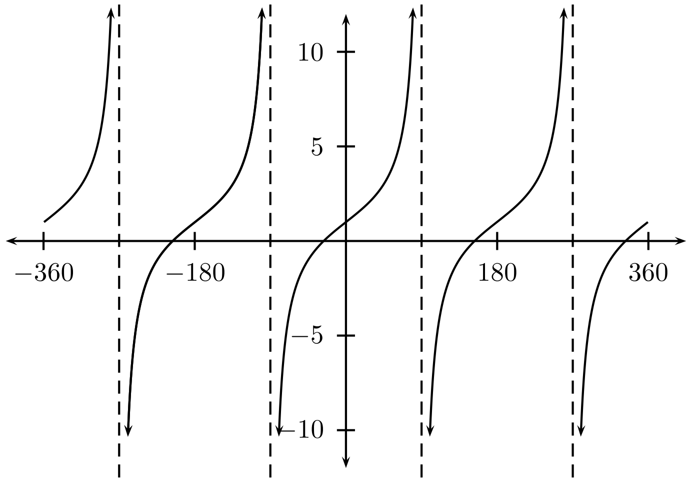


Figure 14.8: The graph of $2 \tan \theta + 1$.

Activity :: Functions of the Form $y = a \tan(\theta) + q$:

1. **On the same set of axes, plot the following graphs:**
   - **(a)** $a(\theta) = \tan \theta - 2$
   - **(b)** $b(\theta) = \tan \theta - 1$
   - **(c)** $c(\theta) = \tan \theta$
   - **(d)** $d(\theta) = \tan \theta + 1$
   - **(e)** $e(\theta) = \tan \theta + 2$

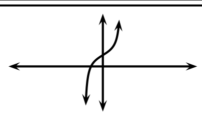
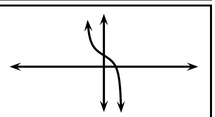
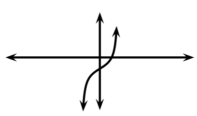
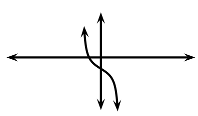
   Use your results to deduce the effect of q.

2. **On the same set of axes, plot the following graphs:**
   - **(a)** $f(\theta) = -2 \cdot \tan \theta$
   - **(b)** $g(\theta) = -1 \cdot \tan \theta$
   - **(c)** $h(\theta) = 0 \cdot \tan \theta$
   - **(d)** $j(\theta) = 1 \cdot \tan \theta$
   - **(e)** $k(\theta) = 2 \cdot \tan \theta$

   Use your results to deduce the effect of a.

You should have found that the value of a affects the steepness of each of the branches. You should have also found that the value of q affects the vertical shift as for $\sin \theta$ and $\cos \theta$. These different properties are summarised in Table 14.3.

Table 14.3: Table summarising general shapes and positions of graphs of functions of the form $y = a \tan(x) + q$.

	$a > 0$	$a < 0$
$q > 0$		
$q < 0$		

Domain and Range

The domain of $f(\theta) = a\tan(\theta) + q$ is all the values of θ such that $\cos\theta$ is not equal to 0. We have already seen that when $\cos\theta = 0$, $\tan\theta = \frac{\sin\theta}{\cos\theta}$ is undefined, as we have division by zero. We know that $\cos\theta = 0$ for all $\theta = 90° + 180°n$, where n is an integer. So the domain of $f(\theta) = a\tan(\theta) + q$ is all values of θ, except the values $\theta = 90° + 180°n$.

The range of $f(\theta) = a\tan\theta + q$ is $\{f(\theta) : f(\theta) \in (-\infty,\infty)\}$.

Intercepts

The y-intercept, y_{int}, of $f(\theta) = a\tan(x) + q$ is again simply the value of $f(\theta)$ at $\theta = 0°$.

$$\begin{aligned} y_{int} &= f(0°) \\ &= a\tan(0°) + q \\ &= a(0) + q \\ &= q \end{aligned}$$

Asymptotes

As θ approaches $90°$, $\tan\theta$ approaches infinity. But as θ is undefined at $90°$, θ can only approach $90°$, but never equal it. Thus the $\tan\theta$ curve gets closer and closer to the line $\theta = 90°$, without ever touching it. Thus the line $\theta = 90°$ is an asymptote of $\tan\theta$. $\tan\theta$ also has asymptotes at $\theta = 90° + 180°n$, where n is an integer.

?

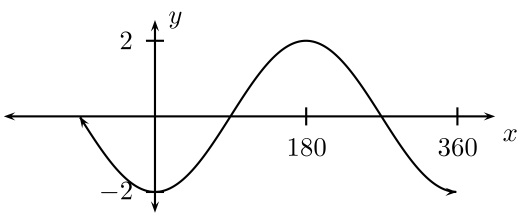
Exercise: Graphs of Trigonometric Functions

1. Using your knowldge of the effects of a and q, sketch each of the following graphs, without using a table of values, for $\theta \in [0°; 360°]$

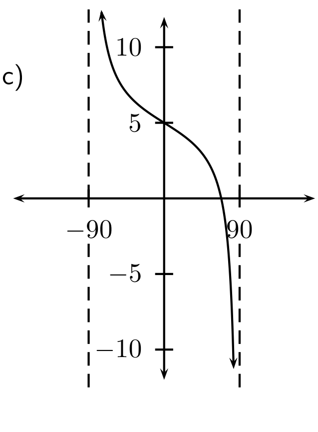
   (a) $y = 2\sin\theta$
   (b) $y = -4\cos\theta$
   (c) $y = -2\cos\theta + 1$
   (d) $y = \sin\theta - 3$
   (e) $y = \tan\theta - 2$
   (f) $y = 2\cos\theta - 1$

2. Give the equations of each of the following graphs:

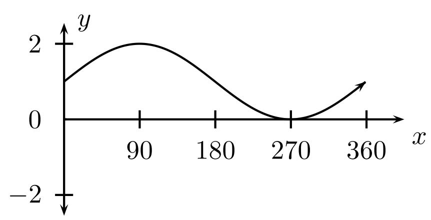
a)



c)

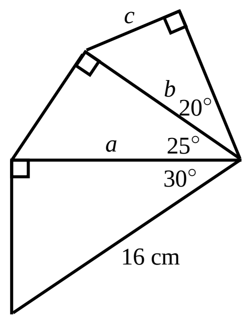


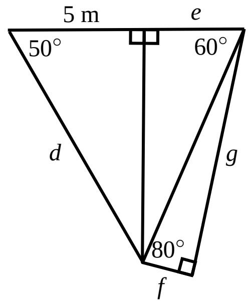
b)



14.7 End of Chapter Exercises

1. Calculate the unknown lengths





2. In the triangle PQR, $PR = 20$ cm, $QR = 22$ cm and $P\hat{R}Q = 30°$. The perpendicular line from P to QR intersects QR at X. Calculate

   A the length XR,

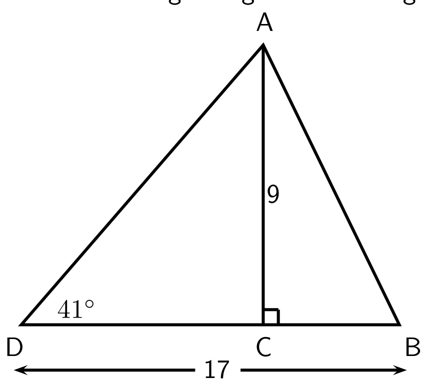
   B the length PX, and

   C the angle $Q\hat{P}X$

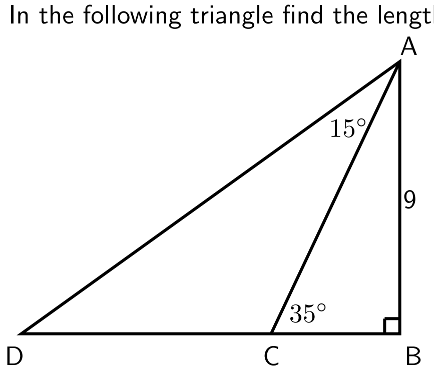
3. A ladder of length 15 m is resting against a wall, the base of the ladder is 5 m from the wall. Find the angle between the wall and the ladder?

4. A ladder of length 25 m is resting against a wall, the ladder makes an angle $37°$ to the wall. Find the distance between the wall and the base of the ladder?

5. In the following triangle find the angle $A\hat{B}C$



6. In the following triangle find the length of side CD



7. $A(5; 0)$ and $B(11; 4)$. Find the angle between the line through A and B and the x-axis.

8. $C(0; -13)$ and $D(-12; 14)$. Find the angle between the line through C and D and the y-axis.

9. $E(5; 0)$, $F(6; 2)$ and $G(8; -2)$. Find the angle $F\hat{E}G$.

10. A $5\,\mathrm{m}$ ladder is placed $2\,\mathrm{m}$ from the wall. What is the angle the ladder makes with the wall?

11. An isosceles triangle has sides $9\,\mathrm{cm}$, $9\,\mathrm{cm}$ and $2\,\mathrm{cm}$. Find the size of the smallest angle of the triangle.

12. A right-angled triangle has hypotenuse $13\,\mathrm{mm}$ Find the length of the other two sides if one of the angles of the triangle is $50°$.

13. One of the angles of a rhombus (**rhombus** - A four-sided polygon, each of whose sides is of equal length.) with perimeter $20\,\mathrm{cm}$ is $30°$.

    A Find the sides of the rhombus.

    B Find the length of both diagonals.

14. Captain Hook was sailing towards a lighthouse of height $10\,\mathrm{m}$.

    A If the top of the lighthouse is $30\,\mathrm{m}$ away, what is the angle of elevation of the boat to the nearest integer?

    B If the boat moves another $7\,\mathrm{m}$ towards the lighthouse, what is the new angle of elevation of the boat to the nearest integer?

15. (Tricky) A triangle with angles $40°$, $40°$ and $100°$ has a perimeter of $20\,\mathrm{cm}$. Find the length of each side of the triangle.

Chapter 15

Statistics - Grade 10

15.1 Introduction

Information in the form of numbers, graphs and tables is all around us; on television, on the radio or in the newspaper. We are exposed to crime rates, sports results, rainfall, government spending, rate of HIV/AIDS infection, population growth and economic growth.

This chapter demonstrates how Mathematics can be used to manipulate data, to represent or misrepresent trends and patterns and to provide solutions that are directly applicable to the world around us.

Skills relating to the collection, organisation, display, analysis and interpretation of information that were introduced in earlier grades are developed further.

15.2 Recap of Earlier Work

The collection of data has been introduced in earlier grades as a method of obtaining answers to questions about the world around us. This work will be briefly reviewed.

15.2.1 Data and Data Collection

Data

Definition: Data
Data refers to the pieces of information that have been observed and recorded, from an experiment or a survey. There are two types of data: primary and secondary. The word "data" is the plural of the word "datum", and therefore one should say, "the data are" and not "the data is".

Data can be classified as *primary* or *secondary*, and primary data can be classified as *qualitative* or *quantitative*. Figure 15.1 summarises the classifications of data.

Primary data describes the original data that have been collected. This type of data is also known as *raw* data. Often the primary data set is very large and is therefore summarised or processed to extract meaningful information.

Qualitative data is information that cannot be written as numbers.

Quantitative data is information that can be written as numbers.

Secondary data is primary data that has been summarised or processed.

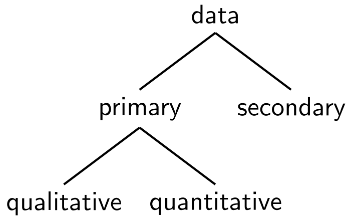


Figure 15.1: Classes of data.

Purpose of Data Collection

Data is collected to provide answers that help with understanding a particular situation. For example:

- The local government might want to know how many residents have electricity and might ask the question: "Does your home have a safe, independent supply of electricity?"
- A supermarket manager might ask the question: "What flavours of soft drink should be stocked in my supermarket?" The question asked of customers might be "What is your favourite soft drink?" Based on the customers' responses, the manager can make an informed decision as to what soft drinks to stock.
- A company manufacturing medicines might ask "How effective is our pill at relieving a headache?" The question asked of people using the pill for a headache might be: "Does taking the pill relieve your headache?" Based on responses, the company learns how effective their product is.
- A motor car company might want to improve their customer service, and might ask their customers: "How can we improve our customer service?"
- A cell phone manufacturing company might collect data about how often people buy new cell phones and what factors affect their choice, so that the cell phone company can focus on those features that would make their product more attractive to buyers.
- A town councillor might want to know how many accidents have occurred at a particular intersection, to decide whether a robot should be installed. The councillor would visit the local police station to research their records to collect the appropriate data.

However, it is important to note that different questions reveal different features of a situation, and that this affects the ability to understand the situation. For example, if the first question in the list was re-phrased to be: "Does your home have electricity?" then if you answered yes, but you were getting your electricity from a neighbour, then this would give the wrong impression that you did not need an independent supply of electricity.

15.2.2 Methods of Data Collection

The method of collecting the data must be appropriate to the question being asked. Some examples of data collecting methods are:

1. Questionnaires, surveys and interviews
2. Experiments
3. Other sources (friends, family, newspapers, books, magazines and the Internet)

The most important aspect of each method of data collecting is to clearly formulate the question that is to be answered. The details of the data collection should therefore be structured to take your question into account.

For example, questionnaires, interviews or surveys would be most appropriate for the list of questions in Section 15.2.1.

15.2.3 Samples and Populations

Before the data collecting starts, an important point to decide upon, is how much data is needed to make sure that the results give an accurate reflection to the answers that are required for the study. Ideally, the study should be designed to maximise the amount of information collected while minimising the effort. The concepts of *populations* and *samples* is vital to minimising effort.

The following terms should be familiar:

Population describes the entire group under consideration in a study. For example, if you wanted to know how many learners in your school got the flu each winter, then your population would be all the learners in your school.

Sample describes a group chosen to represent the population under consideration in a study. For example, for the survey on winter flu, you might select a sample of learners, maybe one from each class.

Random sample describes a sample chosen from a population in such a way that each member of the population has an equal chance of being chosen.

Choosing a representative sample is crucial to obtaining results that are unbiased. For example, if we wanted to determine whether peer pressure affects the decision to start smoking, then the results would be different if only boys were interviewed, compared to if only girls were interviewed, compared to both boys and girls being interviewed.

Therefore questions like: "How many interviews are needed?" and "How do I select the subjects for the interviews?" must be asked during the design stage of the interview process.

The most accurate results are obtained if the entire population is sampled for the survey, but this is expensive and time-consuming. The next best method is to *randomly* select a sample of subjects for the interviews. This means that whatever the method used to select subjects for the interviews, each subject has an equal chance of being selected. There are various methods of doing this but all start with a complete list of each member of the population. Then names can be picked out of a hat or can be selected by using a random number generator. Most modern scientific calculators have a random number generator or you can find one on a spreadsheet program on a computer.

If the subjects for the interviews, are randomly selected then it does not matter too much how many interviews are conducted. So, if you had a total population of 1 000 learners in your school and you randomly selected 100, then that would be the sample that is used to conduct your survey.

15.3 Example Data Sets

The remainder of this chapter deals with the mathematical details that are required to analyse the data collected.

The following are some example sets of data which can be used to apply the methods that are being explained.

15.3.1 Data Set 1: Tossing a Coin

A fair coin was tossed 100 times and the values on the top face were recorded.

15.3.2 Data Set 2: Casting a die

A fair die was cast 100 times and the values on the top face were recorded. The data are recorded in Table 15.3.2.

H	T	T	H	H	T	H	H	H	H
H	H	H	H	T	H	H	T	T	T
T	T	H	T	T	H	T	H	T	H
H	H	T	T	H	T	T	H	T	T
T	H	H	H	T	T	H	T	T	H
H	T	T	T	T	H	T	T	H	H
T	T	H	T	T	H	T	T	H	T
H	T	T	H	T	T	T	T	H	T
T	H	T	T	H	H	H	T	H	T
T	T	T	H	H	T	T	T	H	T

Table 15.1: Results of 100 tosses of a fair coin. H means that the coin landed heads-up and T means that the coin landed tails-up.

3	5	3	6	2	6	6	5	5	6	6	4	2	1	5	3	2	4	5	4
1	4	3	2	6	6	4	6	2	6	5	1	5	1	2	4	4	2	4	4
4	2	6	4	5	4	3	5	5	4	6	1	1	4	6	6	4	5	3	5
2	6	3	2	4	5	3	2	2	6	3	4	3	2	6	4	5	2	1	5
5	4	1	3	1	3	5	1	3	6	5	3	4	3	4	5	1	2	1	2
1	3	2	3	6	3	1	6	3	6	6	1	4	5	2	2	6	3	5	3
1	1	6	4	5	1	6	5	3	2	6	2	3	2	5	6	3	5	5	6
2	6	6	3	5	4	1	4	5	1	4	1	3	4	3	6	2	4	3	6
6	1	1	2	4	5	2	5	3	4	3	4	5	3	3	3	1	1	4	3
5	2	1	4	2	5	2	2	1	5	4	5	1	5	3	2	2	5	1	1

Table 15.2: Results of 200 casts of a fair die.

15.3.3 Data Set 3: Mass of a Loaf of Bread

A loaf of bread should weigh 800g. The masses of 10 different loaves of bread were measured at a store for 1 week. The data is shown in Table 15.3.

"The Trade Metrology Act requires that if a loaf of bread is not labelled, it must weigh 800g, with the leeway of five percent under or 10 percent over. However, an average of 10 loaves must be an exact match to the mass stipulated. - Sunday Tribune of 10 October 2004 on page 10"

Monday	Tuesday	Wednesday	Thursday	Friday	Saturday	Sunday
802.39	787.78	815.74	807.41	801.48	786.59	799.01
796.76	798.93	809.68	798.72	818.26	789.08	805.99
802.50	793.63	785.37	809.30	787.65	801.45	799.35
819.59	812.62	809.05	791.13	805.28	817.76	801.01
801.21	795.86	795.21	820.39	806.64	819.54	796.67
789.00	796.33	787.87	799.84	789.45	802.05	802.20
788.99	797.72	776.71	790.69	803.16	801.24	807.32
808.80	780.38	812.61	801.82	784.68	792.19	809.80
802.37	790.83	792.43	789.24	815.63	799.35	791.23
796.20	817.57	799.05	825.96	807.89	806.65	780.23

Table 15.3: Masses (in g) of 10 different loaves of bread, from the same manufacturer, measured at the same store over a period of 1 week.

15.3.4 Data Set 4: Global Temperature

The mean global temperature from 1861 to 1996 is listed in Table 15.4. The data, obtained from `http://www.cgd.ucar.edu/stats/Data/Climate/`, was converted to mean temperature in degrees Celsius.

`http://lib.stat.cmu.edu/DASL/`

Year	Temperature	Year	Temperature	Year	Temperature	Year	Temperature
1861	12.66	1901	12.871	1941	13.152	1981	13.228
1862	12.58	1902	12.726	1942	13.147	1982	13.145
1863	12.799	1903	12.647	1943	13.156	1983	13.332
1864	12.619	1904	12.601	1944	13.31	1984	13.107
1865	12.825	1905	12.719	1945	13.153	1985	13.09
1866	12.881	1906	12.79	1946	13.015	1986	13.183
1867	12.781	1907	12.594	1947	13.006	1987	13.323
1868	12.853	1908	12.575	1948	13.015	1988	13.34
1869	12.787	1909	12.596	1949	13.005	1989	13.269
1870	12.752	1910	12.635	1950	12.898	1990	13.437
1871	12.733	1911	12.611	1951	13.044	1991	13.385
1872	12.857	1912	12.678	1952	13.113	1992	13.237
1873	12.802	1913	12.671	1953	13.192	1993	13.28
1874	12.68	1914	12.85	1954	12.944	1994	13.355
1875	12.669	1915	12.962	1955	12.935	1995	13.483
1876	12.687	1916	12.727	1956	12.836	1996	13.314
1877	12.957	1917	12.584	1957	13.139		
1878	13.092	1918	12.7	1958	13.208		
1879	12.796	1919	12.792	1959	13.133		
1880	12.811	1920	12.857	1960	13.094		
1881	12.845	1921	12.902	1961	13.124		
1882	12.864	1922	12.787	1962	13.129		
1883	12.783	1923	12.821	1963	13.16		
1884	12.73	1924	12.764	1964	12.868		
1885	12.754	1925	12.868	1965	12.935		
1886	12.826	1926	13.014	1966	13.035		
1887	12.723	1927	12.904	1967	13.031		
1888	12.783	1928	12.871	1968	13.004		
1889	12.922	1929	12.718	1969	13.117		
1890	12.703	1930	12.964	1970	13.064		
1891	12.767	1931	13.041	1971	12.903		
1892	12.671	1932	12.992	1972	13.031		
1893	12.631	1933	12.857	1973	13.175		
1894	12.709	1934	12.982	1974	12.912		
1895	12.728	1935	12.943	1975	12.975		
1896	12.93	1936	12.993	1976	12.869		
1897	12.936	1937	13.092	1977	13.148		
1898	12.759	1938	13.187	1978	13.057		
1899	12.874	1939	13.111	1979	13.154		
1900	12.959	1940	13.055	1980	13.195		

Table 15.4: Global temperature changes over the past x years. Is there a warming of the planet?

15.3.5 Data Set 5: Price of Petrol

The price of petrol in South Africa from August 1998 to July 2000 is shown in Table 15.5.

15.4 Grouping Data

One of the first steps to processing a large set of raw data is to arrange the data values together into a smaller number of groups, and then count how many of each data value there are in each group. The groups are usually based on some sort of interval of data values, so data values that fall into a specific interval, would be grouped together. The grouped data is often presented graphically or in a frequency table. (Frequency means "how many times")

Table 15.5: Petrol prices

Date	Price (R/l)
August 1998	R 2.37
September 1998	R 2.38
October 1998	R 2.35
November 1998	R 2.29
December 1998	R 2.31
January 1999	R 2.25
February 1999	R 2.22
March 1999	R 2.25
April 1999	R 2.31
May 1999	R 2.49
June 1999	R 2.61
July 1999	R 2.61
August 1999	R 2.62
September 1999	R 2.75
October 1999	R 2.81
November 1999	R 2.86
December 1999	R 2.85
January 2000	R 2.86
February 2000	R 2.81
March 2000	R 2.89
April 2000	R 3.03
May 2000	R 3.18
June 2000	R 3.22
July 2000	R 3.36

Worked Example 61: Grouping Data

Question: Group the elements of Data Set 1 to determine how many times the coin landed heads-up and how many times the coin landed tails-up.

Answer

Step 1 : Identify the groups

There are two unique data values: H and T. Therefore there are two groups, one for the H-data values and one for the T-data values.

Step 2 : Count how many data values fall into each group.

Data Value	Frequency
H	44
T	56

Step 3 : Check that the total of the frequency column is equal to the total number of data values.

There are 100 data values and the total of the frequency column is 44+56=100.

15.4.1 Exercises - Grouping Data

1. The height of 30 learners are given below. Fill in the grouped data below. (Tally is a convenient way to count in 5's. We use IIII to indicate 5.)

142	163	169	132	139	140	152	168	139	150
161	132	162	172	146	152	150	132	157	133
141	170	156	155	169	138	142	160	164	168

Group	Tally	Frequency
$130 \leq h < 140$		
$140 \leq h < 150$		
$150 \leq h < 160$		
$160 \leq h < 170$		
$170 \leq h < 180$		

2. An experiment was conducted in class and 50 learners were asked to guess the number of sweets in a jar. The following guesses were recorded.

56	49	40	11	33	33	37	29	30	59
21	16	38	44	38	52	22	24	30	34
42	15	48	33	51	44	33	17	19	44
47	23	27	47	13	25	53	57	28	23
36	35	40	23	45	39	32	58	22	40

A Draw up a grouped frequency table using intervals 11-20, 21-30, 31-40, etc.

15.5 Graphical Representation of Data

Once the data has been collected, it must be organised in a manner that allows for the information to be extracted most efficiently. One method of organisation is to display the data in the form of graphs. Functions and graphs have been studied in Chapter **??**, and similar techniques will be used here. However, instead of drawing graphs from equations as was done in Chapter **??**, bar graphs, histograms and pie charts will be drawn directly from the data.

15.5.1 Bar and Compound Bar Graphs

A bar chart is used to present data where each observation falls into a specific category and where the categories are unrelated. The frequencies (or percentages) are listed along the y-axis and the categories are listed along the x-axis. The heights of the bars correspond to the frequencies. The bars are of equal width and should not touch neighbouring bars.

A compound bar chart (also called component bar chart) is a variant: here the bars are cut into various components depending on what is being shown. If percentages are used for various components of a compound bar, then the total bar height must be 100%. The compound bar chart is a little more complex but if this method is used sensibly, a lot of information can be quickly shown in an attractive fashion.

Examples of a bar and a compound bar graph, for Data Set 1 Table 15.1, are shown in Figure 15.2. According to the frequency table for Data Set 1, the coin landed heads-up 44 times and tails-up 56 times.

15.5.2 Histograms and Frequency Polygons

It is often useful to look at the frequency with which certain values fall in pre-set groups or classes of specified sizes. The choice of the groups should be such that they help highlight features in the data. If these grouped values are plotted in a manner similar to a bar graph, then the resulting graph is known as a histogram. Examples of histograms are shown in Figure 15.3 for Data Set 2, with group sizes of 1 and 2.

Groups	$0 < n \leq 1$	$1 < n \leq 2$	$2 < n \leq 3$	$3 < n \leq 4$	$4 < n \leq 5$	$5 < n \leq 6$
Frequency	30	32	35	34	37	32

Table 15.6: Frequency table for Data Set 2, with a group size of 1.

The same data used to plot a histogram are used to plot a frequency polygon, except the pair of data values are plotted as a point and the points are joined with straight lines. The frequency polygons for the histograms in Figure 15.3 are shown in Figure 15.4.

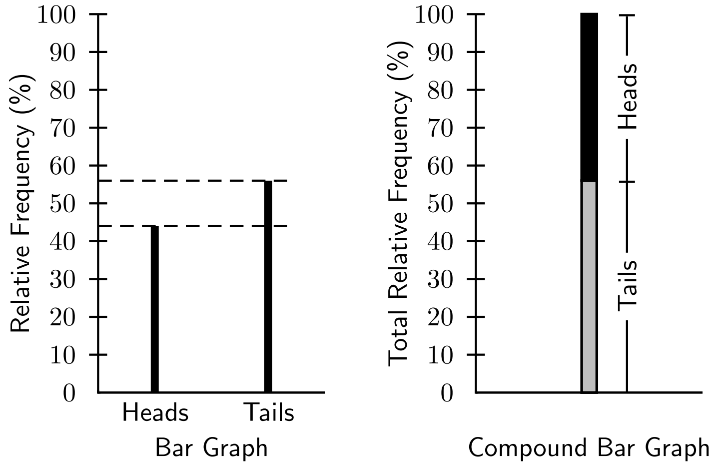


Figure 15.2: Examples of a bar graph (left) and compound bar graph (right) for Data Set 1. The compound bar graph extends from 0% to 100%.

Groups	$0 < n \leq 2$	$2 < n \leq 4$	$4 < n \leq 6$
Frequency	62	69	69

Table 15.7: Frequency table for Data Set 2, with a group size of 2.

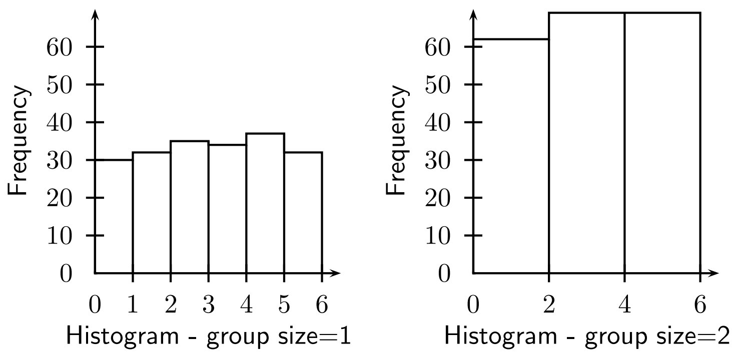


Figure 15.3: Examples of histograms for Data Set 2, with a group size = 1 (left) and a group size = 2 (right). The scales on the y-axis for each graph are the same, and the values in the graph on the right are higher than the values of the graph on the left.

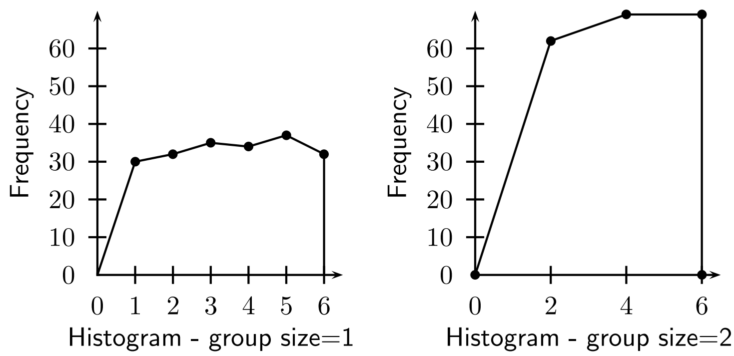


Figure 15.4: Examples of histograms for Data Set 2, with a group size = 1 (left) and a group size = 2 (right). The scales on the y-axis for each graph are the same, and the values in the graph on the right are higher than the values of the graph on the left.

Unlike histograms, many frequency polygons can be plotted together to compare several frequency distributions, provided that the data has been grouped in the same way.

15.5.3 Pie Charts

A pie chart is a graph that is used to show what categories make up a specific section of the data, and what the contribution each category makes to the entire set of data. A pie chart is based on a circle, and each category is represented as a wedge of the circle or alternatively as a slice of the pie. The area of each wedge is proportional to the ratio of that specific category to the total number of data values in the data set. The wedges are usually shown in different colours to make the distinction between the different categories easier.

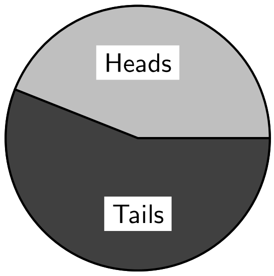


Figure 15.5: Example of a pie chart for Data Set 1. Pie charts show what contribution each group makes to the total data set.

Method: Drawing a pie-chart

1. Draw a circle that represents the entire data set.
2. Calculate what proportion of 360° each category corresponds to according to

$$\text{Angular Size} = \frac{\text{Frequency}}{\text{Total}} \times 360°$$

3. Draw a wedge corresponding to the angular contribution.
4. Check that the total degrees for the different wedges adds up to close to $360°$.

Worked Example 62: Pie Chart

Question: Draw a pie chart for Data Set 2, showing the relative proportions of each data value to the total.

Answer

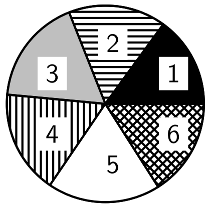
Step 1 : Determine the frequency table for Data Set 2.

							Total
Data Value	1	2	3	4	5	6	–
Frequency	30	32	35	34	37	32	200

Step 2 : Calculate the angular size of the wedge for each data value

Data Value	Angular Size of Wedge
1	$\frac{\text{Frequency}}{\text{Total}} \times 360° = \frac{30}{200} \times 360 = 54°$
2	$\frac{\text{Frequency}}{\text{Total}} \times 360° = \frac{32}{200} \times 360 = 57{,}6°$
3	$\frac{\text{Frequency}}{\text{Total}} \times 360° = \frac{35}{200} \times 360 = 63°$
4	$\frac{\text{Frequency}}{\text{Total}} \times 360° = \frac{34}{200} \times 360 = 61{,}2°$
5	$\frac{\text{Frequency}}{\text{Total}} \times 360° = \frac{37}{200} \times 360 = 66{,}6°$
6	$\frac{\text{Frequency}}{\text{Total}} \times 360° = \frac{32}{200} \times 360 = 57{,}6°$

Step 3 : Draw the pie, with the size of each wedge as calculated above.



Pie Chart for Data Set 2

Note that the total angular size of the wedges may not add up to exactly 360° because of rounding.

15.5.4 Line and Broken Line Graphs

All graphs that have been studied until this point (bar, compound bar, histogram, frequency polygon and pie) are drawn from grouped data. The graphs that will be studied in this section are drawn from the ungrouped or raw data.

Line and broken line graphs are plots of a dependent variable as a function of an independent variable, e.g. the average global temperature as a function of time, or the average rainfall in a country as a function of season.

Usually a line graph is plotted after a table has been provided showing the relationship between the two variables in the form of pairs. Just as in (x,y) graphs, each of the pairs results in a specific point on the graph, and being a LINE graph these points are connected to one another by a LINE.

Many other line graphs exist; they all CONNECT the points by LINES, not necessarily straight lines. Sometimes polynomials, for example, are used to describe approximately the basic relationship between the given pairs of variables, and between these points.

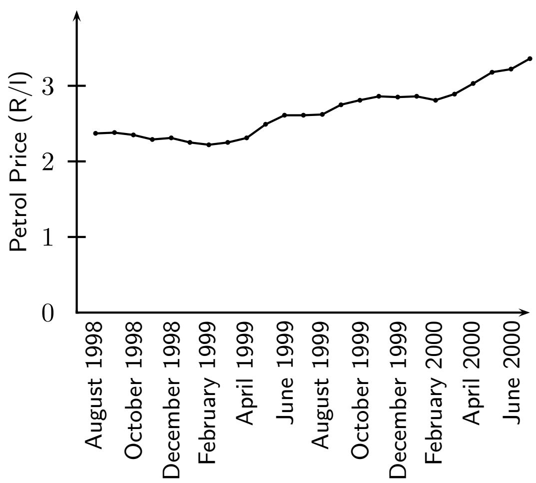


Figure 15.6: Example of a line graph for Data Set 5.

Worked Example 63: Line Graphs

Question: Clawde the cat is overweight and her owners have decided to put her on a restricted eating plan. Her mass is measured once a month and is tabulated below. Draw a line graph of the data to determine whether the restricted eating plan is working.

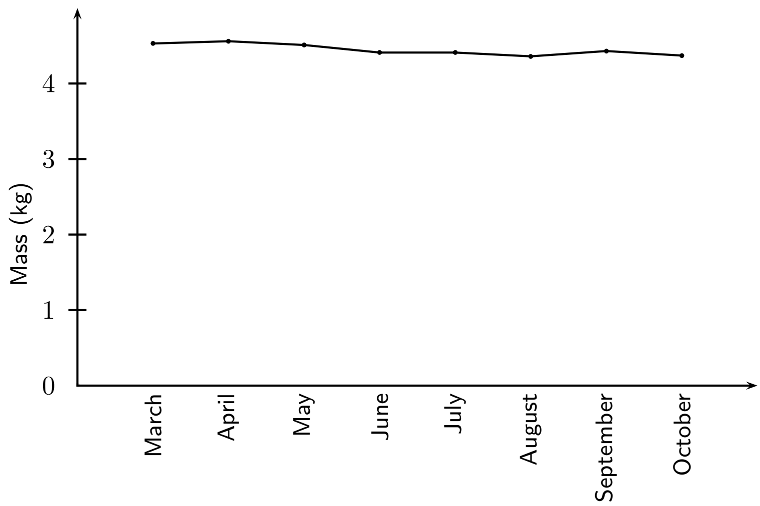
Month	**Mass** (kg)
March	4,53
April	4,56
May	4,51
June	4,41
July	4,41
August	4,36
September	4,43
October	4,37

Answer

Step 1 : Determine what is required

We are required to plot a line graph to determine whether the restricted eating plan is helping Clawde the cat lose weight. We are given all the information that we need to plot the graph.

Step 2 : Plot the graph



Step 3 : Analyse Graph

There is a slight decrease of mass from March to October, so the restricted eating plan is working, but very slowly.

15.5.5 Exercises - Graphical Representation of Data

1. Represent the following information on a pie chart.

Walk	15
Cycle	24
Train	18
Bus	8
Car	35
Total	100

2. Represent the following information using a broken line graph.

Time	07h00	08h00	09h00	10h00	11h00	12h00
Temp (°C)	16	16,5	17	19	20	24

3. Represent the following information on a histogram. Using a coloured pen, draw a frequency polygon on this histogram.

Time in seconds	Frequency
16 - 25	5
26 - 35	10
36 - 45	26
46 - 55	30
56 - 65	15
66 - 75	12
76 - 85	10

4. The maths marks of a class of 30 learners are given below, represent this information using a suitable graph.

82	75	66	54	79	78	29	55	68	91
43	48	90	61	45	60	82	63	72	53
51	32	62	42	49	62	81	49	61	60

5. Use a compound bar graph to illustrate the following information

Year	2003	2004	2005	2006	2007
Girls	18	15	13	12	15
Boys	15	11	18	16	10

15.6 Summarising Data

If the data set is very large, it is useful to be able to summarise the data set by calculating a few quantities that give information about how the data values are spread and about the central values in the data set.

15.6.1 Measures of Central Tendency

An average is simply a number that is representative of a set of data. Specifically, it is a *measure of central tendency* which means that it gives an indication of the main tendency of the set of data. Averages are useful for comparing data, especially when sets of different sizes are being compared.

There are several types of average. Perhaps the simplest and most commonly used average is the *mean* of a set of data. Other common types of average are the *median* and the *mode*.

Mean

The mean, (also known as arithmetic mean), is simply the arithmetic average of a group of numbers (or data set) and is shown using the bar symbol $\bar{}$. So the mean of the variable x is $\bar{x}$ pronounced "x-bar". The mean of a set of values is calculated by adding up all the values in the set and dividing by the number of items in that set. The mean is calculated from the raw, ungrouped data.

Definition: Mean
The mean of a data set, x, denoted by $\bar{x}$, is the average of the data values, and is calculated as:

$$\bar{x} = \frac{\text{sum of all values}}{\text{number of values}} \tag{15.1}$$

Method: Calculating the mean

1. Find the total of the data values in the data set.
2. Count how many data values there are in the data set.
3. Divide the total by the number of data values.

Worked Example 64: Mean

Question: What is the mean of $x = \{10,20,30,40,50\}$?
Answer
Step 1 : Find the total of the data values

$$10 + 20 + 30 + 40 + 50 = 150$$

Step 2 : Count the number of data values in the data set
There are 5 values in the data set.
Step 3 : Divide the total by the number of data values.

$$150 \div 5 = 30$$

Step 4 : Answer
$\therefore$ the mean of the data set $x = \{10,20,30,40,50\}$ is 30.

Median

Definition: Median
The median of a set of data is the data value in the central position, when the data set has been arranged from highest to lowest or from lowest to highest. There are an equal number of data values on either side of the median value.

The median is calculated from the raw, ungrouped data, as follows.

Method: Calculating the median

1. Order the data from smallest to largest or from largest to smallest.
2. Count how many data values there are in the data set.
3. Find the data value in the central position of the set.

Worked Example 65: Median

Question: What is the median of $\{10,14,86,2,68,99,1\}$?
Answer
Step 1 : Order the data set from lowest to highest
1,2,10,14,68,85,99
Step 2 : Count the number of data values in the data set
There are 7 points in the data set.
Step 3 : Find the central position of the data set
The central position of the data set is 4.
Step 4 : Find the data value in the central position of the ordered data set.
14 is in the central position of the data set.
Step 5 : Answer
$\therefore$ 14 is the median of the data set $\{1,2,10,14,68,85,99\}$.

This example has highlighted a potential problem with determining the median. It is very easy to determine the median of a data set with an odd number of data values, but what happens when there is an even number of data values in the data set?

When there is an even number of data values, the median is the mean of the two middle points.

Important: Finding the Central Position of a Data Set

An easy way to determine the central position or positions for any ordered data set is to take the total number of data values, add 1, and then divide by 2. If the number you get is a whole number, then that is the central position. If the number you get is a fraction, take the two whole numbers on either side of the fraction, as the positions of the data values that must be averaged to obtain the median.

Worked Example 66: Median

Question: What is the median of $\{11,10,14,86,2,68,99,1\}$?
Answer
Step 1 : Order the data set from lowest to highest
1,2,10,11,14,68,85,99
Step 2 : Count the number of data values in the data set
There are 8 points in the data set.
Step 3 : Find the central position of the data set
The central position of the data set is between positions 4 and 5.
Step 4 : Find the data values around the central position of the ordered data set.
11 is in position 4 and 14 is in position 5.
Step 5 : Answer
$\therefore$ the median of the data set $\{1,2,10,11,14,68,85,99\}$ is

$$(11 + 14) \div 2 = 12{,}5$$

Mode

Definition: Mode
The mode is the data value that occurs most often, i.e. it is the most frequent value or most common value in a set.

Method: Calculating the mode Count how many times each data value occurs. The mode is the data value that occurs the most.

The mode is calculated from grouped data, or single data items.

Worked Example 67: Mode

Question: Find the mode of the data set $x = \{1, 2, 3, 4, 4, 4, 5, 6, 7, 8, 8, 9,10,10\}$
Answer
Step 1 : Count how many times each data value occurs.

data value	frequency	data value	frequency
1	1	6	1
2	1	7	1
3	1	8	2
4	3	9	1
5	1	10	2

Step 2 : Find the data value that occurs most often.
4 occurs most often.
Step 3 : Answer
The mode of the data set $x = \{1, 2, 3, 4, 4, 4, 5, 6, 7, 8, 8, 9,10,10\}$ is 4.

A data set can have more than one mode. For example, both 2 and 3 are modes in the set 1, 2, 2, 3, 3. If all points in a data set occur with equal frequency, it is equally accurate to describe the data set as having many modes or no mode.

15.6.2 Measures of Dispersion

The mean, median and mode are measures of central tendency, i.e. they provide information on the central data values in a set. When describing data it is sometimes useful (and in some cases necessary) to determine the spread of a distribution. Measures of dispersion provide information on how the data values in a set are distributed around the mean value. Some measures of dispersion are range, percentiles and quartiles.

Range

Definition: Range
The range of a data set is the difference between the lowest value and the highest value in the set.

Method: Calculating the range

1. Find the highest value in the data set.
2. Find the lowest value in the data set.
3. Subtract the lowest value from the highest value. The difference is the range.

Worked Example 68: Range

Question: Find the range of the data set $x = \{1, 2, 3, 4, 4, 4, 5, 6, 7, 8, 8, 9,10,10\}$
Answer
Step 1 : Find the highest and lowest values.
10 is the highest value and 1 is the lowest value.
Step 2 : Subtract the lowest value from the highest value to calculate the range.

$$10 - 1 = 9$$

Step 3 : Answer
For the data set $x = \{1, 2, 3, 4, 4, 4, 5, 6, 7, 8, 8, 9,10,10\}$, the range is 9.

Quartiles

Definition: Quartiles
Quartiles are the three data values that divide an ordered data set into four groups containing equal numbers of data values. The median is the second quartile.

The quartiles of a data set are formed by the two boundaries on either side of the median, which divide the set into four equal sections. The lowest 25% of the data being found below the first quartile value, also called the lower quartile. The median, or second quartile divides the set into two equal sections. The lowest 75% of the data set should be found below the third quartile, also called the upper quartile. For example:

Data Items										
22	24	48	51	60	72	73	75	80	88	90
		↓ Lower quartile (Q_1)			↓ Median (Q_2)			↓ Upper quartile (Q_3)		

Method: Calculating the quartiles

1. Order the data from smallest to largest or from largest to smallest.
2. Count how many data values there are in the data set.
3. Divide the number of data values by 4. The result is the number of data values per group.
4. Determine the data values corresponding to the first, second and third quartiles using the number of data values per quartile.

Worked Example 69: Quartiles

Question: What are the quartiles of $\{3{,}5{,}1{,}8{,}9{,}12{,}25{,}28{,}24{,}30{,}41{,}50\}$?
Answer
Step 1 : Order the data set from lowest to highest
$\{1, 3, 5, 8, 9, 12, 24, 25, 28, 30, 41, 50\}$
Step 2 : Count the number of data values in the data set
There are 12 values in the data set.
Step 3 : Divide the number of data values by 4 to find the number of data values per quartile.

$$12 \div 4 = 3$$

Step 4 : Find the data values corresponding to the quartiles.

1	3	5	‖	8	9	12	‖	24	25	28	‖	30	41	50
			Q_1				Q_2				Q_3			

The first quartile occurs between data position 3 and 4 and is the average of data values 5 and 8. The second quartile occurs between positions 6 and 7 and is the average of data values 12 and 24. The third quartile occurs between positions 9 and 10 and is the average of data values 28 and 30.
Step 5 : Answer
The first quartile = 6,5. (Q_1)
The second quartile = 18. (Q_2)
The third quartile = 29. (Q_3)

Inter-quartile Range

Definition: Inter-quartile Range
The inter quartile range is a measure which provides information about the spread of a data set, and is calculated by subtracting the first quartile from the third quartile, giving the range of the middle half of the data set, trimming off the lowest and highest quarters, i.e. $Q_3 - Q_1$.

The semi-interquartile range is half the interquartile range, i.e. $\frac{Q_3 - Q_1}{2}$

Worked Example 70: Medians, Quartiles and the Interquartile Range

Question: A class of 12 students writes a test and the results are as follows: 20, 39, 40, 43, 43, 46, 53, 58, 63, 70, 75, 91. Find the range, quartiles and the Interquartile Range.
Answer
Step 1 :

20	39	40	‖	43	43	46	‖	53	58	63	‖	70	75	91
			Q_1				M				Q_3			

Step 2 : The Range
The range = 91 - 20 = 71. This tells us that the marks are quite widely spread.
Step 3 : The median lies between the 6th and 7th mark
i.e. $M = \frac{46+53}{2} = \frac{99}{2} = 49{,}5$
Step 4 : The lower quartile lies between the 3rd and 4th mark
i.e. $Q_1 = \frac{40+43}{2} = \frac{83}{2} = 41{,}5$
Step 5 : The upper quartile lies between the 9th and 10th mark
i.e. $Q_3 = \frac{63+70}{2} = \frac{133}{2} = 66{,}5$
Step 6 : Analysing the quartiles
The quartiles are 41,5, 49,5 and 66,5. These quartiles tell us that 25% of the marks are less than 41,5; 50% of the marks are less than 49,5 and 75% of the marks are less than 66,5. They also tell us that 50% of the marks lie between 41,5 and 66,5.
Step 7 : The Interquartile Range
The Interquartile Range = 66,5 - 41,5 = 25. This tells us that the width of the middle 50% of the data values is 25.
Step 8 : The Semi-interquatile Range
The Semi-interquartile Range = $\frac{25}{2}$ = 12,5

Percentiles

Definition: Percentiles
Percentiles are the 99 data values that divide a data set into 100 groups.

The calculation of percentiles is identical to the calculation of quartiles, except the aim is to divide the data values into 100 groups instead of the 4 groups required by quartiles.

Method: Calculating the percentiles

1. Order the data from smallest to largest or from largest to smallest.
2. Count how many data values there are in the data set.

3. Divide the number of data values by 100. The result is the number of data values per group.
4. Determine the data values corresponding to the first, second and third quartiles using the number of data values per quartile.

15.6.3 Exercises - Summarising Data

1. Three sets of data are given:

   A **Data set 1:** 9 12 12 14 16 22 24

   B **Data set 2:** 7 7 8 11 13 15 16 16

   C **Data set 3:** 11 15 16 17 19 19 22 24 27

   For each one find:

   i. the range
   ii. the lower quartile
   iii. the interquartile range
   iv. the semi-interquartile range
   v. the median
   vi. the upper quartile

2. There is 1 sweet in one jar, and 3 in the second jar. The mean number of sweets in the first two jars is 2.

   A If the mean number in the first three jars is 3, how many are there in the third jar?

   B If the mean number in the first four jars is 4, how many are there in the fourth jar?

   C If the mean number in the first n jars is n, how many are there in the n jar?

3. Find a set of five ages for which the mean age is 5, the modal age is 2 and the median age is 3 years.
4. Four friends each have some marbles. They work out that the mean number of marbles they have is 10. One of them leaves. She has 4 marbles. How many marbles do the remaining friends have together?

Worked Example 71: Mean, Median and Mode for Grouped Data

Question:
Consider the following grouped data and calculate the mean, the modal group and the median group.

Mass (kg)	Frequency
41 - 45	7
46 - 50	10
51 - 55	15
56 - 60	12
61 - 65	6
	Total = 50

Answer
Step 1 : Calculating the mean
To calculate the mean we need to add up all the masses and divide by 50. We do not know actual masses, so we approximate by choosing the midpoint of each group. We then multiply those midpoint numbers by the frequency. Then we add these numbers together to find the approximate total of the masses. This is show in the table below.

Mass (kg)	Midpoint	Frequency	Midpt × Freq
41 - 45	(41+45)/2 = 43	7	43 × 7 = 301
46 - 50	48	10	480
51 - 55	53	15	795
56 - 60	58	12	696
61 - 65	63	6	378
		Total = 50	Total = 2650

Step 2 : Answer

The mean $= \frac{2650}{50} = 53$.

The modal group is the group 51 - 53 because it has the highest frequency.

The median group is the group 51 - 53, since the 25th and 26th terms are contained within this group.

?

Exercise: More mean, modal and median group exercises.

In each data set given, find the mean, the modal group and the median group.

1. Times recorded when learners played a game.

Time in seconds	Frequency
36 - 45	5
46 - 55	11
56 - 65	15
66 - 75	26
76 - 85	19
86 - 95	13
96 - 105	6

2. The following data were collected from a group of learners.

Mass in kilograms	Frequency
41 - 45	3
46 - 50	5
51 - 55	8
56 - 60	12
61 - 65	14
66 - 70	9
71 - 75	7
76 - 80	2

15.7 Misuse of Statistics

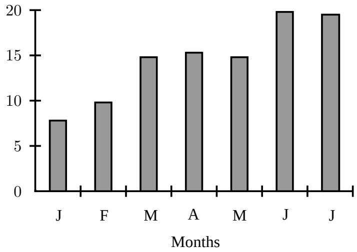
In many cases groups can gain an advantage by misleading people with the misuse of statistics.

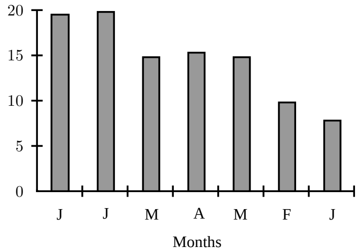
Common techniques used include:

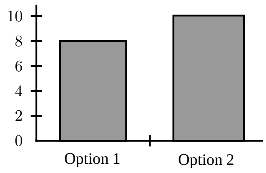
- Three dimensional graphs.
- Axes that do not start at zero.

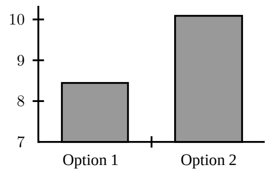
- Axes without scales.
- Graphic images that convey a negative or positive mood.
- Assumption that a correlation shows a necessary causality.
- Using statistics that are not truly representative of the entire population.
- Using misconceptions of mathematical concepts

For example, the following pairs of graphs show identical information but look very different. Explain why.



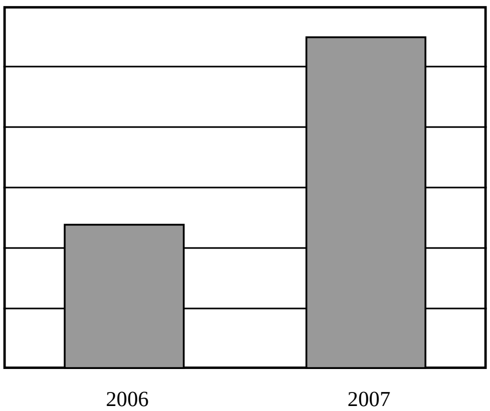






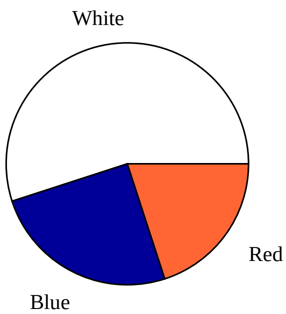
15.7.1 Exercises - Misuse of Statistics

1. A company has tried to give a visual representation of the increase in their earnings from one year to the next. Does the graph below convince you? Critically analyse the graph.



2. In a study conducted on a busy highway, data was collected about drivers breaking the speed limit and the colour of the car they were driving. The data were collected during a 20 minute time interval during the middle of the day, and are presented in a table and pie chart below.

Colour of car	Frequency of drivers speeding
White	22
Blue	10
Red	8



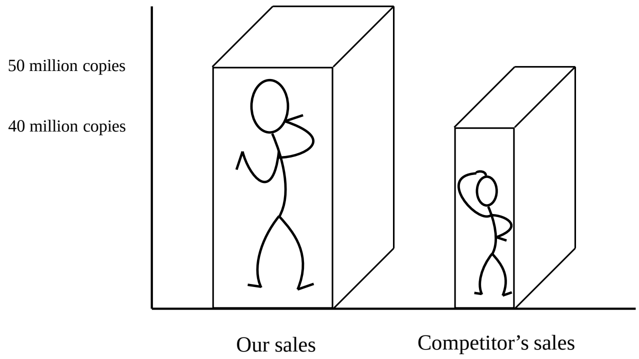
Conclusions made by a novice based on the data are summarised as follows:

"People driving white cars are more likely to break the speed limit."

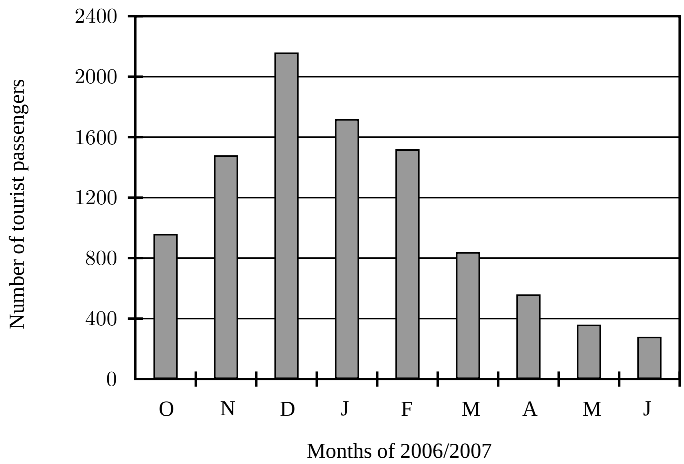
"Drivers in blue and red cars are more likely to stick to the speed limit."

Do you agree with these conclusions? Explain.

3. A record label produces a graphic, showing their advantage in sales over their competitors. Identify at least three devices they have used to influence and mislead the readers impression.



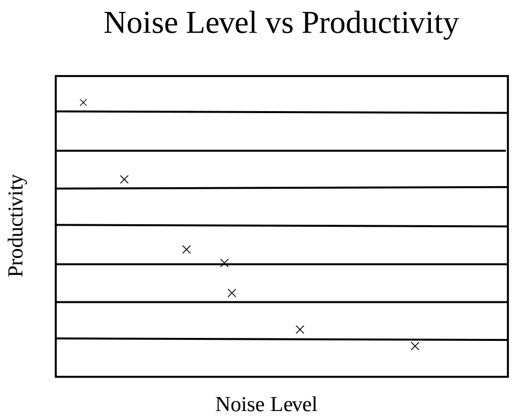
4. In an effort to discredit their competition, a tour bus company prints the graph shown below. Their claim is that the competitor is losing business. Can you think of a better explanation?



5. The caption from a newspaper article quoted below, demonstrates a misuse of statistical concepts. Explain.

"More than 40% of learners in South Africa are below average in mathematics."

6. To test a theory, 8 different offices were monitored for noise levels and productivity of the employees in the office. The results are graphed below.



The following statement was then made:
"If an office environment is noisy, this leads to poor productivity."

Explain the flaws in this thinking.

15.8 Summary of Definitions

mean The mean of a data set, x, denoted by $\bar{x}$, is the average of the data values, and is calculated as:

$$\bar{x} = \frac{\text{sum of values}}{\text{number of values}} \tag{15.2}$$

median The median is the centre data value in a data set that has been ordered from lowest to highest

mode The mode is the data value that occurs most often in a data set.

15.9 Exercises

1. "Using the median size as a reference, you would be able to fit four 1 cent coins and a car into a match box." Explain why this statement is true.
2. Calculate the mean, median, and mode of Data Set 3.
3. The tallest 7 trees in a park have heights in metres of 41, 60, 47, 42, 44, 42, and 47. Find the median of their heights.
4. The students in Bjorn's class have the following ages: 5, 9, 1, 3, 4, 6, 6, 6, 7, 3. Find the mode of their ages.
5. The masses (in kg, correct to the nearest 0,1 kg) of thirty people were measured as follows:

45,1	57,9	67,9	57,4	50,7	61,1	63,9	67,5	69,7	71,7
68,0	63,2	58,7	56,9	78,5	59,7	54,4	66,4	51,6	47,7
70,9	54,8	59,1	60,3	60,1	52,6	74,9	72,1	49,5	49,8

A Copy the frequency table below, and complete it.

Mass (in kg)	Tally	Number of people
$45{,}0 \leq m < 50{,}0$		
$50{,}0 \leq m < 55{,}0$		
$55{,}0 \leq m < 60{,}0$		
$60{,}0 \leq m < 65{,}0$		
$65{,}0 \leq m < 70{,}0$		
$70{,}0 \leq m < 75{,}0$		
$75{,}0 \leq m < 80{,}0$		

B Draw a frequency polygon for this information.

C What can you conclude from looking at the graph?

6. An engineering company has designed two different types of engines for motorbikes. The two different motorbikes are tested for the time it takes (in seconds) for them to accelerate from 0 km/h to 60 km/h.

	Test 1	Test 2	Test 3	Test 4	Test 5	Test 6	Test 7	Test 8	Test 9	Test 10	Average
Bike 1	1.55	1.00	0.92	0.80	1.49	0.71	1.06	0.68	0.87	1.09	
Bike 2	0.9	1.0	1.1	1.0	1.0	0.9	0.9	1.0	0.9	1.1	

A What kind of average should be used for this information?

B Calculate the average you chose in the previous question for each motorbike.

C Which motorbike would you choose based on this information? Take note of accuracy of the numbers from each set of tests.

D How far will a motorbike travelling at 60 km/h travel in 1 second?

7. The heights of 40 learners are given below.

154	140	145	159	150	132	149	150	138	152
141	132	169	173	139	161	163	156	157	171
168	166	151	152	132	142	170	162	146	152
142	150	161	138	170	131	145	146	147	160

A Set up a frequency table using 6 intervals.

B Calculate the approximate mean.

C Determine the mode.

D Determin the modal class.

E How many learners are taller than your approximate average in (b)?

8. In a traffic survey, a random sample of 50 motorists were asked the distance they drove to work daily. This information is shown in the table below.

Distance in km	1-5	6-10	11-15	16-20	21-25	26-30	31-35	36-40	41-45
Frequency	4	5	9	10	7	8	3	2	2

A Find the approximate mean.

B Find the modal class.

C What percentage of samples drove

i. less than 16 km?

ii. more than 30 km?

iii. between 16 km and 30 km daily?

9. A company wanted to evaluate the training programme in its factory. They gave the same task to trained and untrained employees and timed each one in seconds.

Trained	121	137	131	135	130
	128	130	126	132	127
	129	120	118	125	134
Untrained	135	142	126	148	145
	156	152	153	149	145
	144	134	139	140	142

A Draw a back-to-back stem and leaf diagram to show the two sets of data.

B Find the medians and quartiles for both sets of data.

C Find the Interquartile Range for both sets of data.

D Comment on the results.

10. A small firm employs nine people. The annual salaries of the employers are:

R600 000	R250 000	R200 000
R120 000	R100 000	R100 000
R100 000	R90 000	R80 000

A Find the mean of these salaries.

B Find the mode.

C Find the median.

D Of these three figures, which would you use for negotiating salary increases if you were a trade union official? Why?

11. The marks for a particular class test are listed here:

67	58	91	67	58	82	71	51	60	84
31	67	96	64	78	71	87	78	89	38
69	62	60	73	60	87	71	49		

A Complete the frequency table using the given class intervals.

Class	Tally	Frequency	Mid-point	Freq $\times$ Midpt
30-39		34,5		
40-49		44,5		
50-59				
60-69				
70-79				
80-89				
90-99				
		Sum =		Sum =

Chapter 16

Probability - Grade 10

16.1 Introduction

Very little in mathematics is truly self-contained. Many branches of mathematics touch and interact with one another, and the fields of probability and statistics are no different. A basic understanding of probability is vital in grasping basic statistics, and probability is largely abstract without statistics to determine the "real world" probabilities.

Probability theory is concerned with predicting statistical outcomes. A simple example of a statistical outcome is observing a head or tail when tossing a coin. Another simple example of a statistical outcome is obtaining the numbers 1, 2, 3, 4, 5, or 6 when rolling a die. (We say one die, many dice.)

For a fair coin, heads should occur for $\frac{1}{2}$ of the number of tosses and for a fair die, each number should occur for $\frac{1}{6}$ of the number of rolls. Therefore, the probability of observing a head on one toss of a fair coin is $\frac{1}{2}$ and that for obtaining a four on one roll of a fair die is $\frac{1}{6}$.

In earlier grades, the idea has been introduced that different situations have different probabilities of occurring and that for many situations there are a finite number of different possible outcomes. In general, events from daily life can be classified as either:

- certain that they will happen; or
- certain that they will not happen; or
- uncertain.

This chapter builds on earlier work and describes how to calculate the probability associated with different situations, and describes how probability is used to assign a number describing the level of chance or the odds associated with aspects of life. The meanings of statements like: 'The HIV test is 85% reliable.' will also be explained.

16.2 Random Experiments

The term *random experiment* or *statistical experiment* is used to describe any repeatable experiment or situation, with individual experiments having one of a set of outcomes or results. A set of outcomes is known as an *event*. For example, the act of tossing a coin or rolling a die can be considered to be simple random experiments. With the results of either a heads or tails or one of $\{1,2,3,4,5,6\}$ being the outcomes. These experiments are repeatable and yield different outcomes each time.

16.2.1 Sample Space of a Random Experiment

The set of all possible outcomes in a random experiment plays an important role in probability theory and is known as the *sample space*. The letter S is used to indicate the sample space.

Using the terminology of set theory, the elements of S are then the outcomes of the random experiment. For example, when tossing a coin the sample space S is made up of $\{heads, tails\}$.

Worked Example 72: Sample Space

Question: What outcomes make up the sample space S when rolling a die.
Answer
Step 3 : Determine all the possible outcomes
The possible outcomes when rolling a die are: 1, 2, 3, 4, 5 and 6.
Step 4 : Define the sample space, S
For rolling a die, the sample space is $S = \{1,2,3,4,5,6\}$.

A set of outcomes is referred to as an *event*. For example, when rolling a die the outcomes that are an even number (i.e. $\{2,4,6\}$) would be referred to as an event. It is clear that outcomes and events are subsets of the sample space, S.

A Venn diagram can be used to show the relationship between the outcomes of a random experiment, the sample space and events associated with the outcomes. The Venn diagram in Figure 16.1 shows the difference between the universal set, a sample space and events and outcomes as subsets of the sample space.

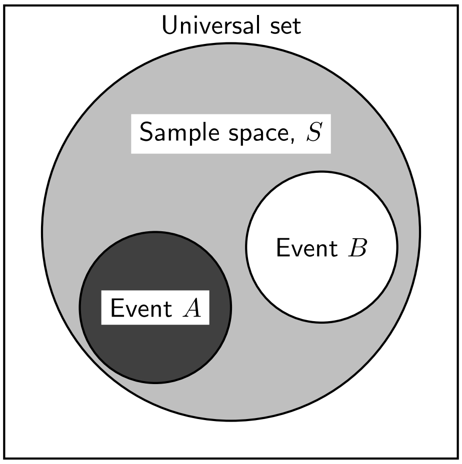


Figure 16.1: Diagram to show difference between the universal set and the sample space. The sample space is made up of all possible outcomes of a statistical experiment and an event is a subset of the sample space.

Venn diagrams can also be used to indicate the union and intersection between events in a sample space (Figure 16.2).

Worked Example 73: Random Experiments

Question: In a box there are pieces of paper with the numbers from 1 to 9 written on them.
$S = \{1; 2; 3; 4; 5; 6; 7; 8; 9\}$
Answer
Step 1 : Consider the events:

- Drawing a prime number; $P = \{2, 3, 5, 7\}$

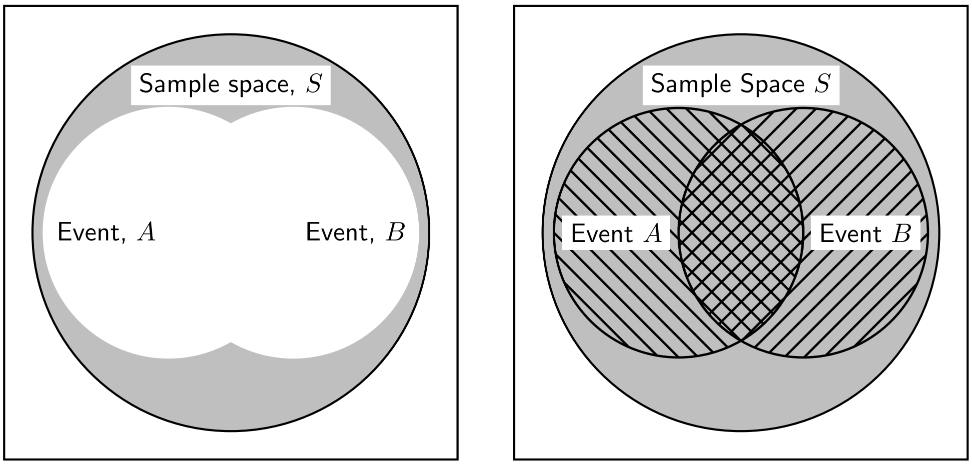
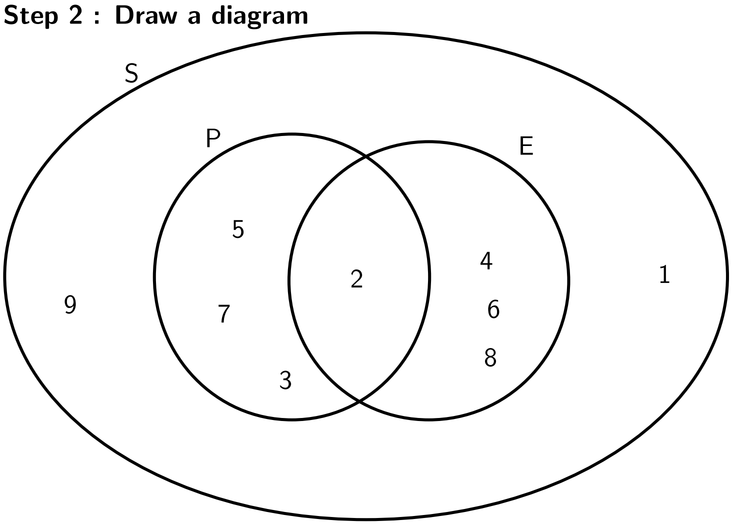


Figure 16.2: Venn diagram to show (left) union of two events, A and B, in the sample space S and (right) intersection of two events A and B, in the sample space S. The crosshatched region indicates the intersection.

- Drawing an even number; $E = \{2, 4, 6, 8\}$

Step 2 : Draw a diagram



Step 3 : Find the union

The *union* of P and E is the set of all elements in P or in E (or in both). P **or** $E = 2, 3, 4, 5, 6, 7, 8$. P **or** E is also written $P \cup E$.

Step 4 : Find the intersection

The *intersection* of P and E is the set of all elements in both P and E. P **and** $E = 2$. P **and** E is also written as $P \cap E$.

Step 5 : Find the number in each set

We use $n(S)$ to refer to the number of elements in a set S, $n(X)$ for the number of elements in X, etc.

$$\begin{aligned} \therefore\ n(S) &= 9 \\ n(P) &= 4 \\ n(E) &= 4 \\ n(P \cup E) &= 7 \\ n(P \cap E) &= 2 \end{aligned}$$

?

Exercise: Random Experiments

1. $S = \{$whole numbers from 1 to 16$\}$, $X = \{$even numbers from 1 to 16$\}$ and $Y = \{$prime numbers from 1 to 16$\}$
   A Draw a Venn diagram S, X and Y.
   B Write down $n(S)$, $n(X)$, $n(Y)$, $n(X \cup Y)$, $n(X \cap Y)$.
2. There are 79 Grade 10 learners at school. All of these take either Maths, Geography or History. The number who take Geography is 41, those who take History is 36, and 30 take Maths. The number who take Maths and History is 16; the number who take Geography and History is 6, and there are 8 who take Maths only and 16 who take only History.
   A Draw a Venn diagram to illustrate all this information.
   B How many learners take Maths and Geography but not History?
   C How many learners take Geography only?
   D How many learners take all three subjects?
3. Pieces of paper labelled with the numbers 1 to 12 are placed in a box and the box is shaken. One piece of paper is taken out and then replaced.
   A What is the sample space, S?
   B Write down the set A, representing the event of taking a piece of paper labelled with a factor 12.
   C Write down the set B, representing the event of taking a piece of paper labelled with a prime number.
   D Represent A, B and S by means of a Venn diagram.
   E Write down
      i. $n(S)$
      ii. $n(A)$
      iii. $n(B)$
      iv. $n(A \cap B)$
      v. $n(A \cup B)$
   F Is $n(A \cup B) = n(A) + n(B) - n(A \cap B)$?

16.3 Probability Models

The word probability relates to uncertain events or knowledge, being closely related in meaning to likely, risky, hazardous, and doubtful. Chance, odds, and bet are other words expressing similar ideas.

Probability is connected with uncertainty. In any statistical experiment, the outcomes that occur may be known, but exactly which one might not be known. Mathematically, probability theory formulates incomplete knowledge pertaining to the likelihood of an occurrence. For example, a meteorologist might say there is a 60% chance that it will rain tomorrow. This means that in 6 of every 10 times when the world is in the current state, it will rain.

A probability is a real number between 0 and 1. In everyday speech, probabilities are usually given as a percentage between 0% and 100%. A probability of 100% means that an event is certain, whereas a probability of 0% is often taken to mean the event is impossible. However, there is a distinction between logically impossible and occurring with zero probability; for example, in selecting a number uniformly between 0 and 1, the probability of selecting 1/2 is 0, but it is not logically impossible. Further, it is certain that whichever number is selected will have had a probability of 0 of being selected.

Another way of referring to probabilities is odds. The odds of an event is defined as the ratio of the probability that the event occurs to the probability that it does not occur. For example, the odds of a coin landing on a given side are $\frac{0.5}{0.5} = 1$, usually written "1 to 1" or "1:1". This means that on average, the coin will land on that side as many times as it will land on the other side.

16.3.1 Classical Theory of Probability

1. Equally likely outcomes are outcomes which have an equal chance of happening. For example when a fair coin is tossed, each outcome in the sample space $S = heads, tails$ is equally likely to occur.

2. When all the outcomes are **equally likely** (in any activity), you can calculate the probability of an event happening by using the following definition:
   P(E)=number of favourable outcomes/total number of possible outcomes
   P(E)=n(E)/n(S)
   For example, when you throw a fair dice the possible outcomes are $S = \{1,2,3,4,5,6\}$ i.e the total number of possible outcomes n(S)=6.

Event 1: get a 4
The only possible outcome is a 4, i.e E=4 i.e number of favourable outcomes: n(E)=1.
Probability of getting a 4 = P(4)=n(E)/n(S)=1/6.
Event 2: get a number greater than 3
Favourable outcomes: $E = \{4,5,6\}$
Number of favourable outcomes: n(E)=3
Probability of getting a number more than 3 = P(more than 3) = n(E)/n(S)=3/6=1/2

Worked Example 74: Classical Probability

Question: Various probabilities relating to a deck of cards
Answer
A standard deck of cards (without jokers) has 52 cards. If we randomly draw a card from the deck, we can think of each card as a possible outcome. Therefore, there are 52 total outcomes. We can now look at various events and calculate their probabilities:

1. Out of the 52 cards, there are 13 clubs. Therefore, if the event of interest is drawing a club, there are 13 favourable outcomes, and the probability of this event is $\frac{13}{52} = \frac{1}{4}$.
2. There are 4 kings (one of each suit). The probability of drawing a king is $\frac{4}{52} = \frac{1}{13}$.
3. What is the probability of drawing a king OR a club? This example is slightly more complicated. We cannot simply add together the number of number of outcomes for each event separately ($4 + 13 = 17$) as this inadvertently counts one of the outcomes twice (the king of clubs). The correct answer is $\frac{16}{52}$.

Exercise: Probability Models

1. A bag contains 6 red, 3 blue, 2 green and 1 white balls. A ball is picked at random. What is the probablity that it is:
   A red
   B blue or white
   C not green
   D not green or red?
2. A card is selected randomly from a pack of 52. What is the probability that it is:

A the 2 of hearts
B a red card
C a picture card
D an ace
E a number less than 4?

3. Even numbers from 2 -100 are written on cards. What is the probability of selecting a multiple of 5, if a card is drawn at random?

16.4 Relative Frequency vs. Probability

There are two approaches to determining the probability associated with any particular event of a random experiment:

1. determining the total number of possible outcomes and calculating the probability of each outcome using the definition of probability
2. performing the experiment and calculating the relative frequency of each outcome

Relative frequency is defined as the number of times an event happens in a statistical experiment divided by the number of trials conducted.

It takes a very large number of trials before the relative frequency of obtaining a head on a toss of a coin approaches the probability of obtaining a head on a toss of a coin. For example, the data in Table 16.1 represent the outcomes of repeating 100 trials of a statistical experiment 100 times, i.e. tossing a coin 100 times.

H	T	T	H	H	T	H	H	H	H
H	H	H	H	T	H	H	T	T	T
T	T	H	T	T	H	T	H	T	H
H	H	T	T	H	T	T	H	T	T
T	H	H	H	T	T	H	T	T	H
H	T	T	T	T	H	T	T	H	H
T	T	H	T	T	H	T	T	H	T
H	T	T	H	T	T	T	T	H	T
T	H	T	T	H	H	H	T	H	T
T	T	T	H	H	T	T	T	H	T

Table 16.1: Results of 100 tosses of a fair coin. H means that the coin landed heads-up and T means that the coin landed tails-up.

The following two worked examples show that the relative frequency of an event is not necessarily equal to the probability of the same event. Relative frequency should therefore be seen as an approximation to probability.

Worked Example 75: Relative Frequency and Probability

Question: Determine the relative frequencies associated with each outcome of the statistical experiment detailed in Table 16.1.
Answer
Step 1 : Identify the different outcomes
There are two unique outcomes: H and T.
Step 2 : Count how many times each outcome occurs.

Outcome	Frequency
H	44
T	56

Step 3 : Determine the total number of trials.
The statistical experiment of tossing the coin was performed 100 times. Therefore, there were 100 trials, in total.
Step 4 : Calculate the relative frequency of each outcome

$$\begin{aligned}\text{Probability of H} &= \frac{\text{frequency of outcome}}{\text{number of trials}}\\ &= \frac{44}{100}\\ &= 0.44\end{aligned}$$

$$\begin{aligned}\text{Relative Frequency of T} &= \frac{\text{frequency of outcome}}{\text{number of trials}}\\ &= \frac{56}{100}\\ &= 0.56\end{aligned}$$

The relative frequency of the coin landing heads-up is 0.44 and the relative frequency of the coin landing tails-up is 0.56.

Worked Example 76: Probability

Question: Determine the probability associated with an evenly weighted coin landing on either of its faces.
Answer
Step 1 : Identify the different outcomes
There are two unique outcomes: H and T.
Step 2 : Determine the total number of outcomes.
There are two possible outcomes.
Step 3 : Calculate the probability of each outcome

$$\begin{aligned}\text{Relative Frequency of H} &= \frac{\text{number of favourable outcomes}}{\text{total number of outcomes}}\\ &= \frac{1}{2}\\ &= 0.5\end{aligned}$$

$$\begin{aligned}\text{Relative Frequency of T} &= \frac{\text{number of favourable outcomes}}{\text{total number of outcomes}}\\ &= \frac{1}{2}\\ &= 0.5\end{aligned}$$

The probability of an evenly weighted coin landing on either face is 0.5.

16.5 Project Idea

Perform an experiment to show that as the number of trials increases, the relative frequency approaches the probability of a coin toss. Perform 10, 20, 50, 100, 200 trials of tossing a coin.

16.6 Probability Identities

The following results apply to probabilities, for the sample space S and two events A and B, within S.

$$P(S) = 1 \tag{16.1}$$

$$P(A \cap B) = P(A) \times P(B) \tag{16.2}$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) \tag{16.3}$$

Worked Example 77: Probabilty identitys

Question: What is the probability of selecting a black or red card from a pack of 52 cards

Answer

P(S)=n(E)/n(S)=52/52=1. because all cards are black or red!

Worked Example 78: Probabilty identitys

Question: What is the probability of drawing a club or an ace with one single pick from a pack of 52 cards

Answer

Step 1 : Identify the identity which describes the situation

$$P(\text{club} \cup \text{ace}) = P(\text{club}) + P(\text{ace}) - P(\text{club} \cap \text{ace})$$

Step 2 : Calculate the answer

$$\begin{aligned} &= \frac{1}{4} + \frac{1}{13} - \left(\frac{1}{4} \times \frac{1}{13}\right) \\ &= \frac{1}{4} + \frac{1}{13} - \frac{1}{52} \\ &= \frac{16}{52} \\ &= \frac{4}{13} \end{aligned}$$

Notice how we have used $P(C \cup A) = P(C) + P(A) - P(C \cap A)$.

?

Exercise: Probability Identities
Answer the following questions

1. Rory is target shooting. His probability of hitting the target is 0.7. He fires five shots. What is the probability that:
   A All five shots miss the center?
   B At least 3 shots hit the center?
2. An archer is shooting arrows at a bullseye. The probability that an arrow hits the bullseye is 0.4. If she fires three arrows, what is the probability that:
   A All the arrows hit the bullseye,
   B only one of the arrows hit the bullseye?
3. A dice with the numbers 1,3,5,7,9,11 on it is rolled. Also a fair coin is tossed.
   A Draw a sample space diagram to show all outcomes.
   B What is the probability that:
      i. A tail is tossed and a 9 rolled?
      ii. A head is tossed and a 3 rolled?
4. Four children take a test. The probability of each one passing is as follows. Sarah: 0.8, Kosma: 0.5, Heather: 0.6, Wendy: 0.9. What is the probability that:
   A all four pass?
   B all four fail?
   C at least one passes?
5. With a single pick from a pack of 52 cards what is the probability that the card will be an ace or a black card?

16.7 Mutually Exclusive Events

Mutually exclusive events are events, which cannot be true at the same time.

Examples of mutually exclusive events are:

1. A die landing on an even number or landing on an odd number.
2. A student passing or failing an exam
3. A tossed coin landing on heads or landing on tails

This means that if we examine the elements of the sets that make up A and B there will be no elements in common. Therefore, $A \cap B = \emptyset$ (where $\emptyset$ refers to the empty set). Since, $P(A \cap B) = 0$, equation 16.3 becomes:

$$P(A \cup B) = P(A) + P(B)$$

for mutually exclusive events.

?

Exercise: Mutually Exclusive Events
Answer the following questions

---

1. A box contains coloured blocks. The number of each colour is given in the following table.

Colour	Purple	Orange	White	Pink
Number of blocks	24	32	41	19

   A block is selected randomly. What is the probability that the block will be:

   A purple
   B purple or white
   C pink and orange
   D not orange?

2. A small private school has a class with children of various ages. The table gies the number of pupils of each age in the class.

3 years female	3 years male	4 years female	4 years male	5 years female	5 years male
6	2	5	7	4	6

   If a pupil is selceted at random what is the probability that the pupil will be:

   A a female
   B a 4 year old male
   C aged 3 or 4
   D aged 3 and 4
   E not 5
   F either 3 or female?

3. Fiona has 85 labeled discs, which are numbered from 1 to 85. If a disc is selected at random what is the probability that the disc number:

   A ends with 5
   B can be multiplied by 3
   C can be multiplied by 6
   D is number 65
   E is not a multiple of 5
   F is a multiple of 4 or 3
   G is a multiple of 2 and 6
   H is number 1?

16.8 Complementary Events

The probability of complementary events refers to the probability associated with events not occurring. For example, if $P(A) = 0.25$, then the probability of A not occurring is the probability associated with all other events in S occurring less the probability of A occurring. This means that

$$P(A') = 1 - P(A)$$

where A' refers to 'not A' In other words, the probability of 'not A' is equal to one minus the probability of A.

Worked Example 79: Probability

Question: If you throw two dice, one red and one blue, what is the probability that at least one of them will be a six?

Answer

Step 1 : Work out probability of event 1

To solve that kind of question, work out the probability that there will be no six.

Step 2 : Work out probability of event 2

The probability that the red dice will not be a six is 5/6, and that the blue one will not be a six is also 5/6.

Step 3 : Probability of neither

So the probability that neither will be a six is $5/6 \times 5/6 = 25/36$.

Step 4 : Probability of one

So the probability that at least one will be a six is $1 - 25/36 = 11/36$.

Worked Example 80: Probability

Question: A bag contains three red balls, five white balls, two green balls and four blue balls:

1. Calculate the probability that a red ball will be drawn from the bag.
2. Calculate the probability that a ball which is not red will be drawn

Answer

Step 1 : Find event 1

Let R be the event that a red ball is drawn:

- P(R)-n(R)/n(S)=3/14
- R and R' are complementary events

Step 2 : Find the probabilitys

∴ P(R') = 1 - P(R) = 1 -3/14 = 11/14

Step 3 : Alternate way to solve it

- Alternately P(R') = P(B) + P(W) + P(G)
- P(R') = 4/14 + 5/14 + 2/14 = 11/14

Extension: Interpretation of Probability Values

The probability of an event is generally represented as a real number between 0 and 1, inclusive. An impossible event has a probability of exactly 0, and a certain event has a probability of 1, but the converses are not always true: probability 0 events are not always impossible, nor probability 1 events certain. The rather subtle distinction between "certain" and "probability 1" is treated at greater length in the article on "almost surely".

Most probabilities that occur in practice are numbers between 0 and 1, indicating the event's position on the continuum between impossibility and certainty. The closer an event's probability is to 1, the more likely it is to occur.

For example, if two mutually exclusive events are assumed equally probable, such as a flipped or spun coin landing heads-up or tails-up, we can express the probability of each event as "1 in 2", or, equivalently, "50%" or "1/2".

Probabilities are equivalently expressed as odds, which is the ratio of the probability of one event to the probability of all other events. The odds of heads-up, for the tossed/spun coin, are (1/2)/(1 - 1/2), which is equal to 1/1. This is expressed as "1 to 1 odds" and often written "1:1".

Odds a:b for some event are equivalent to probability a/(a+b). For example, 1:1 odds are equivalent to probability 1/2, and 3:2 odds are equivalent to probability 3/5.

16.9 End of Chapter Exercises

1. A group of 45 children were asked if they eat Frosties and/or Strawberry Pops. 31 eat both and 6 eat only Frosties. What is the probability that a child chosen at random will eat only Strawberry Pops?
2. In a group of 42 pupils, all but 3 had a packet of chips or a Fanta or both. If 23 had a packet of chips and 7 of these also had a Fanta, what is the probability that one pupil chosen at random has:
   A Both chips and Fanta
   B has only Fanta?
3. Use a Venn diagram to work out the following probabilities from a die being rolled:
   A A multiple of 5 and an odd number
   B a number that is neither a multiple of 5 nor an odd number
   C a number which is not a multiple of 5, but is odd.
4. A packet has yellow and pink sweets. The probability of taking out a pink sweet is 7/12.
   A What is the probability of taking out a yellow sweet
   B If 44 if the sweets are yellow, how many sweets are pink?
5. In a car park with 300 cars, there are 190 Opals. What is the probability that the first car to leave the car park is:
   A an Opal
   B not an Opal
6. Tamara has 18 loose socks in a drawer. Eight of these are orange and two are pink. Calculate the probability that the first sock taken out at random is:
   A Orange
   B not orange
   C pink
   D not pink
   E orange or pink
   F not orange or pink
7. A plate contains 9 shortbread cookies, 4 ginger biscuits, 11 chocolate chip cookies and 18 Jambos. If a biscuit is selected at random, what is the probability that:
   A it is either a ginger biscuit of a Jambo?
   B it is NOT a shortbread cookie.
8. 280 tickets were sold at a raffle. Ingrid bought 15 tickets. What is the probability that Ingrid:
   A Wins the prize
   B Does not win the prize?

9. The children in a nursery school were classified by hair and eye colour. 44 had red hair and not brown eyes, 14 had brown eyes and red hair, 5 had brown eyes but not red hair and 40 did not have brown eyes or red hair.

   A How many children were in the school

   B What is the probility that a child chosen at random has:

   i. Brown eyes

   ii. Red hair

   C A child with brown eyes is chosen randomly. What is the probability that this child will have red hair

10. A jar has purple, blue and black sweets in it. The probability that a sweet, chosen at random, will be purple is 1/7 and the probability that it will be black is 3/5.

    A If I choose a sweet at random what is the probability that it will be:

    i. purple or blue

    ii. Black

    iii. purple

    B If there are 70 sweets in the jar how many purple ones are there?

    C 1/4 if the purple sweets in b) have streaks on them and rest do not. How many purple sweets have streaks?

11. For each of the following, draw a Venn diagram to represent the situation and find an example to illustrate the situation.

    A A sample space in which there are two events that are not mutually exclusive

    B A sample space in which there are two events that are complementary.

12. Use a Venn diagram to prove that the probability of either event A or B occuring is given by: (A and B are not exclusive)
    P(A or B) = P(A) + P(B) - P(A and B)

13. All the clubs are taken out of a pack of cards. The remaining cards are then shuffled and one card chosen. After being chosen, the card is replaced before the next card is chosen.

    A What is the sample space?

    B Find a set to represent the event, P, of drawing a picture card.

    C Find a set for the event, N, of drawing a numbered card.

    D Represent the above events in a Venn diagram

    E What description of the sets P and N is suitable? (Hint: Find any elements of P in N and N in P.)

14. Thuli has a bag containing five orange, three purple and seven pink blocks. The bag is shaken and a block is withdrawn. The colour of the block is noted and the block is replaced.

    A What is the sample space for this experiment?

    B What is the set describing the event of drawing a pink block, P?

    C Write down a set, O or B, to represent the event of drawing either a orange or a purple block.

    D Draw a Venn diagram to show the above information.

Part III

Grade 11

Chapter 17

Exponents - Grade 11

17.1 Introduction

In Grade 10 we studied exponential numbers and learnt that there were six laws that made working with exponential numbers easier. There is one law that we did not study in Grade 10. This will be described here.

17.2 Laws of Exponents

In Grade 10, we worked only with indices that were integers. What happens when the index is not an integer, but is a rational number? This leads us to the final law of exponents,

$$a^{\frac{m}{n}} = \sqrt[n]{a^m} \tag{17.1}$$

17.2.1 Exponential Law 7: $a^{\frac{m}{n}} = \sqrt[n]{a^m}$

We say that x is an nth root of b if $x^n = b$. For example, $(-1)^4 = 1$, so -1 is a 4th root of 1. Using law 6, we notice that

$$(a^{\frac{m}{n}})^n = a^{\frac{m}{n} \times n} = a^m \tag{17.2}$$

therefore $a^{\frac{m}{n}}$ must be an nth root of a^m. We can therefore say

$$a^{\frac{m}{n}} = \sqrt[n]{a^m} \tag{17.3}$$

where $\sqrt[n]{a^m}$ is the nth root of a^m (if it exists).

For example,

$$2^{\frac{2}{3}} = \sqrt[3]{2^2}$$

A number may not always have a real nth root. For example, if $n = 2$ and $a = -1$, then there is no real number such that $x^2 = -1$ because x^2 can never be a negative number.

Extension: Complex Numbers

There are numbers which can solve problems like $x^2 = -1$, but they are beyond the scope of this book. They are called *complex numbers*.

It is also possible for more than one nth root of a number to exist. For example, $(-2)^2 = 4$ and $2^2 = 4$, so both -2 and 2 are 2nd (square) roots of 4. Usually if there is more than one root, we choose the positive real solution and move on.

Worked Example 81: Rational Exponents

Question: Simplify without using a calculator:

$$\left(\frac{5}{4^{-1}-9^{-1}}\right)^{\frac{1}{2}}$$

Answer

Step 1 : Rewrite negative exponents as numbers with postive indices

$$= \left(\frac{5}{\frac{1}{4}-\frac{1}{9}}\right)^{\frac{1}{2}}$$

Step 2 : Simplify inside brackets

$$\begin{aligned} &= \left(\frac{5}{1}\div\frac{9-4}{36}\right)^{\frac{1}{2}} \\ &= \left(\frac{5}{1}\times\frac{36}{5}\right)^{\frac{1}{2}} \\ &= (6^2)^{\frac{1}{2}} \end{aligned}$$

Step 3 : Apply exponential law 6

$$= 6$$

Worked Example 82: More rational Exponents

Question: Simplify:

$$(16x^4)^{\frac{3}{4}}$$

Answer

Step 1 : Covert the number co-efficient to index-form with a prime base

$$= (2^4x^4)^{\frac{3}{4}}$$

Step 2 : Apply exponential laws

$$\begin{aligned} &= 2^{4\times\frac{3}{4}}.x^{4\times\frac{3}{4}} \\ &= 2^3.x^3 \\ &= 8x^3 \end{aligned}$$

Exercise: Applying laws
Use all the laws to:

1. Simplify:

   (a) $(x^0) + 5x^0 - (0{,}25)^{-0{,}5} + 8^{\frac{2}{3}}$
   (b) $s^{\frac{1}{2}} \div s^{\frac{1}{3}}$
   (c) $\frac{12m^{\frac{7}{9}}}{8m^{-\frac{11}{9}}}$
   (d) $(64m^6)^{\frac{2}{3}}$

2. Re-write the expression as a power of x:

$$x\sqrt{x\sqrt{x\sqrt{x\sqrt{x}}}}$$

17.3 Exponentials in the Real-World

In Chapter 8, you used exponentials to calculate different types of interest, for example on a savings account or on a loan and compound growth.

Worked Example 83: Exponentials in the Real world

Question: A type of bacteria has a very high exponential growth rate at 80% every hour. If there are 10 bacteria, determine how many there will be in 5 hours, in 1 day and in 1 week?
Answer
Step 1 : $Population = Initial\ population \times (1 + growth\ percentage)^{time\ period\ in\ hours}$
Therefore, in this case:
$Population = 10(1{,}8)^n$, where n = number of hours
Step 2 : In 5 hours
$Population = 10(1{,}8)^5 = 188$
Step 3 : In 1 day = 24 hours
$Population = 10(1{,}8)^{24} = 13\ 382\ 588$
Step 4 : in 1 week = 168 hours
$Population = 10(1{,}8)^{168} = 7{,}687 \times 10^{43}$
Note this answer is given in scientific notation as it is a very big number.

Worked Example 84: More Exponentials in the Real world

Question: A species of extremely rare, deep water fish has an extremely long lifespan and rarely have children. If there are a total 821 of this type of fish and their growth rate is 2% each month, how many will there be in half of a year? What will be the population be in 10 years and in 100 years ?
Answer
Step 1 : $Population = Initial\ population \times (1 + growth\ percentage)^{time\ period\ in\ months}$

Therefore, in this case:
$Population = 821(1{,}02)^n$, where $n =$ number of months
Step 2 : In half a year = 6 months
$Population = 821(1{,}02)^6 = 924$
Step 3 : In 10 years = 120 months
$Population = 821(1{,}02)^{120} = 8\ 838$
Step 4 : in 100 years = 1 200 months
$Population = 821(1{,}02)^{1\ 200} = 1{,}716 \times 10^{13}$
Note this answer is also given in scientific notation as it is a very big number.

17.4 End of chapter Exercises

1. Simplify as far as possible:

   A $8^{-\frac{2}{3}}$

   B $\sqrt{16} + 8^{-\frac{2}{3}}$

2. Simplify:

   (a) $(x^3)^{\frac{4}{3}}$ (b) $(s^2)^{\frac{1}{2}}$
   (c) $(m^5)^{\frac{5}{3}}$ (d) $(-m^2)^{\frac{4}{3}}$
   (e) $-(m^2)^{\frac{4}{3}}$ (f) $(3y^{\frac{4}{3}})^4$

3. Simplify as much as you can:
$$\frac{3a^{-2}b^{15}c^{-5}}{(a^{-4}b^3c)^{\frac{-5}{2}}}$$

4. Simplify as much as you can:
$$\left(9a^6b^4\right)^{\frac{1}{2}}$$

5. Simplify as much as you can:
$$\left(a^{\frac{3}{2}}b^{\frac{3}{4}}\right)^{16}$$

6. Simplify:
$$x^3\sqrt{x}$$

7. Simplify:
$$\sqrt[3]{x^4b^5}$$

8. Re-write the expression as a power of x:
$$\frac{x\sqrt{x\sqrt{x\sqrt{x\sqrt{x}}}}}{\sqrt[3]{x}}$$

Chapter 18

Surds - Grade 11

18.1 Surd Calculations

There are several laws that make working with surds easier. We will list them all and then explain where each rule comes from in detail.

$$\sqrt[n]{a}\,\sqrt[n]{b} = \sqrt[n]{ab} \tag{18.1}$$

$$\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}} \tag{18.2}$$

$$\sqrt[n]{a^m} = a^{\frac{m}{n}} \tag{18.3}$$

18.1.1 Surd Law 1: $\sqrt[n]{a}\,\sqrt[n]{b} = \sqrt[n]{ab}$

It is often useful to look at a surd in exponential notation as it allows us to use the exponential laws we learnt in section **??**. In exponential notation, $\sqrt[n]{a} = a^{\frac{1}{n}}$ and $\sqrt[n]{b} = b^{\frac{1}{n}}$. Then,

$$\begin{aligned} \sqrt[n]{a}\,\sqrt[n]{b} &= a^{\frac{1}{n}}b^{\frac{1}{n}} \\ &= (ab)^{\frac{1}{n}} \\ &= \sqrt[n]{ab} \end{aligned} \tag{18.4}$$

Some examples using this law:

1. $\sqrt[3]{16} \times \sqrt[3]{4} = \sqrt[3]{64} = 4$
2. $\sqrt{2} \times \sqrt{32} = \sqrt{64} = 8$
3. $\sqrt{a^2b^3} \times \sqrt{b^5c^4} = \sqrt{a^2b^8c^4} = b^4c^2$

18.1.2 Surd Law 2: $\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$

If we look at $\sqrt[n]{\frac{a}{b}}$ in exponential notation and applying the exponential laws then,

$$\begin{aligned} \sqrt[n]{\frac{a}{b}} &= \left(\frac{a}{b}\right)^{\frac{1}{n}} \\ &= \frac{a^{\frac{1}{n}}}{b^{\frac{1}{n}}} \\ &= \frac{\sqrt[n]{a}}{\sqrt[n]{b}} \end{aligned} \tag{18.5}$$

Some examples using this law:

1. $\sqrt{12} \div \sqrt{3} = \sqrt{4} = 2$
2. $\sqrt[3]{24} \div \sqrt[3]{3} = \sqrt[3]{8} = 2$
3. $\sqrt{a^2b^{13}} \div \sqrt{b^5} = \sqrt{a^2b^8} = ab^4$

18.1.3 Surd Law 3: $\sqrt[n]{a^m} = a^{\frac{m}{n}}$

If we look at $\sqrt[n]{a^m}$ in exponential notation and applying the exponential laws then,

$$\begin{aligned} \sqrt[n]{a^m} &= (a^m)^{\frac{1}{n}} \\ &= a^{\frac{m}{n}} \end{aligned} \tag{18.6}$$

For example,

$$\begin{aligned} \sqrt[6]{2^3} &= 2^{\frac{3}{6}} \\ &= 2^{\frac{1}{2}} \\ &= \sqrt{2} \end{aligned}$$

18.1.4 Like and Unlike Surds

Two surds $\sqrt[m]{a}$ and $\sqrt[n]{b}$ are called *like surds* if $m = n$, otherwise they are called *unlike surds*. For example $\sqrt{2}$ and $\sqrt{3}$ are like surds, however $\sqrt{2}$ and $\sqrt[3]{2}$ are unlike surds. An important thing to realise about the surd laws we have just learnt is that the surds in the laws are all like surds.

If we wish to use the surd laws on unlike surds, then we must first convert them into like surds. In order to do this we use the formula

$$\sqrt[n]{a^m} = \sqrt[bn]{a^{bm}} \tag{18.7}$$

to rewrite the unlike surds so that bn is the same for all the surds.

Worked Example 85: Like and Unlike Surds

Question: Simplify to like surds as far as possible, showing all steps: $\sqrt[3]{3} \times \sqrt[5]{5}$

Answer

Step 1 : Find the common root

$$= \sqrt[15]{3^5} \times \sqrt[15]{5^3}$$

Step 2 : Use surd law 1

$$\begin{aligned} &= \sqrt[15]{3^5.5^3} \\ &= \sqrt[15]{243 \times 125} \\ &= \sqrt[15]{30375} \end{aligned}$$

18.1.5 Simplest Surd form

In most cases, when working with surds, answers are given in simplest surd form. For example,

$$\begin{aligned}\sqrt{50} &= \sqrt{25\times 2}\\ &= \sqrt{25}\times\sqrt{2}\\ &= 5\sqrt{2}\end{aligned}$$

$5\sqrt{2}$ is the simplest surd form of $\sqrt{50}$.

Worked Example 86: Simplest surd form

Question: Rewrite $\sqrt{18}$ in the simplest surd form:
Answer
Step 1 : Break the number 18 into its lowest factors

$$\begin{aligned}\sqrt{18} &= \sqrt{2\times 9}\\ &= \sqrt{2\times 3\times 3}\\ &= \sqrt{2}\times\sqrt{3\times 3}\\ &= \sqrt{2}\times\sqrt{3^2}\\ &= 3\sqrt{2}\end{aligned}$$

Worked Example 87: Simplest surd form

Question: Simplify: $\sqrt{147}+\sqrt{108}$
Answer
Step 1 : Simplify each square root seperately

$$\begin{aligned}\sqrt{147}+\sqrt{108} &= \sqrt{49\times 3}+\sqrt{36\times 3}\\ &= \sqrt{7^2\times 3}+\sqrt{6^2\times 3}\end{aligned}$$

Step 2 : Take the values that have ^{2} under the surd to the outside of the square root sign

$$= 7\sqrt{3}+6\sqrt{3}$$

Step 3 : The exact same surds can be treated as "like terms" and may be added

$$= 13\sqrt{3}$$

18.1.6 Rationalising Denominators

It is useful to work with fractions, which have rational denominators instead of surd denominators. It is possible to rewrite any fraction, which has a surd in the denominator as a fraction which has a rational denominator. We will now see how this can be achieved.

Any expression of the form $\sqrt{a}+\sqrt{b}$ (where a and b are rational) can be changed into a rational number by multiplying by $\sqrt{a}-\sqrt{b}$ (similarly $\sqrt{a}-\sqrt{b}$ can be rationalised by multiplying by $\sqrt{a}+\sqrt{b}$). This is because

$$(\sqrt{a}+\sqrt{b})(\sqrt{a}-\sqrt{b}) = a-b \tag{18.8}$$

which is rational (since a and b are rational).

If we have a fraction which has a denominator which looks like $\sqrt{a}+\sqrt{b}$, then we can simply multiply both top and bottom by $\sqrt{a}-\sqrt{b}$ achieving a rational denominator.

$$\begin{aligned} \frac{c}{\sqrt{a}+\sqrt{b}} &= \frac{\sqrt{a}-\sqrt{b}}{\sqrt{a}-\sqrt{b}} \times \frac{c}{\sqrt{a}+\sqrt{b}} \\ &= \frac{c\sqrt{a}-c\sqrt{b}}{a-b} \end{aligned} \tag{18.9}$$

or similarly

$$\begin{aligned} \frac{c}{\sqrt{a}-\sqrt{b}} &= \frac{\sqrt{a}+\sqrt{b}}{\sqrt{a}+\sqrt{b}} \times \frac{c}{\sqrt{a}-\sqrt{b}} \\ &= \frac{c\sqrt{a}+c\sqrt{b}}{a-b} \end{aligned} \tag{18.10}$$

Worked Example 88: Rationalising the Denominator

Question: Rationalise the denominator of: $\frac{5x-16}{\sqrt{x}}$

Answer

Step 1 : Get rid of the square root sign in the denominator

To get rid of $\sqrt{x}$ in the denominator, you can multiply it out by another $\sqrt{x}$. This "rationalises" the surd in the denominator. Note that $\frac{\sqrt{x}}{\sqrt{x}} = 1$, thus the equation becomes rationalised by multiplying by 1 and thus still says the same thing.

$$\frac{5x-16}{\sqrt{x}} \times \frac{\sqrt{x}}{\sqrt{x}}$$

Step 2 : There is no longer a surd in the denominator.

The surd is expressed in the numerator which is the prefered way to write expressions. (That's why denominators get rationalised.)

$$\frac{5x\sqrt{x}-16\sqrt{x}}{x}$$
$$\frac{(\sqrt{x})(5x-16)}{x}$$

Worked Example 89: Rationalising the Denominator

Question: Rationalise the following: $\frac{5x-16}{\sqrt{y}-10}$

Answer

Step 1 : Rationalise this denominator by using a clever form of "1"

$$\frac{5x-16}{\sqrt{y}-10}\times\frac{\sqrt{y}+10}{\sqrt{y}+10}$$

Step 2 : Multiply out the numerators and denominators

$$\frac{5x\sqrt{y}-16\sqrt{y}+50x-160}{y-100}$$

Step 3 : There is no next step in this case.

All the terms in the numerator are different and cannot be simplified and the denominator does not have any surds in it anymore.

Worked Example 90: Rationalise the denominator

Question: Simplify the following: $\frac{y-25}{\sqrt{y}+5}$

Answer

Step 1 : Multiply this equations by a clever form of "1" that would rationalise this denominator

$$\frac{y-25}{\sqrt{y}+5}\times\frac{\sqrt{y}-5}{\sqrt{y}-5}$$

Step 2 : Multiply out the numerators and denominators

$$\begin{aligned}\frac{y\sqrt{y}-25\sqrt{y}-5y+125}{y-25} &= \frac{\sqrt{y}(y-25)-5(y-25)}{(y-25)}\\ &= \frac{(y-25)(\sqrt{y}-25)}{(y-25)}\\ &= \sqrt{y}-25\end{aligned}$$

18.2 End of Chapter Exercises

1. Expand:

$$(\sqrt{x}-\sqrt{2})(\sqrt{x}+\sqrt{2})$$

2. Rationalise the denominator:

$$\frac{10}{\sqrt{x}-\frac{1}{x}}$$

3. Write as a single fraction:

$$\frac{3}{2\sqrt{x}}+\sqrt{x}$$

4. Write in simplest surd form:

(a) $\sqrt{72}$ (b) $\sqrt{45}+\sqrt{80}$
(c) $\frac{\sqrt{48}}{\sqrt{12}}$ (d) $\frac{\sqrt{18}\div\sqrt{72}}{\sqrt{8}}$
(e) $\frac{4}{(\sqrt{8}\div\sqrt{2})}$ (f) $\frac{16}{(\sqrt{20}\div\sqrt{12})}$

5. Expand and simplify:
$$(2+\sqrt{2})^2$$

6. Expand and simplify:
$$(2+\sqrt{2})(1+\sqrt{8})$$

7. Expand and simplify:
$$(1+\sqrt{3})(1+\sqrt{8}+\sqrt{3})$$

8. Rationalise the denominator:
$$\frac{y-4}{\sqrt{y}-2}$$

9. Rationalise the denominator:
$$\frac{2x-20}{\sqrt{y}-\sqrt{10}}$$

10. Proof(without the use of a calculator) that:
$$\sqrt{\frac{8}{3}}+5\sqrt{\frac{5}{3}}-\sqrt{\frac{1}{6}}=\frac{13}{2}\sqrt{\frac{2}{3}}$$

11. Simplify, without use of a calculator:
$$\frac{\sqrt{98}-\sqrt{8}}{\sqrt{50}}$$

12. Simplify, without use of a calculator:
$$\sqrt{5}(\sqrt{45}+2\sqrt{80})$$

13. Write the following with a rational denominator:
$$\frac{\sqrt{5}+2}{\sqrt{5}}$$

14. Simplify:
$$\sqrt{98x^6}+\sqrt{128x^6}$$

15. Evaluate without using a calculator: $\left(2-\frac{\sqrt{7}}{2}\right)^{\frac{1}{2}} . \left(2+\frac{\sqrt{7}}{2}\right)^{\frac{1}{2}}$

16. The use a calculator is not permissible in this question. Simplify completely by showing all your steps: $3^{-\frac{1}{2}}\left[\sqrt{12}+\sqrt[3]{(3\sqrt{3})}\right]$

17. Fill in the blank surd-form numberwhich will make the following equation a true statement: $-3\sqrt{6}\times -2\sqrt{24}=-\sqrt{18}\times$ ....

Chapter 19

Error Margins - Grade 11

We have seen that numbers are either rational or irrational and we have see how to round-off numbers. However, in a calculation that has many steps, it is best to leave the rounding off right until the end.

For example, if you were asked to write

$$3\sqrt{3}+\sqrt{12}$$

as a decimal number correct to two decimal places, there are two ways of doing this as described in Table 19.1.

Table 19.1: Two methods of writing $3\sqrt{3}+\sqrt{12}$ as a decimal number.

☺ **Method 1**	☹ **Method 2**
$3\sqrt{3}+\sqrt{12} = 3\sqrt{3}+\sqrt{4\cdot 3}$	$3\sqrt{3}+\sqrt{12} = 3\times 1{,}73+3{,}46$
$= 3\sqrt{3}+2\sqrt{3}$	$= 5{,}19+3{,}46$
$= 5\sqrt{3}$	$= 8{,}65$
$= 5\times 1{,}732050808\ldots$	
$= 8{,}660254038\ldots$	
$= 8{,}66$	

In the example we see that Method 1 gives 8,66 as an answer while Method 2 gives 8,65 as an answer. The answer of Method 1 is more accurate because the expression was simplified as much as possible before the answer was rounded-off.

In general, it is best to simplify any expression as much as possible, before using your calculator to work out the answer in decimal notation.

Important: Simplification and Accuracy

It is best to simplify all expressions as much as possible before rounding-off answers. This maintains the accuracy of your answer.

Worked Example 91: Simplification and Accuracy

Question: Calculate $\sqrt[3]{54}+\sqrt[3]{16}$. Write the answer to three decimal places.

Answer

Step 1 : Simplify the expression

$$\begin{aligned}\sqrt[3]{54}+\sqrt[3]{16} &= \sqrt[3]{27\cdot 2}+\sqrt[3]{8\cdot 2}\\ &= \sqrt[3]{27}\cdot\sqrt[3]{2}+\sqrt[3]{8}\cdot\sqrt[3]{2}\\ &= 3\sqrt[3]{2}+2\sqrt[3]{2}\\ &= 5\sqrt[3]{2}\\ &= 5\times 1{,}25992105\ldots\end{aligned}$$

Step 2 : Convert any irrational numbers to decimal numbers

$$\begin{aligned}5\sqrt[3]{2} &= 5\times 1{,}25992105\ldots\\ &= 6{,}299605249\ldots\\ &= 6{,}300\end{aligned}$$

Step 3 : Write the final answer to the required number of decimal places.

$$6{,}299605249\ldots = 6{,}300 \quad \text{to three decimal places}$$

$\therefore \quad \sqrt[3]{54}+\sqrt[3]{16} = 6{,}300$ to three decimal places.

Worked Example 92: Simplification and Accuracy 2

Question: Calculate $\sqrt{x+1}+\frac{1}{3}\sqrt{(2x+2)-(x+1)}$ if $x=3{,}6$. Write the answer to two decimal places.

Answer

Step 1 : Simplify the expression

$$\begin{aligned}\sqrt{x+1}+\frac{1}{3}\sqrt{(2x+2)-(x+1)} &= \sqrt{x+1}+\frac{1}{3}\sqrt{2x+2-x-1}\\ &= \sqrt{x+1}+\frac{1}{3}\sqrt{x+1}\\ &= \frac{4}{3}\sqrt{x+1}\end{aligned}$$

Step 2 : Substitute the value of x into the simplified expression

$$\begin{aligned}\frac{4}{3}\sqrt{x+1} &= \frac{4}{3}\sqrt{3{,}6+1}\\ &= \frac{4}{3}\sqrt{4{,}6}\\ &= 2{,}144761059\ldots\times 4\div 3\\ &= 2{,}859681412\ldots\end{aligned}$$

Step 3 : Write the final answer to the required number of decimal places.

$$2{,}859681412\ldots = 2{,}86 \quad \text{To two decimal places}$$

$\therefore \sqrt{x+1}+\frac{1}{3}\sqrt{(2x+2)-(x+1)} = 2{,}86$ (to two decimal places) if $x=3{,}6$.

Extension: Significant Figures

In a number, each non-zero digit is a significant figure. Zeroes are only counted if they are between two non-zero digits or are at the end of the decimal part. For example, the number 2000 has 1 significant figure (the 2), but $2000{,}0$ has 5 significant figures. Estimating a number works by removing significant figures from your number (starting from the right) until you have the desired number of significant figures, rounding as you go. For example $6{,}827$ has 4 significant figures, but if you wish to write it to 3 significant figures it would mean removing the 7 and rounding up, so it would be $6{,}83$. It is important to know when to estimate a number and when not to. It is usually good practise to only estimate numbers when it is absolutely necessary, and to instead use symbols to represent certain irrational numbers (such as π); approximating them only at the very end of a calculation. If it is necessary to approximate a number in the middle of a calculation, then it is often good enough to approximate to a few decimal places.

Chapter 20

Quadratic Sequences - Grade 11

20.1 Introduction

In Grade 10, you learned about arithmetic sequences, where the difference between consecutive terms was constant. In this chapter we learn about quadratic sequences.

20.2 What is a quadratic sequence?

Definition: Quadratic Sequence
A quadratic sequence is a sequence of numbers in which the second differences between each consecutive term differ by the same amount, called a common second difference.

For example,

$$1;\ 2;\ 4;\ 7;\ 11;\ \ldots \tag{20.1}$$

is a quadratic sequence. Let us see why ...

If we take the difference between consecutive terms, then:

$$\begin{aligned} a_2 - a_1 &= 2-1 &&= 1 \\ a_3 - a_2 &= 4-2 &&= 2 \\ a_4 - a_3 &= 7-4 &&= 3 \\ a_5 - a_4 &= 11-7 &&= 4 \end{aligned}$$

We then work out the *second differences*, which is simply obtained by taking the difference between the consecutive differences $\{1;\ 2;\ 3;\ 4;\ \ldots\}$ obtained above:

$$\begin{aligned} 2-1 &= 1 \\ 3-2 &= 1 \\ 4-3 &= 1 \\ &\ldots \end{aligned}$$

We then see that the second differences are equal to 1. Thus, (20.1) is a *quadratic sequence*.

Note that the differences between consecutive terms (that is, the first differences) of a quadratic sequence form a sequence where there is a constant difference between consecutive terms. In the above example, the sequence of $\{1;\ 2;\ 3;\ 4;\ \ldots\}$, which is formed by taking the differences between consecutive terms of (20.1), has a linear formula of the kind $ax + b$.

Exercise: Quadratic Sequences

The following are also examples of quadratic sequences:

$$3;\ 6;\ 10;\ 15;\ 21;\ \ldots$$
$$4;\ 9;\ 16;\ 25;\ 36;\ \ldots$$
$$7;\ 17;\ 31;\ 49;\ 71;\ \ldots$$
$$2;\ 10;\ 26;\ 50;\ 82;\ \ldots$$
$$31;\ 30;\ 27;\ 22;\ 15;\ \ldots$$

Can you calculate the common second difference for each of the above examples?

Worked Example 93: Quadratic sequence

Question: Write down the next two terms and find a formula for the n^{th} term of the sequence $5, 12, 23, 38$,..., ...,

Answer

Step 1 : Find the first differences between the terms.

i.e. $7, 11, 15$

Step 2 : Find the 2nd differences between the terms.

the second difference is 4.

So continuing the sequence, the differences between each term will be:

$15 + 4 = 19$

$19 + 4 = 23$

Step 3 : Finding the next two terms.

So the next two terms in the sequence willl be:

$38 + 19 = 57$

$57 + 23 = 80$

So the sequence will be: $5, 12, 23, 38, 57, 80$

Step 4 : We now need to find the formula for this sequence.

We know that the first difference is 4. The start of the formula will therefore be $2n^2$.

Step 5 : We now need to work out the next part of the sequence.

If $n = 1$, you have to get the value of term 1, which is 5 in this particular sequence. The difference between $2n^2 = 2$ and original number (5) is 3, which leads to $n + 2$. Check is it works for the second term, i.e. when $n = 2$.

Then $2n^2 = 8$. The difference between term 2(12) and 8 is 4, which is can be written as $n + 2$.

So for the sequence $5, 12, 23, 38$,... the formula for the n^th term is $2n^2 + n + 2$.

General Case

If the sequence is quadratic, the n^th term should be $T_n = an^2 + bn + c$

TERMS	$a+b+c$		$4a+2b+c$		$9a+3b+c$	
1^{st} difference		$3a+b$		$5a+b$		$7a+b$
2^{nd} difference			$2a$		$2a$	

In each case, the 2nd difference is $2a$. This fact can be used to find a, then b then c.

Worked Example 94: Quadratic Sequence

Question: The following sequence is quadratic: $8, 22, 42, 68, ...$ Find the rule.
Answer
Step 1 : Assume that the rule is $an^2 + bn + c$

TERMS	8		22		42		68
1st difference		14		20		26	
2nd difference			6		6		6

Step 2 : Determine values for a, b and c

$$\begin{aligned} Then & \quad 2a = 6 \ which \ gives \ a = 3 \\ And & \quad 3a + b = 14 \rightarrow 9 + b = 14 \rightarrow b = 5 \\ And & \quad a + b + c = 8 \rightarrow 3 + 5 + c = 8 \rightarrow c = 0 \end{aligned}$$

Step 3 : Find the rule
The rule is therefore: $n^th \ term \ = 3n^2 + 5n$
Step 4 : Check answer
For

$$\begin{aligned} n &= 1, 1st \ term = 3(1)^2 + 5(1) = 8 \\ n &= 2, 2nd \ term = 3(2)^2 + 5(2) = 22 \\ n &= 3, 3rd \ term = 3(3)^2 + 5(3) = 42 \end{aligned}$$

Extension: Derivation of the n^{th}-term of a Quadratic Sequence
Let the n^{th}-term for a quadratic sequence be given by

$$a_n = A \cdot n^2 + B \cdot n + C \qquad (20.2)$$

where A, B and C are some constants to be determined.

$$\begin{aligned} a_n &= A \cdot n^2 + B \cdot n + C & (20.3) \\ a_1 &= A(1)^2 + B(1) + C = A + B + C & (20.4) \\ a_2 &= A(2)^2 + B(2) + C = 4A + 2B + C & (20.5) \\ a_3 &= A(3)^2 + B(3) + C = 9A + 3B + C & (20.6) \end{aligned}$$

$$\begin{aligned} \text{Let } d &\equiv a_2 - a_1 \\ \therefore d &= 3A + B \end{aligned}$$

$$\Rightarrow B = d - 3A \qquad (20.7)$$

The common second difference is obtained from

$$\begin{aligned} D &= (a_3 - a_2) - (a_2 - a_1) \\ &= (5A + B) - (3A + B) \\ &= 2A \end{aligned}$$

$$\Rightarrow A = \frac{D}{2} \qquad (20.8)$$

Therefore, from (20.7),

$$B = d - \frac{3}{2} \cdot D \qquad (20.9)$$

From (20.4),

$$C = a_1 - (A + B) = a_1 - \frac{D}{2} - d + \frac{3}{2} \cdot D$$

$$\therefore C = a_1 + D - d \tag{20.10}$$

Finally, the general equation for the n^{th}-term of a quadratic sequence is given by

$$a_n = \frac{D}{2} \cdot n^2 + (d - \frac{3}{2}D) \cdot n + (a_1 - d + D) \tag{20.11}$$

Worked Example 95: Using a set of equations

Question: Study the following pattern: 1; 7; 19; 37; 61; ...

1. What is the next number in the sequence ?
2. Use variables to write an algebraic statement to generalise the pattern.
3. What will the 100th term of the sequence be ?

Answer

Step 1 : The next number in the sequence

The numbers go up in multiples of 6
$1 + 6 = 7$, then $7 + 12 = 19$
Therefore $61 + 6 \times 6 = 97$
The next number in the sequence is 97.

Step 2 : Generalising the pattern

TERMS	1		7		19		37		61
1^{st} difference		6		12		18		24	
2^{nd} difference			6		6		6		6

The pattern will yield a quadratic equation since second difference is constant
Therefore $an^2 + bn + c = y$
For the first term: $n = 1$, then $y = 1$
For the second term: $n = 2$, then $y = 7$
For the third term: $n = 3$, then $y = 19$
etc....

Step 3 : Setting up sets of equations

$$a + b + c = 1 \tag{20.12}$$
$$4a + 2b + c = 7 \tag{20.13}$$
$$9a + 3b + c = 19 \tag{20.14}$$

Step 4 : Solve the sets of equations

$$eqn(2) - eqn(1): \quad 3a + b = 6 \tag{20.15}$$
$$eqn(3) - eqn(2): \quad 5a + b = 12 \tag{20.16}$$
$$eqn(5) - eqn(4): \quad 2a = 6 \tag{20.17}$$
$$Therefore \quad a = 3, b = -3 \; and \; c = 1 \tag{20.18}$$

Step 5 : Final answer

The general formula for the pattern is $3n^2 - 3n + 1$

Step 6 : Term 100

Substitude n with 100:
$3(100)^2 - 3(100) + 1 = 29\ 701$
The value for term 100 is 29 701.

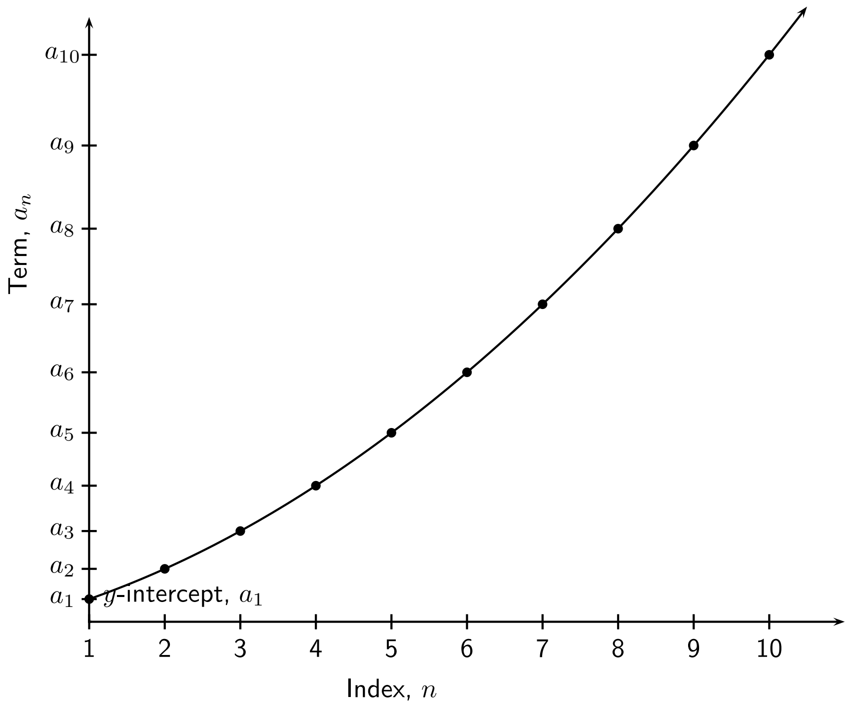
Extension: Plotting a graph of terms of a quadratic sequence

Plotting a_n vs. n for a quadratic sequence yields a parabolic graph.

Given the quadratic sequence,

$$3;\ 6;\ 10;\ 15;\ 21;\ \ldots$$

If we plot each of the terms vs. the corresponding index, we obtain a graph of a parabola.



20.3 End of chapter Exercises

1. Find the first 5 terms of the quadratic sequence defined by:

$$a_n = n^2 + 2n + 1$$

2. Determine which of the following sequences is a quadratic sequence by calculating the common second difference:

   A $6, 9, 14, 21, 30, \ldots$

   B $1, 7, 17, 31, 49, \ldots$

   C $8, 17, 32, 53, 80, \ldots$

   D $9, 26, 51, 84, 125, \ldots$

   E $2, 20, 50, 92, 146, \ldots$

   F $5, 19, 41, 71, 109, \ldots$

   G $2, 6, 10, 14, 18, \ldots$

   H $3, 9, 15, 21, 27, \ldots$

   I $10, 24, 44, 70, 102, \ldots$

   J $1, 2.5, 5, 8.5, 13, \ldots$

   K $2.5, 6, 10.5, 16, 22.5, \ldots$

L $0.5, 9, 20.5, 35, 52.5, \ldots$

3. Given $a_n = 2n^2$, find for which value of n, $a_n = 242$?
4. Given $a_n = (n-4)^2$, find for which value of n, $a_n = 36$?
5. Given $a_n = n^2 + 4$, find for which value of n, $a_n = 85$?
6. Given $a_n = 3n^2$, find a_{11}?
7. Given $a_n = 7n^2 + 4n$, find a_9?
8. Given $a_n = 4n^2 + 3n - 1$, find a_5?
9. Given $a_n = 1{,}5n^2$, find a_{10}?
10. For each of the quadratic sequences, find the common second difference, the formula for the general term and then use the formula to find a_{100}.

    A $4{,}7{,}12{,}19{,}28, \ldots$
    B $2{,}8{,}18{,}32{,}50, \ldots$
    C $7{,}13{,}23{,}37{,}55, \ldots$
    D $5{,}14{,}29{,}50{,}77, \ldots$
    E $7{,}22{,}47{,}82{,}127, \ldots$
    F $3{,}10{,}21{,}36{,}55, \ldots$
    G $3{,}7{,}13{,}21{,}31, \ldots$
    H $1{,}8{,}27{,}64{,}125, \ldots$
    I $6{,}13{,}32{,}69{,}130, \ldots$
    J $2{,}9{,}17{,}27{,}39, \ldots$

Chapter 21

Finance - Grade 11

21.1 Introduction

In Grade 10, the ideas of simple and compound interest was introduced. In this chapter we will be extending those ideas, so it is a good idea to go back to Chapter 8 and revise what you learnt in Grade 10. If you master the techniques in this chapter, you will understand about depreciation and will learn how to determine which bank is offering the better interest rate.

21.2 Depreciation

It is said that when you drive a new car out of the dealership, it loses 20% of its value, because it is now "second-hand". And from there on the value keeps falling, or *depreciating*. Second hand cars are cheaper than new cars, and the older the car, usually the cheaper it is. If you buy a second hand (or should we say *pre-owned*!) car from a dealership, they will base the price on something called *book value*.

The book value of the car is the value of the car taking into account the loss in value due to wear, age and use. We call this loss in value *depreciation*, and in this section we will look at two ways of how this is calculated. Just like interest rates, the two methods of calculating depreciation are *simple* and *compound* methods.

The terminology used for simple depreciation is **straight-line depreciation** and for compound depreciation is **reducing-balance depreciation**. In the straight-line method the value of the asset is reduced by the same constant amount each year. In the compound depreciation method the value of the asset is reduced by the same percentage each year. This means that the value of an asset does not decrease by a constant amount each year, but the decrease is most in the first year, then by a smaller amount in the second year and by even a smaller amount in the third year, and so on.

Extension: Depreciation

You may be wondering why we need to calculate depreciation. Determining the value of assets (as in the example of the second hand cars) is one reason, but there is also a more financial reason for calculating depreciation - tax! Companies can take depreciation into account as an expense, and thereby reduce their taxable income. A lower taxable income means that the company will pay less income tax to the Revenue Service.

21.3 Simple Depreciation (it really is simple!)

Let us go back to the second hand cars. One way of calculating a depreciation amount would be to assume that the car has a limited useful life. Simple depreciation assumes that the value of the car decreases by an equal amount each year. For example, let us say the limited useful life of a car is 5 years, and the cost of the car today is R60 000. What we are saying is that after 5 years you will have to buy a new car, which means that the old one will be valueless at that point in time. Therefore, the amount of depreciation is calculated:

$$\frac{\text{R60 000}}{\text{5 years}} = \text{R12 000} \quad \text{per year.}$$

The value of the car is then:

End of Year 1	R60 000 - 1×(R12 000)	= R48 000
End of Year 2	R60 000 - 2×(R12 000)	= R36 000
End of Year 3	R60 000 - 3×(R12 000)	= R24 000
End of Year 4	R60 000 - 4×(R12 000)	= R12 000
End of Year 5	R60 000 - 5×(R12 000)	= R0

This looks similar to the formula for simple interest:

$$\text{Total Interest after } n \text{ years} = n \times (P \times i)$$

where i is the annual percentage interest rate and P is the principal amount.

If we replace the word *interest* with the word *depreciation* and the word *principal* with the words *initial value* we can use the same formula:

$$\text{Total depreciation after } n \text{ years} = n \times (P \times i)$$

Then the book value of the asset after n years is:

$$\begin{aligned} \text{Initial Value - Total depreciation after } n \text{ years} &= P - n \times (P \times i) \\ &= P(1 - n \times i) \end{aligned}$$

For example, the book value of the car after two years can be simply calculated as follows:

$$\begin{aligned} \text{Book Value after 2 years} &= P(1 - n \times i) \\ &= \text{R60 000}(1 - 2 \times 20\%) \\ &= \text{R60 000}(1 - 0{,}4) \\ &= \text{R60 000}(0{,}6) \\ &= \text{R36 000} \end{aligned}$$

as expected.

Note that the difference between the simple interest calculations and the simple depreciation calculations is that while the interest adds value to the principal amount, the depreciation amount reduces value!

Worked Example 96: Simple Depreciation method

Question: A car is worth $R240\ 000$ now. If it depreciates at a rate of 15% p.a. on a staight-line depreciation, what is it worth in 5 years' time ?

Answer

Step 1 : Determine what has been provided and what is required

$$\begin{aligned} P &= R240\ 000 \\ i &= 0{,}15 \\ n &= 5 \\ A &\ is\ required \end{aligned}$$

Step 2 : Determine how to approach the problem

$$A = 240\ 000(1 - 0{,}15 \times 5)$$

Step 3 : Solve the problem

$$\begin{aligned} A &= 240\ 000(1 - 0{,}75) \\ &= 240\ 000 \times 0{,}25 \\ &= 60\ 000 \end{aligned}$$

Step 4 : Write the final answer
In 5 years' time the car is worth $R60\ 000$

Worked Example 97: Simple Depreciation

Question: A small business buys a photocopier for R 12 000. For the tax return the owner depreciates this asset over 3 years using a straight-line depreciation method. What amount will he fill in on his tax form after 1 year, after 2 years and then after 3 years ?
Answer
Step 1 : Understanding the question
The owner of the business wants the photocopier to depreciates to R0 after 3 years. Thus, the value of the photocopier will go down by $12\ 000 \div 3 = R4\ 000$ per year.
Step 2 : Value of the photocopier after 1 year
$12\ 000 - 4\ 000 = R8\ 000$
Step 3 : Value of the machine after 2 years
$8\ 000 - 4\ 000 = R4\ 000$
Step 4 : Write the final answer
$4\ 000 - 4\ 000 = 0$
After 3 years the photocopier is worth nothing

Extension: Salvage Value
Looking at the same example of our car with an initial value of R60 000, what if we suppose that we think we would be able to sell the car at the end of the 5 year period for R10 000? We call this amount the "Salvage Value"

We are still assuming simple depreciation over a useful life of 5 years, but now instead of depreciating the full value of the asset, we will take into account the salvage value, and will only apply the depreciation to the value of the asset that we expect not to recoup, i.e. R60 000 - R10 000 = R50 000.

The annual depreciation amount is then calculated as (R60 000 - R10 000) / 5 = R10 000

In general, the for simple (straight line) depreciation:

$$\text{Annual Depreciation} = \frac{\text{Initial Value - Salvage Value}}{\text{Useful Life}}$$

Exercise: Simple Depreciation

1. A business buys a truck for R560 000. Over a period of 10 years the value of the truck depreciates to R0 (using the straight-line method). What is the value of the truck after 8 years ?
2. Shrek wants to buy his grandpa's donkey for R800. His grandpa is quite pleased with the offer, seeing that it only depreciated at a rate of 3% per year using the straight-line method. Grandpa bought the donkey 5 years ago. What did grandpa pay for the donkey then ?
3. Seven years ago, Rocco's drum kit cost him R 12 500. It has now been valued at R2 300. What rate of simple depreciation does this represent ?
4. Fiona buys a DsTV satellite dish for R3 000. Due to weathering, its value depreciates simply at 15% per annum. After how long will the satellite dish be worth nothing ?

21.4 Compound Depreciation

The second method of calculating depreciation is to assume that the value of the asset decreases at a certain annual rate, but that the initial value of the asset this year, is the book value of the asset at the end of last year.

For example, if our second hand car has a limited useful life of 5 years and it has an initial value of R60 000, then the interest rate of depreciation is 20% (100%/5 years). After 1 year, the car is worth:

$$\begin{aligned}
\text{Book Value after first year} &= P(1 - n \times i) \\
&= \text{R60 000}(1 - 1 \times 20\%) \\
&= \text{R60 000}(1 - 0{,}2) \\
&= \text{R60 000}(0{,}8) \\
&= \text{R48 000}
\end{aligned}$$

At the beginning of the second year, the car is now worth R48 000, so after two years, the car is worth:

$$\begin{aligned}
\text{Book Value after second year} &= P(1 - n \times i) \\
&= \text{R48 000}(1 - 1 \times 20\%) \\
&= \text{R48 000}(1 - 0{,}2) \\
&= \text{R48 000}(0{,}8) \\
&= \text{R38 400}
\end{aligned}$$

We can tabulate these values.

End of first year	R60 000$(1 - 1 \times 20\%)$=R60 000$(1 - 1 \times 20\%)^1$	= R48 000,00
End of second year	R48 000$(1 - 1 \times 20\%)$=R60 000$(1 - 1 \times 20\%)^2$	= R38 400,00
End of third year	R38 400$(1 - 1 \times 20\%)$=R60 000$(1 - 1 \times 20\%)^3$	= R30 720,00
End of fourth year	R30 720$(1 - 1 \times 20\%)$=R60 000$(1 - 1 \times 20\%)^4$	= R24 576,00
End of fifth year	R24 576$(1 - 1 \times 20\%)$=R60 000$(1 - 1 \times 20\%)^5$	= R19 608,80

We can now write a general formula for the book value of an asset if the depreciation is compounded.

$$\text{Initial Value - Total depreciation after } n \text{ years} = P(1 - i)^n \qquad (21.1)$$

For example, the book value of the car after two years can be simply calculated as follows:

$$\begin{aligned}
\text{Book Value after 2 years} &= P(1-i)^n \\
&= \text{R60 000}(1-20\%)^2 \\
&= \text{R60 000}(1-0{,}2)^2 \\
&= \text{R60 000}(0{,}8)^2 \\
&= \text{R38 400}
\end{aligned}$$

as expected.

Note that the difference between the compound interest calculations and the compound depreciation calculations is that while the interest adds value to the principal amount, the depreciation amount reduces value!

Worked Example 98: Compound Depreciation

Question: The Flamingo population of the Bergriver mouth is depreciating on a reducing balance at a rate of 12% p.a. If there is now 3 200 flamingos in the wetlands of the Bergriver mouth, how many will there be in 5 years' time ? Answer to three significant numbers.

Answer

Step 1 : Determine what has been provided and what is required

$$\begin{aligned}
P &= R3\ 200 \\
i &= 0{,}12 \\
n &= 5 \\
A & \ is \ required
\end{aligned}$$

Step 2 : Determine how to approach the problem

$$A = 3\ 200(1-0{,}12)^5$$

Step 3 : Solve the problem

$$\begin{aligned}
A &= 3\ 200(0{,}88)^5 \\
&= 3\ 200 \times 0{,}527731916 \\
&= 1688{,}742134
\end{aligned}$$

Step 4 : Write the final answer

There would be approximately 1 690 flamingos in 5 years' time.

Worked Example 99: Compound Depreciation

Question: Farmer Brown buys a tractor for R250 000 and depreciates it by 20% per year using the compound depreciation method. What is the depreciated value of the tractor after 5 years ?

Answer

Step 1 : Determine what has been provided and what is required

$$
\begin{aligned}
P &= R250\ 000 \\
i &= 0{,}2 \\
n &= 5 \\
A &\ is\ required
\end{aligned}
$$

Step 2 : Determine how to approach the problem

$$A = 250\ 000(1 - 0{,}2)^5$$

Step 3 : Solve the problem

$$
\begin{aligned}
A &= 250\ 000(0{,}8)^5 \\
&= 250\ 000 \times 0{,}32768 \\
&= 81\ 920
\end{aligned}
$$

Step 4 : Write the final answer
Depreciated value after 5 years is R 81 920

?

Exercise: Compound Depreciation

1. On January 1, 2008 the value of my Kia Sorento is R320 000. Each year after that, the cars value will decrease 20% of the previous years value. What is the value of the car on January 1, 2012.
2. The population of Bonduel decreases at a rate of 9,5% per annum as people migrate to the cities. Calculate the decrease in population over a period of 5 years if the initial population was 2 178 000.
3. A 20 kg watermelon consists of 98% water. If it is left outside in the sun it loses 3% of its water each day. How much does in weigh after a month of 31 days ?
4. A computer depreciates at $x\%$ per annum using the reducing-balance method. Four years ago the value of the computer was R10 000 and is now worth R4 520. Calculate the value of x correct to two decimal places.

21.5 Present Values or Future Values of an Investment or Loan

21.5.1 Now or Later

When we studied simple and compound interest we looked at having a sum of money now, and calculating what it will be worth in the future. Whether the money was borrowed or invested, the calculations examined what the total money would be at some future date. We call these *future values*.

It is also possible, however, to look at a sum of money in the future, and work out what it is worth now. This is called a *present value*.

For example, if R1 000 is deposited into a bank account now, the future value is what that amount will accrue to by some specified future date. However, if R1 000 is needed at some future time, then the present value can be found by working backwards - in other words, how much must be invested to ensure the money grows to R1 000 at that future date?

The equation we have been using so far in compound interest, which relates the open balance (P), the closing balance (A), the interest rate (i as a rate per annum) and the term (n in years) is:

$$A = P \cdot (1 + i)^n \tag{21.2}$$

Using simple algebra, we can solve for P instead of A, and come up with:

$$P = A \cdot (1 + i)^{-n} \tag{21.3}$$

This can also be written as follows, but the first approach is usually preferred.

$$P = A/(1 + i)^n \tag{21.4}$$

Now think about what is happening here. In Equation 21.2, we start off with a sum of money and we let it grow for n years. In Equation 21.3 we have a sum of money which we know in n years time, and we "unwind" the interest - in other words we take off interest for n years, until we see what it is worth right now.

We can test this as follows. If I have R1 000 now and I invest it at 10% for 5 years, I will have:

$$\begin{aligned} A &= P \cdot (1 + i)^n \\ &= \text{R1 000}(1 + 10\%)^5 \\ &= \text{R1 610,51} \end{aligned}$$

at the end. BUT, if I know I have to have R1 610,51 in 5 years time, I need to invest:

$$\begin{aligned} P &= A \cdot (1 + i)^{-n} \\ &= \text{R1 610,51}(1 + 10\%)^{-5} \\ &= \text{R1 000} \end{aligned}$$

We end up with R1 000 which - if you think about it for a moment - is what we started off with. Do you see that?

Of course we could apply the same techniques to calculate a present value amount under simple interest rate assumptions - we just need to solve for the opening balance using the equations for simple interest.

$$A = P(1 + i \times n) \tag{21.5}$$

Solving for P gives:

$$P = A/(1 + i \times n) \tag{21.6}$$

Let us say you need to accumulate an amount of R1 210 in 3 years time, and a bank account pays *Simple Interest* of 7%. How much would you need to invest in this bank account today?

$$\begin{aligned} P &= \frac{A}{1 + n \cdot i} \\ &= \frac{\text{R1 210}}{1 + 3 \times 7\%} \\ &= \text{R1 000} \end{aligned}$$

Does this look familiar? Look back to the simple interest worked example in Grade 10. There we started with an amount of R1 000 and looked at what it would grow to in 3 years' time using simple interest rates. Now we have worked backwards to see what amount we need as an opening balance in order to achieve the closing balance of R1 210.

In practice, however, present values are usually always calculated assuming compound interest. So unless you are explicitly asked to calculate a present value (or opening balance) using simple interest rates, make sure you use the compound interest rate formula!

?

Exercise: Present and Future Values

1. After a 20-year period Josh's lump sum investment matures to an amount of R313 550. How much did he invest if his money earned interest at a rate of 13,65% p.a.compounded half yearly for the first 10 years, 8,4% p.a. compounded quarterly for the next five years and 7,2% p.a. compounded monthly for the remaining period ?
2. A loan has to be returned in two equal semi-annual instalments. If the rate of interest is 16% per annum, compounded semi-annually and each instalment is R1 458, find the sum borrowed.

21.6 Finding i

By this stage in your studies of the mathematics of finance, you have always known what interest rate to use in the calculations, and how long the investment or loan will last. You have then either taken a known starting point and calculated a future value, or taken a known future value and calculated a present value.

But here are other questions you might ask:

1. I want to borrow R2 500 from my neighbour, who said I could pay back R3 000 in 8 months time. What interest is she charging me?
2. I will need R450 for some university textbooks in 1,5 years time. I currently have R400. What interest rate do I need to earn to meet this goal?

Each time that you see something different from what you have seen before, start off with the basic equation that you should recognise very well:

$$A = P \cdot (1+i)^n$$

If this were an algebra problem, and you were told to "solve for i", you should be able to show that:

$$\begin{aligned} A/P &= (1+i)^n \\ (1+i) &= (A/P)^{1/n} \\ i &= (A/P)^{1/n} - 1 \end{aligned}$$

You do not need to memorise this equation, it is easy to derive any time you need it!

So let us look at the two examples mentioned above.

1. Check that you agree that P=R2 500, A=R3 000, n=8/12=0,666667. This means that:

$$\begin{aligned} i &= (\text{R3 000}/\text{R2 500})^{1/0,666667} - 1 \\ &= 31,45\% \end{aligned}$$

Ouch! That is not a very generous neighbour you have.

2. Check that P=R400, A=R450, n=1,5

$$\begin{aligned} i &= (\text{R450}/\text{R400})^{1/1{,}5} - 1 \\ &= 8{,}17\% \end{aligned}$$

This means that as long as you can find a bank which pays more than 8,17% interest, you should have the money you need!

Note that in both examples, we expressed n as a number of years ($\frac{8}{12}$ years, not 8 because that is the number of months) which means i is the annual interest rate. Always keep this in mind - keep years with years to avoid making silly mistakes.

Exercise: Finding i

1. A machine costs R45 000 and has a scrap value of R9 000 after 10 years. Determine the annual rate of depreciation if it is calculated on the reducing balance method.
2. After 5 years an investment doubled in value. At what annual rate was interest compounded ?

21.7 Finding n - Trial and Error

By this stage you should be seeing a pattern. We have our standard formula, which has a number of variables:

$$A = P \cdot (1 + i)^n$$

We have solved for A (in section 8.5), P (in section 21.5) and i (in section 21.6). This time we are going to solve for n. In other words, if we know what the starting sum of money is and what it grows to, and if we know what interest rate applies - then we can work out how long the money needs to be invested for all those other numbers to tie up.

This section will calculate n by trial and error and by using a calculator. The proper algebraic solution will be learnt in Grade 12.

Solving for n, we can write:

$$\begin{aligned} A &= P(1 + i)^n \\ \frac{A}{P} &= (1 + i)^n \end{aligned}$$

Now we have to examine the numbers involved to try to determine what a possible value of n is. Refer to Table 5.1 (on page 38) for some ideas as to how to go about finding n.

Worked Example 100: Term of Investment - Trial and Error

Question: If we invest R3 500 into a savings account which pays 7,5% compound interest for an unknown period of time, at the end of which our account is worth R4 044,69. How long did we invest the money?

Answer

Step 1 : Determine what is given and what is required

- P=R3 500
- i=7,5%
- A=R4 044,69

We are required to find n.

Step 2 : Determine how to approach the problem

We know that:

$$\begin{aligned} A &= P(1+i)^n \\ \frac{A}{P} &= (1+i)^n \end{aligned}$$

Step 3 : Solve the problem

$$\begin{aligned} \frac{\text{R4 044,69}}{\text{R3 500}} &= (1+7{,}5\%)^n \\ 1{,}156 &= (1{,}075)^n \end{aligned}$$

We now use our calculator and try a few values for n.

Possible n	$1{,}075^n$
1,0	1,075
1,5	1,115
2,0	1,156
2,5	1,198

We see that n is close to 2.

Step 4 : Write final answer

The R3 500 was invested for about 2 years.

?

Exercise: Finding n - Trial and Error

1. A company buys two typs of motor cars: The Acura costs R80 600 and the Brata R101 700 VAT included. The Acura depreciates at a rate, compunded annually of 15,3% per year and the Brata at 19,7&, also compunded annually, per year. After how many years will the book value of the two models be the same ?
2. The fuel in the tank of a truck decreases every minute by 5,5% of the amount in the tank at that point in time. Calculate after how many minutes there will be less than $30l$ in the tank if it originally held $200l$.

21.8 Nominal and Effective Interest Rates

So far we have discussed annual interest rates, where the interest is quoted as a per annum amount. Although it has not been explicitly stated, we have assumed that when the interest is quoted as a per annum amount it means that the interest is once a year.

Interest however, may be paid more than just once a year, for example we could receive interest on a monthly basis, i.e. 12 times per year. So how do we compare a monthly interest rate, say, to an annual interest rate? This brings us to the concept of the effective annual interest rate.

One way to compare different rates and methods of interest payments would be to compare the Closing Balances under the different options, for a given Opening Balance. Another, more widely used, way is to calculate and compare the "effective annual interest rate" on each option. This way, regardless of the differences in how frequently the interest is paid, we can compare apples-with-apples.

For example, a savings account with an opening balance of R1 000 offers a compound interest rate of 1% per month which is paid at the end of every month. We can calculate the accumulated balance at the end of the year using the formulae from the previous section. But be careful our interest rate has been given as a monthly rate, so we need to use the same units (months) for our time period of measurement.

So we can calculate the amount that would be accumulated by the end of 1-year as follows:

$$\begin{aligned} \text{Closing Balance after 12 months} &= P \times (1+i)^n \\ &= \text{R1 000} \times (1+1\%)^{12} \\ &= \text{R1 126,83} \end{aligned}$$

Remember, the trick to using the formulae is to define the time period, and use the interest rate relevant to the time period.

Note that because we are using a monthly time period, we have used $n = 12$ months to calculate the balance at the end of one year.

The effective annual interest rate is an annual interest rate which represents the equivalent per annum interest rate assuming compounding.

It is the annual interest rate in our Compound Interest equation that equates to the same accumulated balance after one year. So we need to solve for the effective annual interest rate so that the accumulated balance is equal to our calculated amount of R1 126,83.

We use $i12$ to denote the monthly interest rate. We have introduced this notation here to distinguish between the annual interest rate, i. Specifically, we need to solve for i in the following equation:

$$\begin{aligned} P \times (1+i)^1 &= P \times (1+i12)^{12} \\ (1+i) &= (1+i12)^{12} \quad \text{divide both sides by } P \\ i &= (1+i12)^{12} - 1 \quad \text{subtract 1 from both sides} \end{aligned}$$

For the example, this means that the effective annual rate for a monthly rate $i12 = 1\%$ is:

$$\begin{aligned} i &= (1+i12)^{12} - 1 \\ &= (1+1\%)^{12} - 1 \\ &= 0{,}12683 \\ &= 12{,}683\% \end{aligned}$$

If we recalculate the closing balance using this annual rate we get:

$$\begin{aligned} \text{Closing Balance after 1 year} &= P \times (1+i)^n \\ &= \text{R1 000} \times (1+12{,}683\%)^1 \\ &= \text{R1 126,83} \end{aligned}$$

which is the same as the answer obtained for 12 months.

Note that this is greater than simply multiplying the monthly rate by 12 ($12 \times 1\% = 12\%$) due to the effects of compounding. The difference is due to interest on interest. We have seen this before, but it is an important point!

21.8.1 The General Formula

So we know how to convert a monthly interest rate into an effective annual interest. Similarly, we can convert a quarterly interest, or a semi-annual interest rate or an interest rate of any frequency for that matter into an effective annual interest rate.

For a quarterly interest rate of say 3% per quarter, the interest will be paid four times per year (every three month). We can calculate the effective annual interest rate by solving for i:

$$P(1+i) = P(1+i4)^4$$

where $i4$ is the quarterly interest rate.

So $(1+i) = (1{,}03)^4$, and so $i = 12{,}55\%$. This is the effective annual interest rate.

In general, for interest paid at a frequency of T times per annum, the follow equation holds:

$$P(1+i) = P(1+iT)^T \qquad (21.7)$$

where iT is the interest rate paid T times per annum.

21.8.2 De-coding the Terminology

Market convention however, is not to state the interest rate as say 1% per month, but rather to express this amount as an annual amount which in this example would be paid monthly. This annual amount is called the nominal amount.

The market convention is to quote a nominal interest rate of "12% per annum paid monthly" instead of saying (an effective) 1% per month. We know from a previous example, that a nominal interest rate of 12% per annum paid monthly, equates to an effective annual interest rate of 12,68%, and the difference is due to the effects of interest-on-interest.

So if you are given an interest rate expressed as an annual rate but paid more frequently than annual, we first need to calculate the actual interest paid per period in order to calculate the effective annual interest rate.

$$\text{monthly interest rate} = \frac{\text{Nominal interest Rate per annum}}{\text{number of periods per year}} \qquad (21.8)$$

For example, the monthly interest rate on 12% interest per annum paid monthly, is:

$$\begin{aligned}\text{monthly interest rate} &= \frac{\text{Nominal interest Rate per annum}}{\text{number of periods per year}} \\ &= \frac{12\%}{12 \text{ months}} \\ &= 1\% \text{ per month}\end{aligned}$$

The same principle apply to other frequencies of payment.

Worked Example 101: Nominal Interest Rate

Question: Consider a savings account which pays a nominal interest at 8% per annum, paid quarterly. Calculate (a) the interest amount that is paid each quarter, and (b) the effective annual interest rate.

Answer

Step 1 : Determine what is given and what is required

We are given that a savings account has a nominal interest rate of 8% paid quarterly. We are required to find:

- the quarterly interest rate, $i4$
- the effective annual interest rate, i

Step 2 : Determine how to approach the problem

We know that:

$$\text{quarterly interest rate} = \frac{\text{Nominal interest Rate per annum}}{\text{number of quarters per year}}$$

and

$$P(1+i) = P(1+iT)^T$$

where T is 4 because there are 4 payments each year.

Step 3 : Calculate the monthly interest rate

$$\begin{aligned}
\text{quarterly interest rate} &= \frac{\text{Nominal interest Rate per annum}}{\text{number of periods per year}} \\
&= \frac{8\%}{4 \text{ quarters}} \\
&= 2\% \text{ per quarter}
\end{aligned}$$

Step 4 : Calculate the effective annual interest rate

The effective annual interest rate (i) is calculated as:

$$\begin{aligned}
(1+i) &= (1+i4)^4 \\
(1+i) &= (1+2\%)^4 \\
i &= (1+2\%)^4 - 1 \\
&= 8{,}24\%
\end{aligned}$$

Step 5 : Write the final answer

The quarterly interest rate is 2% and the effective annual interest rate is 8,24%, for a nominal interest rate of 8% paid quarterly.

Worked Example 102: Nominal Interest Rate

Question: On their saving accounts, Echo Bank offers an interest rate of 18% nominal, paid monthly. If you save R100 in such an account now, how much would the amount have accumulated to in 3 years' time?

Answer

Step 1 : Determine what is given and what is required

Interest rate is 18% nominal paid monthly. There are 12 months in a year. We are working with a yearly time period, so $n = 3$. The amount we have saved is $R100$, so $P = 100$. We need the accumulated value, A.

Step 2 : Recall relevant formulae

We know that

$$\text{monthly interest rate} = \frac{\text{Nominal interest Rate per annum}}{\text{number of periods per year}}$$

for converting from nominal interest rate to effective interest rate, we have

$$1 + i = (1+iT)^T$$

and for cacluating accumulated value, we have

$$A = P \times (1+i)^n$$

Step 3 : Calculate the effective interest rate

There are 12 month in a year, so

$$\begin{aligned}
i12 &= \frac{\text{Nominal annual interest rate}}{12} \\
&= \frac{18\%}{12} \\
&= 1{,}5\% \text{ per month}
\end{aligned}$$

and then, we have

$$\begin{aligned} 1+i &= (1+i12)^{1}2 \\ i &= (1+i12)^{1}2-1 \\ &= (1+1{,}5\%)^{1}2-1 \\ &= (1{,}015)^{1}2-1 \\ &= 19{,}56\% \end{aligned}$$

Step 4 : Reach the final answer

$$\begin{aligned} A &= P \times (1+i)^n \\ &= 100 \times (1+19{,}56\%)^3 \\ &= 100 \times 1{,}7091 \\ &= 170{,}91 \end{aligned}$$

Step 5 : Write the final answer
The accumulated value is $R170{,}91$. (Remember to round off to the the nearest cent.)

Exercise: Nominal and Effect Interest Rates

1. Calculate the effective rate equivalent to a nominal interest rate of 8,75% p.a. compounded monthly.
2. Cebela is quoted a nominal interest rate of 9,15% per annum compounded every four months on her investment of R 85 000. Calculate the effective rate per annum.

21.9 Formulae Sheet

As an easy reference, here are the key formulae that we derived and used during this chapter. While memorising them is nice (there are not many), it is the application that is useful. Financial experts are not paid a salary in order to recite formulae, they are paid a salary to use the right methods to solve financial problems.

21.9.1 Definitions

P	Principal (the amount of money at the starting point of the calculation)
i	interest rate, normally the effective rate per annum
n	period for which the investment is made
iT	the interest rate paid T times per annum, i.e. $iT = \frac{\text{Nominal Interest Rate}}{T}$

21.9.2 Equations

$$
\begin{aligned}
Simple\ Increase : A &= P(1 + i \times n) \\
Compound\ Increase : A &= P(1 + i)^n \\
Simple\ Decrease : A &= P(1 - i \times n) \\
Compound\ Decrease : A &= P(1 - i)^n \\
Effective\ Annual\ Interest\ Rate(i) : (1 + i) &= (1 + iT)^T
\end{aligned}
$$

21.10 End of Chapter Exercises

1. Shrek buys a Mercedes worth R385 000 in 2007. What will the value of the Mercedes be at the end of 2013 if

   A the car depreciates at 6% p.a. straight-line depreciation

   B the car depreciates at 12% p.a. reducing-balance depreciation.

2. Greg enters into a 5-year hire-purchase agreement to buy a computer for R8 900. The interest rate is quoted as 11% per annum based on simple interest. Calculate the required monthly payment for this contract.

3. A computer is purchased for R16 000. It depreciates at 15% per annum.

   A Determine the book value of the computer after 3 years if depreciation is calculated according to the straight-line method.

   B Find the rate, according to the reducing-balance method, that would yield the same book value as in 3a after 3 years.

4. Maggie invests R12 500,00 for 5 years at 12% per annum compounded monthly for the first 2 years and 14% per annum compounded semi-annually for the next 3 years. How much will Maggie receive in total after 5 years?

5. Tintin invests R120 000. He is quoted a nominal interest rate of 7,2% per annum compounded monthly.

   A Calculate the effective rate per annum correct to THREE decimal places.

   B Use the effective rate to calculate the value of Tintin's investment if he invested the money for 3 years.

   C Suppose Tintin invests his money for a total period of 4 years, but after 18 months makes a withdrawal of R20 000, how much will he receive at the end of the 4 years?

6. Paris opens accounts at a number of clothing stores and spends freely. She gets heself into terrible debt and she cannot pay off her accounts. She owes Hilton Fashion world R5 000 and the shop agrees to let Paris pay the bill at a nominal interest rate of 24% compounded monthly.

   A How much money will she owe Hilton Fashion World after two years ?

   B What is the effective rate of interest that Hilton Fashion World is charging her ?

Chapter 22

Solving Quadratic Equations - Grade 11

22.1 Introduction

In grade 10, the basics of solving linear equations, quadratic equations, exponential equations and linear inequalities were studied. This chapter extends on that work. We look at different methods of solving quadratic equations.

22.2 Solution by Factorisation

The solving of quadratic equations by factorisation was discussed in Grade 10. Here is an example to remind you of what is involved.

Worked Example 103: Solution of Quadratic Equations

Question: Solve the equation $2x^2 - 5x - 12 = 0$.

Answer

Step 1 : Determine whether the equation has common factors

This equation has no common factors.

Step 2 : Determine if the equation is in the form $ax^2 + bx + c$ with $a > 0$

The equation is in the required form, with $a = 2$, $b = -5$ and $c = -12$.

Step 3 : Factorise the quadratic

$2x^2 - 5x - 12$ has factors of the form:

$$(2x + s)(x + v)$$

with s and v constants to be determined. This multiplies out to

$$2x^2 + (s + 2v)x + sv$$

We see that $sv = -12$ and $s + 2v = -5$. This is a set of simultaneous equations in s and v, but it is easy to solve numerically. All the options for s and v are considered below.

s	v	$s+2v$
2	-6	-10
-2	6	10
3	-4	-5
-3	4	5
4	-3	-2
-4	3	2
6	-2	2
-6	2	-2

We see that the combination $s = 3$ and $v = -4$ gives $s + 2v = -5$.

Step 4 : Write the equation with factors

$$(2x + 3)(x - 4) = 0$$

Step 5 : Solve the equation

If two brackets are multiplied together and give 0, then one of the brackets must be 0, therefore

$$2x + 3 = 0$$

or

$$x - 4 = 0$$

Therefore, $x = -\frac{3}{2}$ or $x = 4$

Step 6 : Write the final answer

The solutions to $2x^2 - 5x - 12 = 0$ are $x = -\frac{3}{2}$ or $x = 4$.

It is important to remember that a quadratic equation has to be in the form $ax^2 + bx + c = 0$ before one can solve it using these methods.

Worked Example 104: Solving quadratic equation by factorisation

Question: Solve for a: $a(a - 3) = 10$

Answer

Step 1 : Rewrite the equation in the form $ax^2 + bx + c = 0$

Remove the brackets and move all terms to one side.

$$a^2 - 3a - 10 = 0$$

Step 2 : Factorise the trinomial

$$(a + 2)(a - 5) = 0$$

Step 3 : Solve the equation

$$a + 2 = 0$$

or

$$a - 5 = 0$$

Solve the two linear equations and check the solutions in the original equation.

Step 4 : Write the final answer

Therefore, $a = -2$ or $a = 5$

Worked Example 105: Solving fractions that lead to a quadratic equation

Question: Solve for b: $\frac{3b}{b+2} + 1 = \frac{4}{b+1}$

Answer

Step 1 : Put both sides over the LCM

$$\frac{3b(b+1) + (b+2)(b+1)}{(b+2)(b+1)} = \frac{4(b+2)}{(b+2)(b+1)}$$

Step 2 : Determine the restrictions

The denominators are the same, therefore the numerators must be the same.
However, $b \neq -2$ and $b \neq -1$

Step 3 : Simplify equation to the standard form

$$\begin{aligned} 3b^2 + 3b + b^2 + 3b + 2 &= 4b + 8 \\ 4b^2 + 2b - 6 &= 0 \\ 2b^2 + b - 3 &= 0 \end{aligned}$$

Step 4 : Factorise the trinomial and solve the equation

$$\begin{aligned} (2b+3)(b-1) &= 0 \\ 2b+3=0 \quad &or \quad b-1=0 \\ b=\frac{-3}{2} \quad &or \quad b=1 \end{aligned}$$

Step 5 : Check solutions in original equation

Both solutions are valid
Therefore, $b = \frac{-3}{2}$ or $b = 1$

Exercise: Solution by Factorisation

Solve the following quadratic equations by factorisation. Some answers may be left in surd form.

1. $2y^2 - 61 = 101$
2. $2y^2 - 10 = 0$
3. $y^2 - 4 = 10$
4. $2y^2 - 8 = 28$
5. $7y^2 = 28$
6. $y^2 + 28 = 100$
7. $7y^2 + 14y = 0$
8. $12y^2 + 24y + 12 = 0$
9. $16y^2 - 400 = 0$
10. $y^2 - 5y + 6 = 0$
11. $y^2 + 5y - 36 = 0$
12. $y^2 + 2y = 8$
13. $-y^2 - 11y - 24 = 0$
14. $13y - 42 = y^2$

15. $y^2 + 9y + 14 = 0$
16. $y^2 - 5ky + 4k^2 = 0$
17. $y(2y + 1) = 15$
18. $\frac{5y}{y-2} + \frac{3}{y} + 2 = \frac{-6}{y^2-2y}$
19. $\frac{y-2}{y+1} = \frac{2y+1}{y-7}$

22.3 Solution by Completing the Square

We have seen that expressions of the form:

$$a^2x^2 - b^2$$

are known as differences of squares and can be factorised as follows:

$$(ax - b)(ax + b).$$

This simple factorisation leads to another technique to solve quadratic equations known as *completing the square*.

We demonstrate with a simple example, by trying to solve for x in:

$$x^2 - 2x - 1 = 0. \tag{22.1}$$

We cannot easily find factors of this term, but the first two terms look similar to the first two terms of the perfect square:

$$(x - 1)^2 = x^2 - 2x + 1.$$

However, we can cheat and create a perfect square by adding 2 to both sides of the equation in (22.1) as:

$$\begin{aligned} x^2 - 2x - 1 &= 0 \\ x^2 - 2x - 1 + 2 &= 0 + 2 \\ x^2 - 2x + 1 &= 2 \\ (x - 1)^2 &= 2 \\ (x - 1)^2 - 2 &= 0 \end{aligned}$$

Now we know that:

$$2 = (\sqrt{2})^2$$

which means that:

$$(x - 1)^2 - 2$$

is a difference of squares. Therefore we can write:

$$(x - 1)^2 - 2 = [(x - 1) - \sqrt{2}][(x - 1) + \sqrt{2}] = 0.$$

The solution to $x^2 - 2x - 1 = 0$ is then:

$$(x - 1) - \sqrt{2} = 0$$

or

$$(x - 1) + \sqrt{2} = 0.$$

This means $x = 1 + \sqrt{2}$ or $x = 1 - \sqrt{2}$. This example demonstrates the use of *completing the square* to solve a quadratic equation.

Method: Solving Quadratic Equations by Completing the Square

1. Write the equation in the form $ax^2 + bx + c = 0$. e.g. $x^2 + 2x - 3 = 0$
2. Take the constant over to the right hand side of the equation. e.g. $x^2 + 2x = 3$
3. If necessary, make the coefficient of the x^2 term $= 1$, by dividing through by the existing coefficient.
4. Take half the coefficient of the x term, square it and add it to both sides of the equation. e.g. in $x^2 + 2x = 3$, half of the x term is 1. $1^1 = 1$. Therefore we add 1 to both sides to get: $x^2 + 2x + 1 = 3 + 1$.
5. Write the left hand side as a perfect square: $(x + 1)^2 - 4 = 0$
6. You should then be able to factorise the equation in terms of difference of squares and then solve for x: $(x + 1 - 2)(x + 1 + 2) = 0$

Worked Example 106: Solving Quadratic Equations by Completing the Square

Question: Solve:

$$x^2 - 10x - 11 = 0$$

by completing the square

Answer

Step 1 : Write the equation in the form $ax^2 + bx + c = 0$

$$x^2 - 10x - 11 = 0$$

Step 2 : Take the constant over to the right hand side of the equation

$$x^2 - 10x = 11$$

Step 3 : Check that the coefficient of the x^2 term is 1.

The coefficient of the x^2 term is 1.

Step 4 : Take half the coefficient of the x term, square it and add it to both sides

The coefficient of the x term is -10. $\frac{(-10)}{2} = -5$. $(-5)^2 = 25$. Therefore:

$$x^2 - 10x + 25 = 11 + 25$$

Step 5 : Write the left hand side as a perfect square

$$(x - 5)^2 - 36 = 0$$

Step 6 : Factorise equation as difference of squares

$$(x - 5)^2 - 36 = 0$$

$$[(x - 5) + 6][(x - 5) - 6] = 0$$

Step 7 : Solve for the unknown value

$$\begin{aligned} [x + 1][x - 11] &= 0 \\ \therefore x = -1 \quad &\text{or} \quad x = 11 \end{aligned}$$

Worked Example 107: Solving Quadratic Equations by Completing the Square

Question: Solve:

$$2x^2 - 8x - 16 = 0$$

by completing the square

Answer

Step 1 : Write the equation in the form $ax^2 + bx + c = 0$

$$2x^2 - 8x - 16 = 0$$

Step 2 : Take the constant over to the right hand side of the equation

$$2x^2 - 8x = 16$$

Step 3 : Check that the coefficient of the x^2 term is 1.

The coefficient of the x^2 term is 2. Therefore, divide both sides by 2:

$$x^2 - 4x = 8$$

Step 4 : Take half the coefficient of the x term, square it and add it to both sides

The coefficient of the x term is -4. $\frac{(-4)}{2} = -2$. $(-2)^2 = 4$. Therefore:

$$x^2 - 4x + 4 = 8 + 4$$

Step 5 : Write the left hand side as a perfect square

$$(x - 2)^2 - 12 = 0$$

Step 6 : Factorise equation as difference of squares

$$[(x - 2) + \sqrt{12}][(x - 2) - \sqrt{12}] = 0$$

Step 7 : Solve for the unknown value

$$\begin{aligned} [x - 2 + \sqrt{12}][x - 2 - \sqrt{12}] &= 0 \\ \therefore x = 2 - \sqrt{12} &\quad \text{or} \quad x = 2 + \sqrt{12} \end{aligned}$$

Step 8 : The last three steps can also be done in a different the way

Leave left hand side written as a perfect square

$$(x - 2)^2 = 12$$

Step 9 : Take the square root on both sides of the equation

$$x - 2 = \pm\sqrt{12}$$

Step 10 : Solve for x

Therefore $x = 2 - \sqrt{12}$ or $x = 2 + \sqrt{12}$

Compare to answer in step 7.

Exercise: Solution by Completing the Square

Solve the following equations by completing the square:

1. $x^2 + 10x - 2 = 0$
2. $x^2 + 4x + 3 = 0$
3. $x^2 + 8x - 5 = 0$
4. $2x^2 + 12x + 4 = 0$
5. $x^2 + 5x + 9 = 0$
6. $x^2 + 16x + 10 = 0$
7. $3x^2 + 6x - 2 = 0$
8. $z^2 + 8z - 6 = 0$
9. $2z^2 - 11z = 0$
10. $5 + 4z - z^2 = 0$

22.4 Solution by the Quadratic Formula

It is not always possible to solve a quadratic equation by factorising and it is lengthy and tedious to solve a quadratic equations by completing the square. In these situations, you can use the *quadratic formula* that gives the solutions to any quadratic equation.

Consider the general form of the quadratic function:

$$f(x) = ax^2 + bx + c.$$

Factor out the a to get:

$$f(x) = a(x^2 + \frac{b}{a}x + \frac{c}{a}). \tag{22.2}$$

Now we need to do some detective work to figure out how to turn (22.2) into a perfect square plus some extra terms. We know that for a perfect square:

$$(m + n)^2 = m^2 + 2mn + n^2$$

and

$$(m - n)^2 = m^2 - 2mn + n^2$$

The key is the middle term, which is $2\times$ the first term $\times$ the second term. In (22.2), we know that the first term is x so $2\times$ the second term is $\frac{b}{a}$. This means that the second term is $\frac{b}{2a}$. So,

$$(x + \frac{b}{2a})^2 = x^2 + 2\frac{b}{2a}x + (\frac{b}{2a})^2.$$

In general if you add a quantity and subtract the same quantity, nothing has changed. This means if we add and subtract $\left(\frac{b}{2a}\right)^2$ from the right hand side of (22.2) we will get:

$$\begin{aligned} f(x) &= a(x^2 + \frac{b}{a}x + \frac{c}{a}) && (22.3)\\ &= a\left(x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2 + \frac{c}{a}\right) && (22.4)\\ &= a\left(\left[x + \left(\frac{b}{2a}\right)\right]^2 - \left(\frac{b}{2a}\right)^2 + \frac{c}{a}\right) && (22.5)\\ &= a\left(\left[x + \left(\frac{b}{2a}\right)\right]^2\right) + c - \frac{b^2}{4a} && (22.6) \end{aligned}$$

We set $f(x) = 0$ to find its roots, which yields:

$$a(x + \frac{b}{2a})^2 = \frac{b^2}{4a} - c \tag{22.7}$$

Now dividing by a and taking the square root of both sides gives the expression

$$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2}{4a^2} - \frac{c}{a}} \tag{22.8}$$

Finally, solving for x implies that

$$\begin{aligned} x &= -\frac{b}{2a} \pm \sqrt{\frac{b^2}{4a^2} - \frac{c}{a}} \\ &= -\frac{b}{2a} \pm \sqrt{\frac{b^2 - 4ac}{4a^2}} \end{aligned}$$

which can be further simplified to:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{22.9}$$

These are the solutions to the quadratic equation. Notice that there are two solutions in general, but these may not always exists (depending on the sign of the expression $b^2 - 4ac$ under the square root). These solutions are also called the *roots* of the quadratic equation.

Worked Example 108: Using the quadratic formula

Question: Solve for the roots of the function $f(x) = 2x^2 + 3x - 7$.

Answer

Step 1 : Determine whether the equation can be factorised

The expression cannot be factorised. Therefore, the general quadratic formula must be used.

Step 2 : Identify the coefficients in the equation for use in the formula

From the equation:

$$a = 2$$

$$b = 3$$

$$c = -7$$

Step 3 : Apply the quadratic formula

Always write down the formula first and then substitute the values of a, b and c.

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} && (22.10) \\ &= \frac{-(3) \pm \sqrt{(3)^2 - 4(2)(-7)}}{2(2)} && (22.11) \\ &= \frac{-3 \pm \sqrt{56}}{4} && (22.12) \\ &= \frac{-3 \pm 2\sqrt{14}}{4} && (22.13) \end{aligned}$$

Step 4 : Write the final answer

The two roots of $f(x) = 2x^2 + 3x - 7$ are $x = \frac{-3+2\sqrt{14}}{4}$ and $\frac{-3-2\sqrt{14}}{4}$.

Worked Example 109: Using the quadratic formula but no solution

Question: Solve for the solutions to the quadratic equation $x^2 - 5x + 8 = 0$.

Answer

Step 1 : Determine whether the equation can be factorised

The expression cannot be factorised. Therefore, the general quadratic formula must be used.

Step 2 : Identify the coefficients in the equation for use in the formula

From the equation:

$$a = 1$$

$$b = -5$$

$$c = 8$$

Step 3 : Apply the quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad (22.14)$$

$$= \frac{-(-5) \pm \sqrt{(-5)^2 - 4(1)(8)}}{2(1)} \quad (22.15)$$

$$= \frac{5 \pm \sqrt{-7}}{2} \quad (22.16)$$

(22.17)

Step 4 : Write the final answer

Since the expression under the square root is negative these are not real solutions ($\sqrt{-7}$ is not a real number). Therefore there are no real solutions to the quadratic equation $x^2 - 5x + 8 = 0$. This means that the graph of the quadratic function $f(x) = x^2 - 5x + 8$ has no x-intercepts, but that the entire graph lies above the x-axis.

?

Exercise: Solution by the Quadratic Formula

Solve for t using the quadratic formula.

1. $3t^2 + t - 4 = 0$
2. $t^2 - 5t + 9 = 0$
3. $2t^2 + 6t + 5 = 0$
4. $4t^2 + 2t + 2 = 0$
5. $-3t^2 + 5t - 8 = 0$
6. $-5t^2 + 3t - 3 = 0$
7. $t^2 - 4t + 2 = 0$
8. $9t^2 - 7t - 9 = 0$
9. $2t^2 + 3t + 2 = 0$
10. $t^2 + t + 1 = 0$

!

Important:

- In all the examples done so far, the solutions were left in surd form. Answers can also be given in decimal form, using the calculator. Read the instructions when answering questions in a test or exam whether to leave answers in surd form, or in decimal form to an appropriate number of decimal places.
- Completing the square as a method to solve a quadratic equation is only done when specifically asked.

Exercise: Mixed Exercises

Solve the quadratic equations by either factorisation, completing the square or by using the quadratic formula:

- Always try to factorise first, then use the formula if the trinomial cannot be factorised.
- Do some of them by completing the square and then compare answers to those done using the other methods.

1. $24y^2 + 61y - 8 = 0$
2. $-8y^2 - 16y + 42 = 0$
3. $-9y^2 + 24y - 12 = 0$
4. $-5y^2 + 0y + 5 = 0$
5. $-3y^2 + 15y - 12 = 0$
6. $49y^2 + 0y - 25 = 0$
7. $-12y^2 + 66y - 72 = 0$
8. $-40y^2 + 58y - 12 = 0$
9. $-24y^2 + 37y + 72 = 0$
10. $6y^2 + 7y - 24 = 0$
11. $2y^2 - 5y - 3 = 0$
12. $-18y^2 - 55y - 25 = 0$
13. $-25y^2 + 25y - 4 = 0$
14. $-32y^2 + 24y + 8 = 0$
15. $9y^2 - 13y - 10 = 0$
16. $35y^2 - 8y - 3 = 0$
17. $-81y^2 - 99y - 18 = 0$
18. $14y^2 - 81y + 81 = 0$
19. $-4y^2 - 41y - 45 = 0$
20. $16y^2 + 20y - 36 = 0$
21. $42y^2 + 104y + 64 = 0$
22. $9y^2 - 76y + 32 = 0$
23. $-54y^2 + 21y + 3 = 0$
24. $36y^2 + 44y + 8 = 0$
25. $64y^2 + 96y + 36 = 0$
26. $12y^2 - 22y - 14 = 0$
27. $16y^2 + 0y - 81 = 0$
28. $3y^2 + 10y - 48 = 0$
29. $-4y^2 + 8y - 3 = 0$
30. $-5y^2 - 26y + 63 = 0$
31. $x^2 - 70 = 11$
32. $2x^2 - 30 = 2$
33. $x^2 - 16 = 2 - x^2$
34. $2y^2 - 98 = 0$
35. $5y^2 - 10 = 115$
36. $5y^2 - 5 = 19 - y^2$

22.5 Finding an equation when you know its roots

We have mentioned before that the *roots* of a quadratic equation are the solutions or answers you get from solving the quadatic equation. Working back from the answers, will take you to an equation.

Worked Example 110: Find an equation when roots are given

Question: Find an equation with roots 13 and -5

Answer

Step 1 : Write down as the product of two brackets

The step before giving the solutions would be:

$$(x - 13)(x + 5) = 0$$

Notice that the signs in the brackets are opposite of the given roots.

Step 2 : Remove brackets

$$x^2 - 8x - 65 = 0$$

Of course, there would be other possibilities as well when each term on each side of the *equal to sign* is multiplied by a constant.

Worked Example 111: Fraction roots

Question: Find an equation with roots $-\frac{3}{2}$ and 4
Answer
Step 1 : Product of two brackets
Notice that if $x = -\frac{3}{2}$ then $2x + 3 = 0$
Therefore the two brackets will be:

$$(2x + 3)(x - 4) = 0$$

Step 2 : Remove brackets
The equation is:

$$2x^2 - 5x - 12 = 0$$

Extension: Theory of Quadratic Equations - Advanced
This section is not in the syllabus, but it gives one a good understanding about some of the solutions of the quadratic equations.

What is the Discriminant of a Quadratic Equation?

Consider a general quadratic function of the form $f(x) = ax^2 + bx + c$. The *discriminant* is defined as:

$$\Delta = b^2 - 4ac. \tag{22.18}$$

This is the expression under the square root in the formula for the roots of this function. We have already seen that whether the roots exist or not depends on whether this factor Δ is negative or positive.

The Nature of the Roots

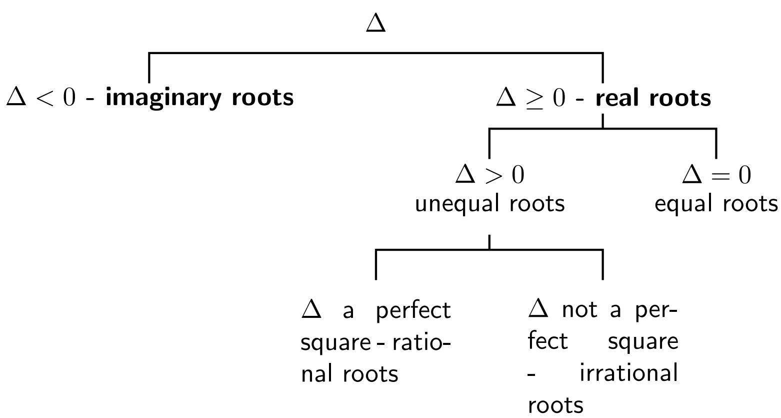
Real Roots ($\Delta \geq 0$)

Consider $\Delta \geq 0$ for some quadratic function $f(x) = ax^2 + bx + c$. In this case there are solutions to the equation $f(x) = 0$ given by the formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-b \pm \sqrt{\Delta}}{2a} \tag{22.19}$$

Since the square roots exists (the expression under the square root is non-negative.) These are the roots of the function $f(x)$.

There various possibilities are summarised in the figure below.



Equal Roots ($\Delta = 0$)

If $\Delta = 0$, then the roots are equal and, from the formula, these are given by

$$x = -\frac{b}{2a} \tag{22.20}$$

Unequal Roots ($\Delta > 0$)

There will be 2 unequal roots if $\Delta > 0$. The roots of $f(x)$ are **rational** if Δ is a perfect square (a number which is the square of a rational number), since, in this case, $\sqrt{\Delta}$ is rational. Otherwise, if Δ is not a perfect square, then the roots are **irrational**.

Imaginary Roots ($\Delta < 0$)

If $\Delta < 0$, then the solution to $f(x) = ax^2 + bx + c = 0$ contains the square root of a negative number and therefore there are no real solutions. We therefore say that the roots of $f(x)$ are *imaginary* (the graph of the function $f(x)$ does not intersect the x-axis).

Extension: Theory of Quadratics - advanced exercises

Exercise: From past papers

1. [IEB, Nov. 2001, HG] Given: $x^2 + bx - 2 + k(x^2 + 3x + 2) = 0 \quad (k \neq -1)$
   A Show that the discriminant is given by:
   $$\Delta = k^2 + 6bk + b^2 + 8$$
   B If $b = 0$, discuss the nature of the roots of the equation.
   C If $b = 2$, find the value(s) of k for which the roots are equal.
2. [IEB, Nov. 2002, HG] Show that $k^2x^2 + 2 = kx - x^2$ has non-real roots for all real values for k.
3. [IEB, Nov. 2003, HG] The equation $x^2 + 12x = 3kx^2 + 2$ has real roots.
   A Find the largest integral value of k.
   B Find one rational value of k, for which the above equation has rational roots.
4. [IEB, Nov. 2003, HG] In the quadratic equation $px^2 + qx + r = 0$, p, q and r are positive real numbers and form a geometric sequence. Discuss the nature of the roots.
5. [IEB, Nov. 2004, HG] Consider the equation:
   $$k = \frac{x^2 - 4}{2x - 5} \qquad \text{where } x \neq \tfrac{5}{2}$$
   A Find a value of k for which the roots are equal.
   B Find an integer k for which the roots of the equation will be rational and unequal.
6. [IEB, Nov. 2005, HG]
   A Prove that the roots of the equation $x^2 - (a+b)x + ab - p^2 = 0$ are real for all real values of a, b and p.
   B When will the roots of the equation be equal?
7. [IEB, Nov. 2005, HG] If b and c can take on only the values 1, 2 or 3, determine all pairs $(b; c)$ such that $x^2 + bx + c = 0$ has real roots.

22.6 End of Chapter Exercises

1. Solve: $x^2 - x - 1 = 0$ (Give your answer correct to two decimal places.)
2. Solve: $16(x+1) = x^2(x+1)$
3. Solve: $y^2 + 3 + \frac{12}{y^2+3} = 7$ (Hint: Let $y^2 + 3 = k$ and solve for k first and use the answer to solve y.)
4. Solve for x: $2x^4 - 5x^2 - 12 = 0$
5. Solve for x:
   - A $x(x-9) + 14 = 0$
   - B $x^2 - x = 3$ (Show your answer correct to ONE decimal place.)
   - C $x + 2 = \frac{6}{x}$ (correct to 2 decimal places)
   - D $\frac{1}{x+1} + \frac{2x}{x-1} = 1$
6. Solve for x by completing the square: $x^2 - px - 4 = 0$
7. The equation $ax^2 + bx + c = 0$ has roots $x = \frac{2}{3}$ and $x = -4$. Find one set of possible values for a, b and c.
8. The two roots of the equation $4x^2 + px - 9 = 0$ differ by 5. Calculate the value of p.
9. An equation of the form $x^2 + bx + c = 0$ is written on the board. Saskia and Sven copy it down incorrectly. Saskia has a mistake in the constant term and obtains the solutions -4 and 2. Sven has a mistake in the coefficient of x and obtains the solutions 1 and -15. Determine the correct equation that was on the board.
10. Bjorn stumbled across the following formula to solve the quadratic equation $ax^2+bx+c = 0$ in a foreign textbook.

$$x = \frac{2c}{-b \pm \sqrt{b^2 - 4ac}}$$

   - A Use this formula to solve the equation:

$$2x^2 + x - 3 = 0$$

   - B Solve the equation again, using factorisation, to see if the formula works for this equation.
   - C Trying to derive this formula to prove that it always works, Bjorn got stuck along the way. His attempt his shown below:

$$\begin{aligned}
ax^2 + bx + c &= 0 \\
a + \frac{b}{x} + \frac{c}{x^2} &= 0 \quad \text{Divided by } x^2 \text{ where } x \neq 0 \\
\frac{c}{x^2} + \frac{b}{x} + a &= 0 \quad \text{Rearranged} \\
\frac{1}{x^2} + \frac{b}{cx} + \frac{a}{c} &= 0 \quad \text{Divided by } c \text{ where } c \neq 0 \\
\frac{1}{x^2} + \frac{b}{cx} &= -\frac{a}{c} \quad \text{Subtracted } \frac{a}{c} \text{ from both sides} \\
\therefore \frac{1}{x^2} + \frac{b}{cx} &+ \ldots \quad \text{Got stuck}
\end{aligned}$$

     Complete his derivation.

Chapter 23

Solving Quadratic Inequalities - Grade 11

23.1 Introduction

Now that you know how to solve quadratic equations, you are ready to learn how to solve quadratic inequalities.

23.2 Quadratic Inequalities

A *quadratic inequality* is an inequality of the form

$$
\begin{aligned}
ax^2 + bx + c &> 0 \\
ax^2 + bx + c &\geq 0 \\
ax^2 + bx + c &< 0 \\
ax^2 + bx + c &\leq 0
\end{aligned}
$$

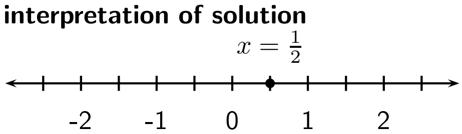
Solving a quadratic inequality corresponds to working out in what region the graph of a quadratic function lies above or below the x-axis.

Worked Example 112: Quadratic Inequality

Question: Solve the inequality $4x^2-4x+1 \leq 0$ and interpret the solution graphically.
Answer
Step 1 : Factorise the quadratic
Let $f(x) = 4x^2-4x+1$. Factorising this quadratic function gives $f(x) = (2x-1)^2$.
Step 2 : Re-write the original equation with factors

$$(2x-1)^2 \leq 0$$

Step 3 : Solve the equation
which shows that $f(x) = 0$ only when $x = \frac{1}{2}$.
Step 4 : Write the final answer
This means that the graph of $f(x) = 4x^2 - 4x + 1$ touches the x-axis at $x = \frac{1}{2}$, but there are no regions where the graph is below the x-axis.
Step 5 : Graphical interpretation of solution



Worked Example 113: Solving Quadratic Inequalities

Question: Find all the solutions to the inequality $x^2 - 5x + 6 \geq 0$.

Answer

Step 1 : Factorise the quadratic

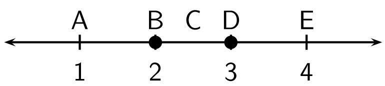
The factors of $x^2 - 5x + 6$ are $(x - 3)(x - 2)$.

Step 2 : Write the inequality with the factors

$$\begin{aligned} x^2 - 5x + 6 &\geq 0 \\ (x - 3)(x - 2) &\geq 0 \end{aligned}$$

Step 3 : Determine which ranges correspond to the inequality

We need to figure out which values of x satisfy the inequality. From the answers we have five regions to consider.



Step 4 : Determine whether the function is negative or positive in each of the regions

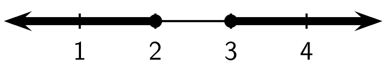
Let $f(x) = x^2 - 5x + 6$. For each region, choose any point in the region and evaluate the function.

		$f(x)$	sign of $f(x)$
Region A	$x < 2$	$f(1) = 2$	$+$
Region B	$x = 2$	$f(2) = 0$	$+$
Region C	$2 < x < 3$	$f(2{,}5) = -2{,}5$	-
Region D	$x = 3$	$f(3) = 0$	$+$
Region E	$x > 3$	$f(4) = 2$	$+$

We see that the function is positive for $x \leq 2$ and $x \geq 3$.

Step 5 : Write the final answer and represent on a number line

We see that $x^2 - 5x + 6 \geq 0$ is true for $x \leq 2$ and $x \geq 3$.



Worked Example 114: Solving Quadratic Inequalities

Question: Solve the quadratic inequality $-x^2 - 3x + 5 > 0$.

Answer

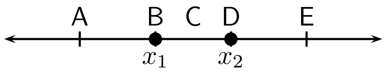
Step 1 : Determine how to approach the problem

Let $f(x) = -x^2 - 3x + 5$. $f(x)$ cannot be factorised so, use the quadratic formula to determine the roots of $f(x)$. The x-intercepts are solutions to the quadratic equation

$$\begin{aligned}
-x^2 - 3x + 5 &= 0 \\
x^2 + 3x - 5 &= 0 \\
\therefore x &= \frac{-3 \pm \sqrt{(3)^2 - 4(1)(-5)}}{2(1)} \\
&= \frac{-3 \pm \sqrt{29}}{2} \\
x_1 &= \frac{-3 - \sqrt{29}}{2} \\
x_2 &= \frac{-3 + \sqrt{29}}{2}
\end{aligned}$$

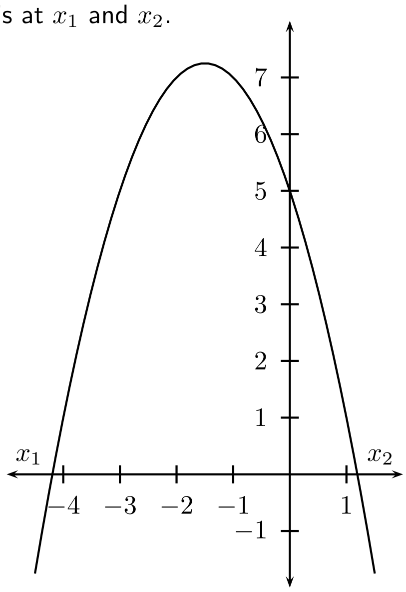
Step 2 : Determine which ranges correspond to the inequality

We need to figure out which values of x satisfy the inequality. From the answers we have five regions to consider.



Step 3 : Determine whether the function is negative or positive in each of the regions

We can use another method to determine the sign of the function over different regions, by drawing a rough sketch of the graph of the function. We know that the roots of the function correspond to the x-intercepts of the graph. Let $g(x) = -x^2 - 3x + 5$. We can see that this is a parabola with a maximum turning point that intersects the x-axis at x_1 and x_2.



It is clear that $g(x) > 0$ for x_1

Step 4 : Write the final answer and represent the solution graphically

$-x^2 - 3x + 5 > 0$ for x_1

When working with an inequality where the variable is in the denominator, a different approach is needed.

Worked Example 115: Non-linear inequality with the variable in the denominator

Question: Solve $\frac{2}{x+3} \leq \frac{1}{x-3}$

Answer

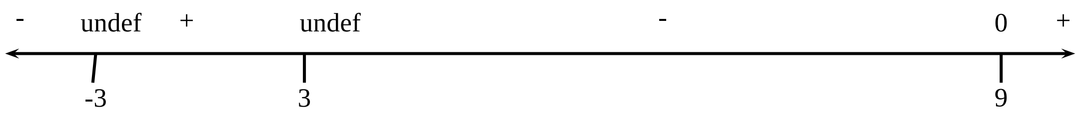
Step 1 : Subtract $\frac{1}{x-3}$ from both sides

$$\frac{2}{x+3} - \frac{1}{x-3} \leq 0$$

Step 2 : Simplify the fraction by finding LCD

$$\frac{2(x-3)-(x+3)}{(x+3)(x-3)} \leq 0$$
$$\frac{x-9}{(x+3)(x-3)} \leq 0$$

Step 3 : Draw a number line for the inequality



We see that the expression is negative for $x < -3$ or $3 < x \leq 9$.

Step 4 : Write the final answer

$$x < -3 \quad or \quad 3 < x \leq 9$$

23.3 End of Chapter Exercises

Solve the following inequalities and show your answer on a number line.

1. Solve: $x^2 - x < 12$.
2. Solve: $3x^2 > -x + 4$
3. Solve: $y^2 < -y - 2$
4. Solve: $-t^2 + 2t > -3$
5. Solve: $s^2 - 4s > -6$
6. Solve: $0 \geq 7x^2 - x + 8$
7. Solve: $0 \geq -4x^2 - x$
8. Solve: $0 \geq 6x^2$
9. Solve: $2x^22 + x + 6 \leq 0$
10. Solve for x if: $\frac{x}{x-3} < 2$ and $x \neq 3$.
11. Solve for x if: $\frac{4}{x-3} \leq 1$.

12. Solve for x if: $\frac{4}{(x-3)^2} < 1$.
13. Solve for x: $\frac{2x-2}{x-3} > 3$
14. Solve for x: $\frac{-3}{(x-3)(x+1)} < 0$
15. Solve: $(2x-3)^2 < 4$
16. Solve: $2x \leq \frac{15-x}{x}$
17. Solve for x: $\frac{x^2+3}{3x-2} \leq 0$
18. Solve: $x - 2 \geq \frac{3}{x}$
19. Solve for x: $\frac{x^2+3x-4}{5+x^4} \leq 0$
20. Determine all real solutions: $\frac{x-2}{3-x} \geq 1$

Chapter 24

Solving Simultaneous Equations - Grade 11

In grade 10, you learnt how to solve sets of simultaneous equations where both equations were linear (i.e. had the highest power equal to 1). In this chapter, you will learn how to solve sets of simultaneous equations where one is linear and one is a quadratic. As in Grade 10, the solution will be found both algebraically and graphically.

The only difference between a system of linear simultaneous equations and a system of simultaneous equations with one linear and one quadratic equation, is that the second system will have at most two solutions.

An example of a system of simultaneous equations with one linear equation and one quadratic equation is:

$$\begin{aligned} y - 2x &= -4 \\ x^2 + y &= 4 \end{aligned} \tag{24.1}$$

24.1 Graphical Solution

The method of graphically finding the solution to one linear and one quadratic equation is identical to systems of linear simultaneous equations.

Method: Graphical solution to a system of simultaneous equations with one linear and one quadratic equation

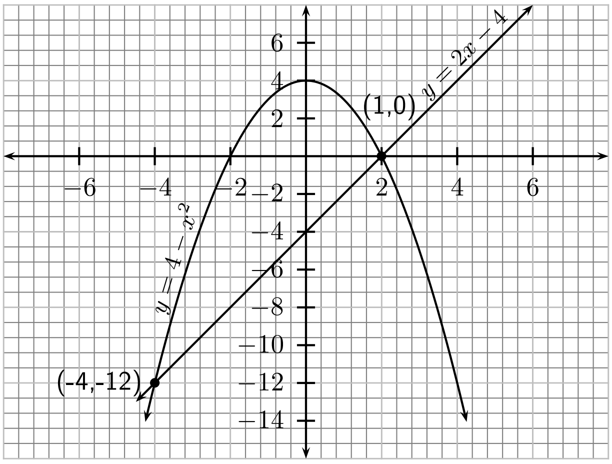
1. Make y the subject of each equation.
2. Draw the graphs of each equation as defined above.
3. The solution of the set of simultaneous equations is given by the intersection points of the two graphs.

For the example, making y the subject of each equation, gives:

$$\begin{aligned} y &= 2x - 4 \\ y &= 4 - x^2 \end{aligned}$$

Plotting the graph of each equation, gives a straight line for the first equation and a parabola for the second equation.

The parabola and the straight line intersect at two points: (2,0) and (-4,-12). Therefore, the solutions to the system of equations in (24.1) is $x = 2, y = 0$ and $x = -4, y = 12$



Worked Example 116: Simultaneous Equations

Question: Solve graphically:

$$
\begin{aligned}
y - x^2 + 9 &= 0 \\
y + 3x - 9 &= 0
\end{aligned}
$$

Answer

Step 1 : Make y the subject of the equation

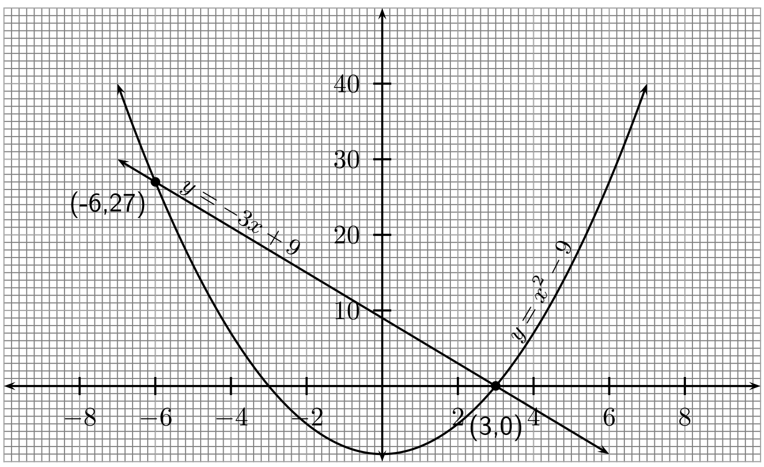
For the first equation:

$$
\begin{aligned}
y - x^2 + 9 &= 0 \\
y &= x^2 - 9
\end{aligned}
$$

and for the second equation:

$$
\begin{aligned}
y + 3x - 9 &= 0 \\
y &= -3x + 9
\end{aligned}
$$

Step 2 : Draw the graphs corresponding to each equation.



Step 3 : Find the intersection of the graphs.

The graphs intersect at $(-6{,}27)$ and at $(3{,}0)$.

Step 4 : Write the solution of the system of simultaneous equations as given by the intersection of the graphs.

The first solution is $x = -6$ and $y = 27$. The second solution is $x = 3$ and $y = 0$.

Exercise: Graphical Solution

Solve the following systems of equations algebraically. Leave your answer in surd form, where appropriate.

1. $b^2 - 1 - a = 0, a + b - 5 = 0$
2. $x + y - 10 = 0, x^2 - 2 - y = 0$
3. $6 - 4x - y = 0, 12 - 2x^2 - y = 0$
4. $x + 2y - 14 = 0, x^2 + 2 - y = 0$
5. $2x + 1 - y = 0, 25 - 3x - x^2 - y = 0$

24.2 Algebraic Solution

The algebraic method of solving simultaneous equations is by substitution.

For example the solution of

$$\begin{aligned} y - 2x &= -4 \\ x^2 + y &= 4 \end{aligned}$$

is:

$$\begin{aligned} y &= 2x - 4 \quad \text{into second equation} \\ x^2 + (2x - 4) &= 4 \\ x^2 + 2x - 8 &= 0 \\ \text{Factorise to get:} \quad (x + 4)(x - 2) &= 0 \\ \therefore \text{ the 2 solutions for } x \text{ are:} \quad x = -4 \quad \text{and} \quad x = 2 & \end{aligned}$$

The corresponding solutions for y are obtained by substitution of the x-values into the first equation

$$\begin{aligned} y = 2(-4) - 4 &= -12 \quad \text{for } x = -4 \\ \text{and:} \quad y = 2(2) - 4 &= 0 \quad \text{for } x = 2 \end{aligned}$$

As expected, these solutions are identical to those obtained by the graphical solution.

Worked Example 117: Simultaneous Equations

Question: Solve algebraically:

$$\begin{aligned} y - x^2 + 9 &= 0 \\ y + 3x - 9 &= 0 \end{aligned}$$

Answer

Step 1 : Make y the subject of the linear equation

$$\begin{aligned} y + 3x - 9 &= 0 \\ y &= -3x + 9 \end{aligned}$$

Step 2 : Substitute into the quadratic equation

$$\begin{aligned} (-3x + 9) - x^2 + 9 &= 0 \\ x^2 + 3x - 18 &= 0 \\ \text{Factorise to get:} \quad (x + 6)(x - 3) &= 0 \\ \therefore \text{ the 2 solutions for } x \text{ are:} \quad x = -6 \quad \text{and} \quad x = 3 \end{aligned}$$

Step 3 : Substitute the values for x into the first equation to calculate the corresponding y-values.

$$\begin{aligned} y = -3(-6) + 9 &= 27 \quad \text{for } x = -6 \\ \text{and:} \quad y = -3(3) + 9 &= 0 \quad \text{for } x = 3 \end{aligned}$$

Step 4 : Write the solution of the system of simultaneous equations.

The first solution is $x = -6$ and $y = 27$. The second solution is $x = 3$ and $y = 0$.

?

Exercise: Algebraic Solution

Solve the following systems of equations algebraically. Leave your answer in surd form, where appropriate.

1. $a+b=5$	$a-b^2+3b-5=0$
2. $a-b+1=0$	$a-b^2+5b-6=0$
3. $a-\frac{(2b+2)}{4}=0$	$a-2b^2+3b+5=0$
4. $a+2b-4=0$	$a-2b^2-5b+3=0$
5. $a-2+3b=0$	$a-9+b^2=0$
6. $a-b-5=0$	$a-b^2=0$
7. $a-b-4=0$	$a+2b^2-12=0$
8. $a+b-9=0$	$a+b^2-18=0$
9. $a-3b+5=0$	$a+b^2-4b=0$
10. $a+b-5=0$	$a-b^2+1=0$
11. $a-2b-3=0$	$a-3b^2+4=0$
12. $a-2b=0$	$a-b^2-2b+3=0$
13. $a-3b=0$	$a-b^2+4=0$
14. $a-2b-10=0$	$a-b^2-5b=0$
15. $a-3b-1=0$	$a-2b^2-b+3=0$
16. $a-3b+1$	$a-b^2=0$
17. $a+6b-5=0$	$a-b^2-8=0$
18. $a-2b+1=0$	$a-2b^2-12b+4$
19. $2a+b-2=0$	$8a+b^2-8=0$
20. $a+4b-19=0$	$8a+5b^2-101=0$
21. $a+4b-18=0$	$2a+5b^2-57$

Chapter 25

Mathematical Models - Grade 11

Up until now, you have only learnt how to solve equations and inequalities, but there has not been much application of what you have learnt. This chapter builds introduces you to the idea of a *mathematical model* which uses mathematical concepts to solve real-world problems.

Definition: Mathematical Model
A mathematical model is a method of using the mathematical language to describe the behaviour of a physical system. Mathematical models are used particularly in the natural sciences and engineering disciplines (such as physics, biology, and electrical engineering) but also in the social sciences (such as economics, sociology and political science); physicists, engineers, computer scientists, and economists use mathematical models most extensively.

A mathematical model is an equation (or a set of equations for the more difficult problems) that describes are particular situation. For example, if Anna receives R3 for each time she helps her mother wash the dishes and R5 for each time she helps her father cut the grass, how much money will Anna earn if she helps her mother 5 times to wash the dishes and helps her father 2 times to wash the car. The first step to modelling is to write the equation, that describes the situation. To calculate how much Anna will earn we see that she will earn :

$$\begin{array}{rrll} & 5 & \times R3 & \text{for washing the dishes} \\ + & 2 & \times R5 & \text{for cutting the grass} \\ = & R15 & +R10 & \\ = & R25 & & \end{array}$$

If however, we say, what is the equation if Anna helps her mother x times and her father y times. Then we have:

$$\text{Total earned} = x \times R3 + y \times R5$$

25.1 Real-World Applications: Mathematical Models

Some examples of where mathematical models are used in the real-world are:

1. To model population growth
2. To model effects of air pollution
3. To model effects of global warming
4. In computer games

5. In the sciences (e.g. physics, chemistry, biology) to understand how the natural world works
6. In simulators that are used to train people in certain jobs, like pilots, doctors and soldiers
7. In medicine to track the progress of a disease

Activity :: Investigation : Simple Models

In order to get used to the idea of mathematical models, try the following simple models. Write an equation that describes the following real-world situations, mathematically:

1. Jack and Jill both have colds. Jack sneezes twice for each sneeze of Jill's. If Jill sneezes x times, write an equation describing how many times they both sneezed?
2. It rains half as much in July as it does in December. If it rains y mm in July, write an expression relating the rainfall in July and December.
3. Zane can paint a room in 4 hours. Billy can paint a room in 2 hours. How long will it take both of them to paint a room together?
4. 25 years ago, Arthur was 5 more than $\frac{1}{3}$ as old as Lee was. Today, Lee is 26 less than twice Arthur's age. How old is Lee?
5. Kevin has played a few games of ten-pin bowling. In the third game, Kevin scored 80 more than in the second game. In the first game Kevin scored 110 less than the third game. His total score for the first two games was 208. If he wants an average score of 146, what must he score on the fourth game?
6. Erica has decided to treat her friends to coffee at the Corner Coffee House. Erica paid R54,00 for four cups of cappuccino and three cups of filter coffee. If a cup of cappuccino costs R3,00 more than a cup of filter coffee, calculate how much each type of coffee costs?
7. The product of two integers is 95. Find the integers if their total is 24.

Worked Example 118: Mathematical Modelling of Falling Objects

Question: When an object is dropped or thrown downward, the distance, d, that it falls in time, t is described by the following equation:

$$s = 5t^2 + v_0 t$$

In this equation, v_0 is the initial velocity, in m·s^{-1}. Distance is measured in meters and time is measured in seconds. Use the equation to find how far an object will fall in 2 s if it is thrown downward at an initial velocity of 10 m·s^{-1}?

Answer

Step 1 : Identify what is given for each problem

We are given an expression to calculate distance travelled by a falling object in terms of initial velocity and time. We are also given the initial velocity and time and are required to calculate the distance travelled.

Step 2 : List all known and unknown information

- $v_0 = 10\,\mathrm{m\cdot s^{-1}}$
- $t = 2\,\mathrm{s}$

- $s = ?\,\text{m}$

Step 3 : Substitute values into expression

$$\begin{aligned} s &= 5t^2 + v_0 t \\ &= 5(2)^2 + (10)(2) \\ &= 5(4) + 20 \\ &= 20 + 20 \\ &= 40 \end{aligned}$$

Step 4 : Write the final answer

The object will fall 40 m in 2 s if it is thrown downward at an initial velocity of 10 m·s^{-1}.

Worked Example 119: Another Mathematical Modelling of Falling Objects

Question: When an object is dropped or thrown downward, the distance, d, that it falls in time, t is described by the following equation:

$$s = 5t^2 + v_0 t$$

In this equation, v_0 is the initial velocity, in m·s^{-1}. Distance is measured in meters and time is measured in seconds. Use the equation find how long it takes for the object to reach the ground if it is dropped from a height of 2000 m. The initial velocity is 0 m·s^{-1}?

Answer

Step 1 : Identify what is given for each problem

We are given an expression to calculate distance travelled by a falling object in terms of initial velocity and time. We are also given the initial velocity and time and are required to calculate the distance travelled.

Step 2 : List all known and unknown information

- $v_0 = 0\,\text{m}\cdot\text{s}^{-1}$
- $t = ?\,\text{s}$
- $s = 2000\,\text{m}$

Step 3 : Substitute values into expression

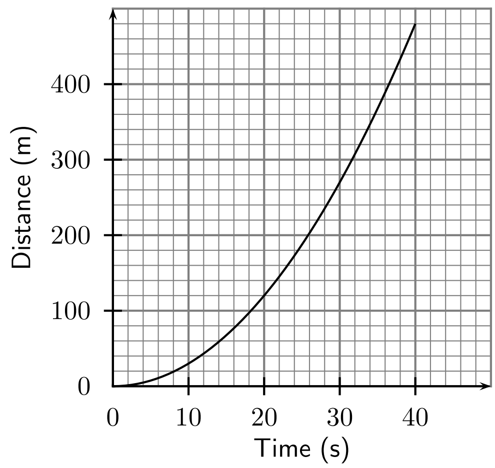
$$\begin{aligned} s &= 5t^2 + v_0 t \\ 2000 &= 5t^2 + (0)(2) \\ 2000 &= 5t^2 \\ t^2 &= \frac{2000}{5} \\ &= 400 \\ \therefore \quad t &= 20\,\text{s} \end{aligned}$$

Step 4 : Write the final answer

The object will take 20 s to reach the ground if it is dropped from a height of 2000 m.

Activity :: Investigation : Mathematical Modelling

The graph below shows the how the distance travelled by a car depends on time. Use the graph to answer the following questions.



1. How far does the car travel in 20 s?
2. How long does it take the car to travel 300 m?

Worked Example 120: More Mathematical Modelling

Question: A researcher is investigating the number of trees in a forest over a period of n years. After investigating numerous data, the following data model emerged:

Year	Number of trees in hundreds
1	1
2	3
3	9
4	27

1. How many trees, in hundreds, are there in the SIXTH year if this pattern is continued?
2. Determine an algebraic expression that describes the number of trees in the n^th year in the forest.
3. Do you think this model, which determines the number of trees in the forest, will continue indefinitely? Give a reason for your answer.

Answer

Step 1 : Find the pattern

The pattern is $3^0; 3^1; 3^2; 3^3; ...$

Therefore, three to the power one less than the year.

Step 2 : Trees in year 6

$$year6 = hundreds = 243hundreds = 24300$$

Step 3 : Algebraic expression for year n

$$number\ of\ trees = 3^{n-1}\ hundreds$$

Step 4 : Conclusion

No

The number of trees will increase without bound to very large numbers, thus the forestry authorities will if necessary cut down some of the trees from time to time.

Worked Example 121: Setting up an equation

Question: Currently the subsription to a gym for a single member is R1 000 annually while family membership is R1 500. The gym is considering raising all membershipfees by the same amount. If this is done then the single membership will cost $\frac{5}{7}$ of the family membership. Determine the proposed increase.

Answer

Step 1 : Summarise the information in a table

Let the proposed increase be x.

	Now	After increase
Single	1 000	1 000+x
Family	1 500	1 500+x

Step 2 : Set up an equation

$$1\ 000 + x = \frac{5}{7}(1\ 500 + x)$$

Step 3 : Solve the equation

$$\begin{aligned} 7\ 000 + 7x &= 7\ 500 + 5x \\ 2x &= 500 \\ x &= 250 \end{aligned}$$

Step 4 : Write down the answer

Therefore the increase is R250.

25.2 End of Chatpter Exercises

1. When an object is dropped or thrown downward, the distance, d, that it falls in time, t is described by the following equation:

   $$s = 5t^2 + v_0 t$$

   In this equation, v_0 is the initial velocity, in m·s^{-1}. Distance is measured in meters and time is measured in seconds. Use the equation to find how long it takes a tennis ball to reach the ground if it is thrown downward from a hot-air balloon that is 500 m high. The tennis ball is thrown at an initial velocity of 5 m·s^{-1}.

2. The table below lists the times that Sheila takes to walk the given distances.

Time (minutes)	5	10	15	20	25	30
Distance (km)	1	2	3	4	5	6

   Plot the points.

   If the relationship between the distances and times are linear, find the equation of the straight line, using any two points. Then use the equation to answer the following questions:

A How long will it take Sheila to walk 21 km?

B How far will Sheila walk in 7 minutes?

If Sheila were to walk half as fast as she is currently walking, what would the graph of her distances and times look like?

3. The power P (in watts) supplied to a circuit by a 12 volt battery is given by the formula $P = 12I - 0{,}5I^2$ where I is the current in amperes.

   A Since both power and current must be greater than 0, find the limits of the current that can be drawn by the circuit.

   B Draw a graph of $P = 12I - 0{,}5I^2$ and use your answer to the first question, to define the extent of the graph.

   C What is the maximum current that can be drawn?

   D From your graph, read off how much power is supplied to the circuit when the current is 10 amperes? Use the equation to confirm your answer.

   E At what value of current will the power supplied be a maximum?

4. You are in the lobby of a business building waiting for the lift. You are late for a meeting and wonder if it will be quicker to take the stairs. There is a fascinating relationship between the number of floors in the building, the number of people in the lift and how often it will stop:

   If N people get into a lift at the lobby and the number of floors in the building is F, then the lift can be expected to stop

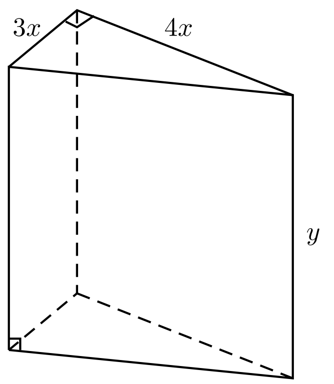
$$F - F\left(\frac{F-1}{F}\right)^N$$

   times.

   A If the building has 16 floors and there are 9 people who get into the lift, how many times is the lift expected to stop?

   B How many people would you expect in a lift, if it stopped 12 times and there are 17 floors?

5. A wooden block is made as shown in the diagram. The ends are right-angled triangles having sides $3x$, $4x$ and $5x$. The length of the block is y. The total surface area of the block is 3 600 cm^2.



Show that

$$y = \frac{300 - x^2}{x}$$

6. A stone is thrown vertically upwards and its height (in metres) above the ground at time t (in seconds) is given by:

$$h(t) = 35 - 5t^2 + 30t$$

   Find its initial height above the ground.

7. After doing some research, a transport company has determined that the rate at which petrol is consumed by one of its large carriers, travelling at an average speed of x km per hour, is given by:

$$P(x) = \frac{55}{2x} + \frac{x}{200} \quad \text{litres per kilometre}$$

   Assume that the petrol costs R4,00 per litre and the driver earns R18,00 per hour (travelling time). Now deduce that the total cost, C, in Rands, for a 2 000 km trip is given by:

$$C(x) = \frac{256000}{x} + 40x$$

8. During an experiment the temperature T (in degrees Celsius), varies with time t (in hours), according to the formula:

$$T(t) = 30 + 4t - \frac{1}{2}t^2 \quad t \in [1; 10]$$

   A Determine an expression for the rate of change of temperature with time.

   B During which time interval was the temperature dropping?

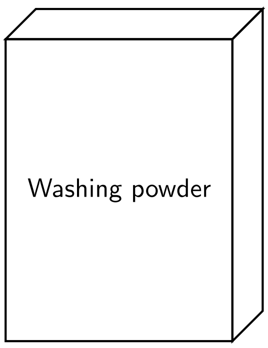
9. In order to reduce the temperature in a room from 28°C, a cooling system is allowed to operate for 10 minutes. The room temperature, T after t minutes is given in °C by the formula:

$$T = 28 - 0{,}008t^3 - 0{,}16t \quad \text{where } t \in [0; 10]$$

   A At what rate (rounded off to TWO decimal places) is the temperature falling when $t = 4$ minutes?

   B Find the lowest room temperature reached during the 10 minutes for which the cooling system operates, by drawing a graph.

10. A washing powder box has the shape of a rectangular prism as shown in the diagram below. The box has a volume of 480 cm^3, a breadth of 4 cm and a length of x cm.



   Show that the total surface area of the box (in cm^2) is given by:

$$A = 8x + 960x^{-1} + 240$$

Extension: Simulations

A simulation is an attempt to model a real-life situation on a computer so that it can be studied to see how the system works. By changing variables, predictions may be made about the behaviour of the system. Simulation is used in many contexts, including the modeling of natural systems or human systems in order to gain insight into their functioning. Other contexts include simulation of technology for performance optimization, safety engineering, testing, training and education. Simulation can be used to show the eventual real effects of alternative conditions and courses of action. **Simulation in education** Simulations in education are somewhat

like training simulations. They focus on specific tasks. In the past, video has been used for teachers and education students to observe, problem solve and role play; however, a more recent use of simulations in education include animated narrative vignettes (ANV). ANVs are cartoon-like video narratives of hypothetical and reality-based stories involving classroom teaching and learning. ANVs have been used to assess knowledge, problem solving skills and dispositions of children, and pre-service and in-service teachers.

Chapter 26

Quadratic Functions and Graphs - Grade 11

26.1 Introduction

In Grade 10, you studied graphs of many different forms. In this chapter, you will learn a little more about the graphs of functions.

26.2 Functions of the Form $y = a(x + p)^2 + q$

This form of the quadratic function is slightly more complex than the form studied in Grade 10, $y = ax^2 + q$. The general shape and position of the graph of the function of the form $f(x) = a(x + p)^2 + q$ is shown in Figure 26.1.

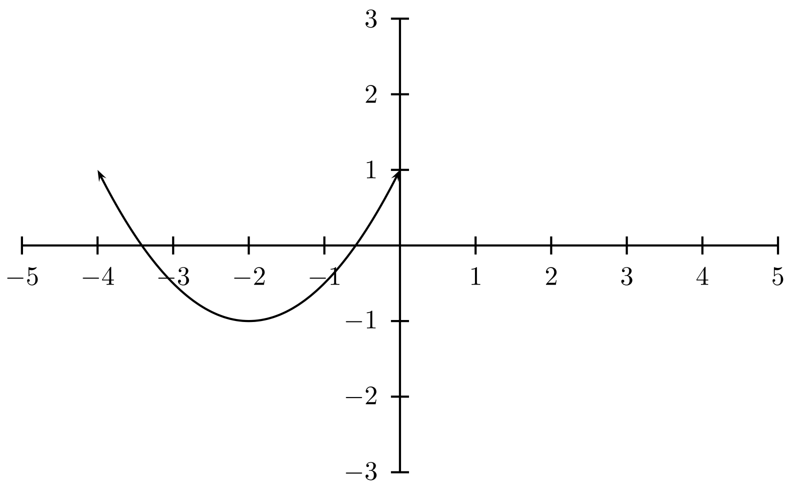


Figure 26.1: Graph of $f(x) = \frac{1}{2}(x + 2)^2 - 1$

Activity :: Investigation : Functions of the Form $y = a(x + p)^2 + q$

1. On the same set of axes, plot the following graphs:
   A $a(x) = (x - 2)^2$
   B $b(x) = (x - 1)^2$
   C $c(x) = x^2$
   D $d(x) = (x + 1)^2$

E $e(x) = (x+2)^2$

Use your results to deduce the effect of p.

2. On the same set of axes, plot the following graphs:
   A $f(x) = (x-2)^2 + 1$
   B $g(x) = (x-1)^2 + 1$
   C $h(x) = x^2 + 1$
   D $j(x) = (x+1)^2 + 1$
   E $k(x) = (x+2)^2 + 1$

   Use your results to deduce the effect of q.
3. Following the general method of the above activities, choose your own values of p and q to plot 5 different graphs (on the same set of axes) of $y = a(x+p)^2 + q$ to deduce the effect of a.

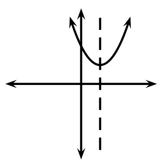
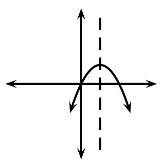
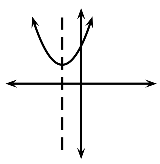
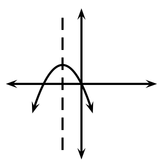
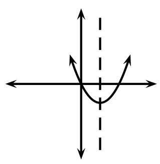
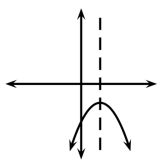
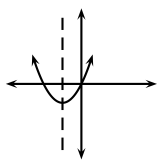
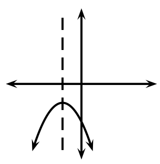
From your graphs, you should have found that a affects whether the graph makes a smile or a frown. If $a < 0$, the graph makes a frown and if $a > 0$ then the graph makes a smile. This is shown in Figure 10.9.

You should have also found that the value of p affects whether the turning point of the graph is above the x-axis ($p < 0$) or below the x-axis ($p > 0$).

You should have also found that the value of q affects whether the turning point is to the left of the y-axis ($q > 0$) or to the right of the y-axis ($q < 0$).

These different properties are summarised in Table 26.1. The axes of symmetry for each graph is shown as a dashed line.

Table 26.1: Table summarising general shapes and positions of functions of the form $y = a(x+p)^2 + q$. The axes of symmetry are shown as dashed lines.

	$p < 0$		$p > 0$	
	$a > 0$	$a < 0$	$a > 0$	$a < 0$
$q \geq 0$				
$q \leq 0$				

26.2.1 Domain and Range

For $f(x) = a(x+p)^2 + q$, the domain is $\{x : x \in \mathbb{R}\}$ because there is no value of $x \in \mathbb{R}$ for which $f(x)$ is undefined.

The range of $f(x) = a(x+p)^2 + q$ depends on whether the value for a is positive or negative. We will consider these two cases separately.

If $a > 0$ then we have:

$$\begin{aligned}
(x+p)^2 &\geq 0 \quad \text{(The square of an expression is always positive)} \\
a(x+p)^2 &\geq 0 \quad \text{(Multiplication by a positive number maintains the nature of the inequality)} \\
a(x+p)^2 + q &\geq q \\
f(x) &\geq q
\end{aligned}$$

This tells us that for all values of x, $f(x)$ is always greater than q. Therefore if $a > 0$, the range of $f(x) = a(x+p)^2 + q$ is $\{f(x) : f(x) \in [q,\infty)\}$.

Similarly, it can be shown that if $a < 0$ that the range of $f(x) = a(x+p)^2+q$ is $\{f(x) : f(x) \in (-\infty,q]\}$. This is left as an exercise.

For example, the domain of $g(x) = (x-1)^2 + 2$ is $\{x : x \in \mathbb{R}\}$ because there is no value of $x \in \mathbb{R}$ for which $g(x)$ is undefined. The range of $g(x)$ can be calculated as follows:

$$\begin{aligned} (x-p)^2 &\geq 0 \\ (x+p)^2 + 2 &\geq 2 \\ g(x) &\geq 2 \end{aligned}$$

Therefore the range is $\{g(x) : g(x) \in [2,\infty)\}$.

Exercise: Domain and Range

1. Given the function $f(x) = (x-4)^2 - 1$. Give the range of $f(x)$.
2. What is the domain the equation $y = 2x^2 - 5x - 18$?

26.2.2 Intercepts

For functions of the form, $y = a(x+p)^2 + q$, the details of calculating the intercepts with the x and y axis is given.

The y-intercept is calculated as follows:

$$y = a(x+p)^2 + q \tag{26.1}$$
$$y_{int} = a(0+p)^2 + q \tag{26.2}$$
$$= ap^2 + q \tag{26.3}$$

If $p = 0$, then $y_{int} = q$.

For example, the y-intercept of $g(x) = (x-1)^2 + 2$ is given by setting $x = 0$ to get:

$$\begin{aligned} g(x) &= (x-1)^2 + 2 \\ y_{int} &= (0-1)^2 + 2 \\ &= (-1)^2 + 2 \\ &= 1 + 2 \\ &= 3 \end{aligned}$$

The x-intercepts are calculated as follows:

$$y = a(x+p)^2 + q \tag{26.4}$$
$$0 = a(x_{int}+p)^2 + q \tag{26.5}$$
$$a(x_{int}+p)^2 = -q \tag{26.6}$$
$$x_{int} + p = \sqrt{-\frac{q}{a}} \tag{26.7}$$
$$x_{int} = \pm\sqrt{-\frac{q}{a}} - p \tag{26.8}$$

However, (26.8) is only valid if $-\frac{q}{a} > 0$ which means that either $q < 0$ or $a < 0$. This is consistent with what we expect, since if $q > 0$ and $a > 0$ then $-\frac{q}{a}$ is negative and in this case the graph lies above the x-axis and therefore does not intersect the x-axis. If however, $q > 0$ and $a < 0$, then $-\frac{q}{a}$ is positive and the graph is hat shaped and should have two x-intercepts. Similarly, if $q < 0$ and $a > 0$ then $-\frac{q}{a}$ is also positive, and the graph should intersect with the x-axis.

For example, the x-intercepts of $g(x) = (x-1)^2 + 2$ is given by setting $y = 0$ to get:

$$\begin{aligned} g(x) &= (x-1)^2 + 2 \\ 0 &= (x_{int} - 1)^2 + 2 \\ -2 &= (x_{int} - 1)^2 \end{aligned}$$

which is not real. Therefore, the graph of $g(x) = (x-1)^2 + 2$ does not have any x-intercepts.

?

Exercise: Intercepts

1. Find the x- and y-intercepts of the function $f(x) = (x-4)^2 - 1$.
2. Find the intercepts with both axes of the graph of $f(x) = x^2 - 6x + 8$.
3. Given: $f(x) = -x^2 + 4x - 3$. Calculate the x- and y-intercepts of the graph of f.

26.2.3 Turning Points

The turning point of the function of the form $f(x) = a(x+p)^2 + q$ is given by examining the range of the function. We know that if $a > 0$ then the range of $f(x) = a(x+p)^2 + q$ is $\{f(x) : f(x) \in [q,\infty)\}$ and if $a < 0$ then the range of $f(x) = a(x+p)^2 + q$ is $\{f(x) : f(x) \in (-\infty,q]\}$.

So, if $a > 0$, then the lowest value that $f(x)$ can take on is q. Solving for the value of x at which $f(x) = q$ gives:

$$\begin{aligned} q &= a(x+p)^2 + q \\ 0 &= a(x+p)^2 \\ 0 &= (x+p)^2 \\ 0 &= x+p \\ x &= -p \end{aligned}$$

$\therefore x = -p$ at $f(x) = q$. The co-ordinates of the (minimal) turning point is therefore $(-p,q)$.

Similarly, if $a < 0$, then the highest value that $f(x)$ can take on is q and the co-ordinates of the (maximal) turning point is $(-p,q)$.

?

Exercise: Turning Points

1. Determine the turning point of the graph of $f(x) = x^2 - 6x + 8$.
2. Given: $f(x) = -x^2 + 4x - 3$. Calculate the co-ordinates of the turning point of f.
3. Find the turning point of the following function by completing the square: $y = \frac{1}{2}(x+2)^2 - 1$.

26.2.4 Axes of Symmetry

There is one axis of symmetry for the function of the form $f(x) = a(x+p)^2 + q$ that passes through the turning point and is parallel to the y-axis. Since the x-coordinate of the turning point is $x = -p$, this is the axis of symmetry.

?

Exercise: Axes of Symmetry

1. Find the equation of the axis of symmetry of the graph $y = 2x^2 - 5x - 18$
2. Write down the equation of the axis of symmetry of the graph of $y = 3(x - 2)^2 + 1$
3. Write down an example of an equation of a parabola where the y-axis is the axis of symmetry.

26.2.5 Sketching Graphs of the Form $f(x) = a(x+p)^2 + q$

In order to sketch graphs of the form, $f(x) = a(x+p)^2 + q$, we need to calculate determine four characteristics:

1. sign of a
2. domain and range
3. turning point
4. y-intercept
5. x-intercept

For example, sketch the graph of $g(x) = -\frac{1}{2}(x+1)^2 - 3$. Mark the intercepts, turning point and axis of symmetry.

Firstly, we determine that $a < 0$. This means that the graph will have a maximal turning point.

The domain of the graph is $\{x : x \in \mathbb{R}\}$ because $f(x)$ is defined for all $x \in \mathbb{R}$. The range of the graph is determined as follows:

$$\begin{aligned} (x+1)^2 &\geq 0 \\ -\frac{1}{2}(x+1)^2 &\leq 0 \\ -\frac{1}{2}(x+1)^2 - 3 &\leq -3 \\ \therefore f(x) &\leq -3 \end{aligned}$$

Therefore the range of the graph is $\{f(x) : f(x) \in (-\infty, -3]\}$.

Using the fact that the maximum value that $f(x)$ achieves is -3, then the y-coordinate of the turning point is -3. The x-coordinate is determined as follows:

$$\begin{aligned} -\frac{1}{2}(x+1)^2 - 3 &= -3 \\ -\frac{1}{2}(x+1)^2 - 3 + 3 &= 0 \\ -\frac{1}{2}(x+1)^2 &= 0 \\ \text{Divide both sides by } -\tfrac{1}{2}\text{:} \quad (x+1)^2 &= 0 \\ \text{Take square root of both sides:} \quad x+1 &= 0 \\ \therefore \quad x &= -1 \end{aligned}$$

The coordinates of the turning point are: $(-1, -3)$.

The y-intercept is obtained by setting $x = 0$. This gives:

$$\begin{aligned} y_{int} &= -\frac{1}{2}(0+1)^2 - 3 \\ &= -\frac{1}{2}(1) - 3 \\ &= -\frac{1}{2} - 3 \\ &= -\frac{1}{2} - 3 \\ &= -\frac{7}{2} \end{aligned}$$

The x-intercept is obtained by setting $y = 0$. This gives:

$$\begin{aligned} 0 &= -\frac{1}{2}(x_{int}+1)^2 - 3 \\ 3 &= -\frac{1}{2}(x_{int}+1)^2 \\ -3 \cdot 2 &= (x_{int}+1)^2 \\ -6 &= (x_{int}+1)^2 \end{aligned}$$

which is not real. Therefore, there are no x-intercepts.

We also know that the axis of symmetry is parallel to the y-axis and passes through the turning point.

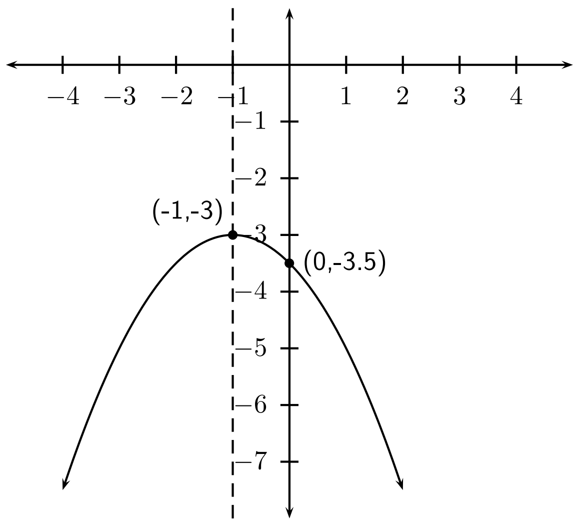


Figure 26.2: Graphs of the function $f(x) = -\frac{1}{2}(x_{int}+1)^2 - 3$

?

Exercise: Sketching the Parabola

1. Draw the graph of $y = 3(x-2)^2 + 1$ showing all the relative intercepts with the axes as well as the coordinates of the turning point.
2. Draw a neat sketch graph of the function defined by $y = ax^2 + bx + c$ if $a > 0$; $b < 0$; $b^2 = 4ac$.

26.2.6 Writing an equation of a shifted parabola

Given a parabola with equation $y = x^2 - 2x - 3$. The graph of the parabola is shifted one unit to the right. Or else the y-axis shifts one unit to the left. Therefore the new equation will become:

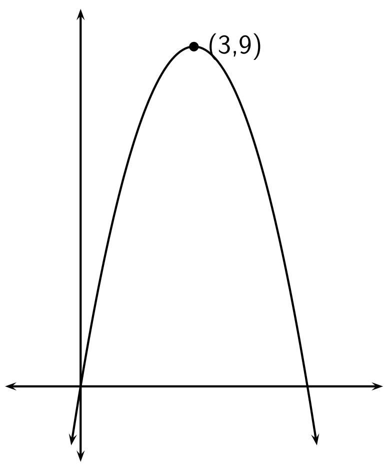
$$\begin{aligned} y &= (x-1)^2 - 2(x-1) - 3 \\ &= x^2 - 2x + 1 - 2x + 2 - 3 \\ &= x^2 - 4x \end{aligned}$$

If the given parabola is shifted 3 units down, the new equation will become:
(Notice the x-axis then moves 3 units upwards)

$$\begin{aligned} y + 3 &= x^2 - 2x - 3 \\ y &= x^2 - 2x - 6 \end{aligned}$$

26.3 End of Chapter Exercises

1. Show that if $a < 0$, then the range of $f(x) = a(x+p)^2 + q$ is $\{f(x) : f(x) \in (-\infty,q]\}$.
2. If (2;7) is the turning point of $f(x) = -2x^2 - 4ax + k$, find the values of the constants a and k.
3. The graph in the figure is represented by the equation $f(x) = ax^2 + bx$. The coordinates of the turning point are (3;9). Show that $a = -1$ and $b = 6$.



4. Given: $f : x = x^2 - 2x3$. Give the equation of the new graph originating if:
   A The graph of f is moved three units to the left.
   B The x - axis is moved down three.
5. A parabola with turning point (-1; -4) is shifted vertically by 4 units upwards. What are the coordinates of the turning point of the shifted parabola ?

Chapter 27

Hyperbolic Functions and Graphs - Grade 11

27.1 Introduction

In Grade 10, you studied graphs of many different forms. In this chapter, you will learn a little more about the graphs of functions.

27.2 Functions of the Form $y = \frac{a}{x+p} + q$

This form of the hyperbolic function is slightly more complex than the form studied in Grade 10.

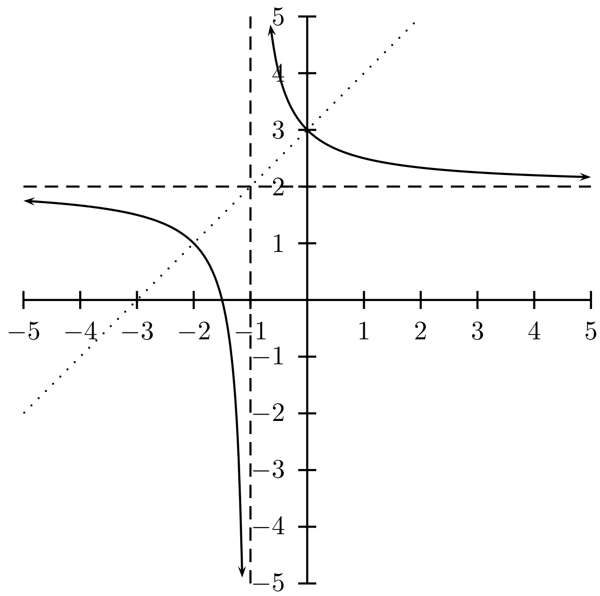


Figure 27.1: General shape and position of the graph of a function of the form $f(x) = \frac{a}{x+p} + q$. The asymptotes are shown as dashed lines.

Activity :: Investigation : Functions of the Form $y = \frac{a}{x+p} + q$

1. On the same set of axes, plot the following graphs:

A $a(x) = \frac{-2}{x+1} + 1$
B $b(x) = \frac{-1}{x+1} + 1$
C $c(x) = \frac{0}{x+1} + 1$
D $d(x) = \frac{+1}{x+1} + 1$
E $e(x) = \frac{+2}{x+1} + 1$

Use your results to deduce the effect of a.

2. On the same set of axes, plot the following graphs:

A $f(x) = \frac{1}{x-2} + 1$
B $g(x) = \frac{1}{x-1} + 1$
C $h(x) = \frac{1}{x+0} + 1$
D $j(x) = \frac{1}{x+1} + 1$
E $k(x) = \frac{1}{x+2} + 1$

Use your results to deduce the effect of p.

3. Following the general method of the above activities, choose your own values of a and p to plot 5 different graphs of $y = \frac{a}{x+p} + q$ to deduce the effect of q.

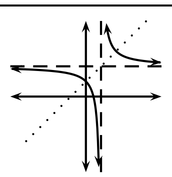
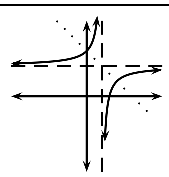
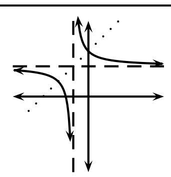
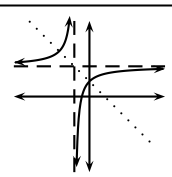
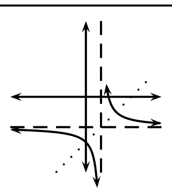
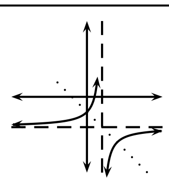
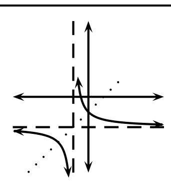
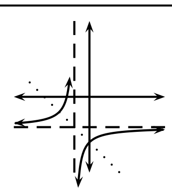
You should have found that the value of a affects whether the graph is located in the first and third quadrants of Cartesian plane.

You should have also found that the value of p affects whether the x-intercept is negative ($p > 0$) or positive ($p < 0$).

You should have also found that the value of q affects whether the graph lies above the x-axis ($q > 0$) or below the x-axis ($q < 0$).

These different properties are summarised in Table 27.1. The axes of symmetry for each graph is shown as a dashed line.

Table 27.1: Table summarising general shapes and positions of functions of the form $y = \frac{a}{x+p} + q$. The axes of symmetry are shown as dashed lines.

	$p < 0$		$p > 0$	
	$a > 0$	$a < 0$	$a > 0$	$a < 0$
$q > 0$				
$q < 0$				

27.2.1 Domain and Range

For $y = \frac{a}{x+p} + q$, the function is undefined for $x = -p$. The domain is therefore $\{x : x \in \mathbb{R}, x \neq -p\}$.

We see that $y = \frac{a}{x+p} + q$ can be re-written as:

$$\begin{aligned} y &= \frac{a}{x+p} + q \\ y - q &= \frac{a}{x+p} \\ \text{If } x \neq -p \text{ then:} \quad (y-q)(x+p) &= a \\ x + p &= \frac{a}{y-q} \end{aligned}$$

This shows that the function is undefined at $y = q$. Therefore the range of $f(x) = \frac{a}{x+p} + q$ is $\{f(x) : f(x) \in (-\infty,q) \cup (q,\infty)\}$.

For example, the domain of $g(x) = \frac{2}{x+1} + 2$ is $\{x : x \in \mathbb{R}, x \neq -1\}$ because $g(x)$ is undefined at $x = -1$.

$$\begin{aligned} y &= \frac{2}{x+1} + 2 \\ (y-2) &= \frac{2}{x+1} \\ (y-2)(x+1) &= 2 \\ (x+1) &= \frac{2}{y-2} \end{aligned}$$

We see that $g(x)$ is undefined at $y = 2$. Therefore the range is $\{g(x) : g(x) \in (-\infty,2) \cup (2,\infty)\}$.

?

Exercise: Domain and Range

Determine the range of $y = \frac{1}{x} + 1$.

Given:$f(x) = \frac{8}{x-8} + 4$. Write down the domain of f.

Determine the domain of $y = -\frac{8}{x+1} + 3$

27.2.2 Intercepts

For functions of the form, $y = \frac{a}{x+p} + q$, the intercepts with the x and y axis is calculated by setting $x = 0$ for the y-intercept and by setting $y = 0$ for the x-intercept.

The y-intercept is calculated as follows:

$$y = \frac{a}{x+p} + q \quad (27.1)$$

$$y_{int} = \frac{a}{0+p} + q \quad (27.2)$$

$$= \frac{a}{p} + q \quad (27.3)$$

For example, the y-intercept of $g(x) = \frac{2}{x+1} + 2$ is given by setting $x = 0$ to get:

$$\begin{aligned} y &= \frac{2}{x+1} + 2 \\ y_{int} &= \frac{2}{0+1} + 2 \\ &= \frac{2}{1} + 2 \\ &= 2 + 2 \\ &= 4 \end{aligned}$$

The x-intercepts are calculated by setting $y = 0$ as follows:

$$y = \frac{a}{x+p} + q \tag{27.4}$$

$$0 = \frac{a}{x_{int}+p} + q \tag{27.5}$$

$$\frac{a}{x_{int}+p} = -q \tag{27.6}$$

$$a = -q(x_{int}+p) \tag{27.7}$$

$$x_{int}+p = \frac{a}{-q} \tag{27.8}$$

$$x_{int} = \frac{a}{-q} - p \tag{27.9}$$

For example, the x-intercept of $g(x) = \frac{2}{x+1} + 2$ is given by setting $x = 0$ to get:

$$\begin{aligned} y &= \frac{2}{x+1} + 2 \\ 0 &= \frac{2}{x_{int}+1} + 2 \\ -2 &= \frac{2}{x_{int}+1} \\ -2(x_{int}+1) &= 2 \\ x_{int}+1 &= \frac{2}{-2} \\ x_{int} &= -1 - 1 \\ x_{int} &= -2 \end{aligned}$$

?

Exercise: Intercepts

Given:$h(x) = \frac{1}{x+4} - 2$. Determine the coordinates of the intercepts of h with the x- and y-axes.

Determine the x-intercept of the graph of $y = \frac{5}{x} + 2$. Give a reason why there is no y-intercept for this function.

27.2.3 Asymptotes

There are two asymptotes for functions of the form $y = \frac{a}{x+p} + q$. They are determined by examining the domain and range.

We saw that the function was undefined at $x = -p$ and for $y = q$. Therefore the asymptotes are $x = -p$ and $y = q$.

For example, the domain of $g(x) = \frac{2}{x+1} + 2$ is $\{x : x \in \mathbb{R}, x \neq -1\}$ because $g(x)$ is undefined at $x = -1$. We also see that $g(x)$ is undefined at $y = 2$. Therefore the range is $\{g(x) : g(x) \in (-\infty,2) \cup (2,\infty)\}$.

From this we deduce that the asymptotes are at $x = -1$ and $y = 2$.

?

Exercise: Asymptotes

Given:$h(x) = \frac{1}{x+4} - 2$.Determine the equations of the asymptotes of h.

Write down the equation of the vertical asymptote of the graph $y = \frac{1}{x-1}$.

27.2.4 Sketching Graphs of the Form $f(x) = \frac{a}{x+p} + q$

In order to sketch graphs of functions of the form, $f(x) = \frac{a}{x+p} + q$, we need to calculate determine four characteristics:

1. domain and range
2. asymptotes
3. y-intercept
4. x-intercept

For example, sketch the graph of $g(x) = \frac{2}{x+1} + 2$. Mark the intercepts and asymptotes.

We have determined the domain to be $\{x : x \in \mathbb{R}, x \neq -1\}$ and the range to be $\{g(x) : g(x) \in (-\infty,2) \cup (2,\infty)\}$. Therefore the asymptotes are at $x = -1$ and $y = 2$.

The y-intercept is $y_{int} = 4$ and the x-intercept is $x_{int} = -2$.

?

Exercise: Graphs

1. Draw the graph of $y = \frac{1}{x} + 2$. Indicate the new horizontal asymptote.
2. Given:$h(x) = \frac{1}{x+4} - 2$. Sketch the graph of h showing clearly the asymptotes and ALL intercepts with the axes.
3. Draw the graph of $y = \frac{1}{x}$ and $y = -\frac{8}{x+1} + 3$ on the same system of axes.
4. Draw the graph of $y = \frac{5}{x-2,5} + 2$. Explain your method.
5. Draw the graph of the function defined by $y = \frac{8}{x-8} + 4$. Indicate the asymptotes and intercepts with the axes.

27.3 End of Chapter Exercises

1. Plot the graph of the hyperbola defined by $y = \frac{2}{x}$ for $-4 \leq x \leq 4$. Suppose the hyperbola is shifted 3 units to the right and 1 unit down. What is the new equation then ?
2. Based on the graph of $y = \frac{1}{x}$, determine the equation of the graph with asymptotes $y = 2$ and $x = 1$ and passing through the point (2; 3).

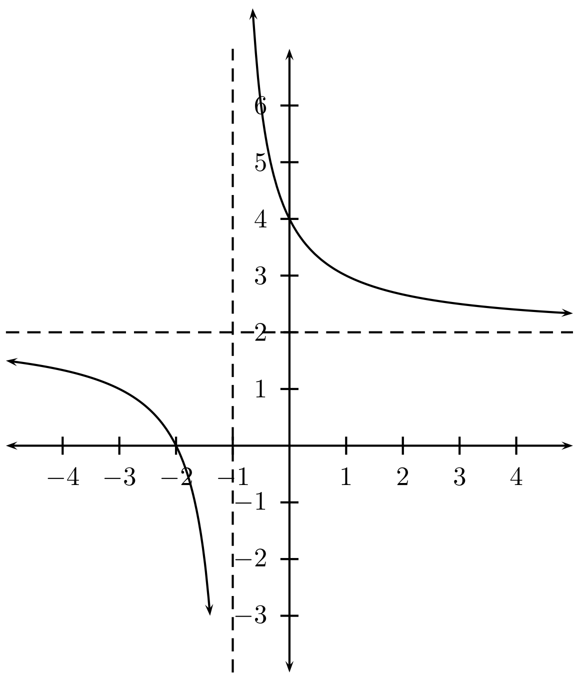


Figure 27.2: Graph of $g(x) = \frac{2}{x+1} + 2$.

Chapter 28

Exponential Functions and Graphs - Grade 11

28.1 Introduction

In Grade 10, you studied graphs of many different forms. In this chapter, you will learn a little more about the graphs of exponential functions.

28.2 Functions of the Form $y = ab^{(x+p)} + q$

This form of the exponential function is slightly more complex than the form studied in Grade 10.

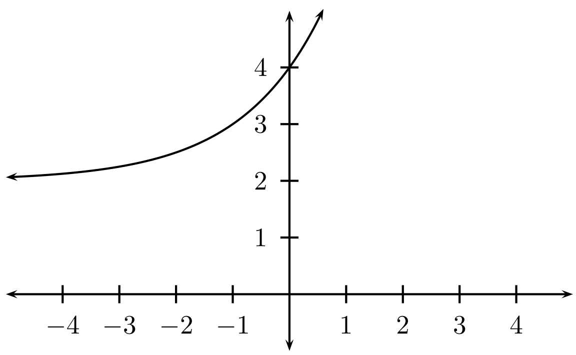


Figure 28.1: General shape and position of the graph of a function of the form $f(x) = ab^{(x+p)} + q$.

Activity :: Investigation : Functions of the Form $y = ab^{(x+p)} + q$

1. On the same set of axes, plot the following graphs:
   A $a(x) = -2 \cdot b^{(x+1)} + 1$
   B $b(x) = -1 \cdot b^{(x+1)} + 1$
   C $c(x) = -0 \cdot b^{(x+1)} + 1$
   D $d(x) = -1 \cdot b^{(x+1)} + 1$
   E $e(x) = -2 \cdot b^{(x+1)} + 1$

   Use your results to deduce the effect of a.
2. On the same set of axes, plot the following graphs:

A $f(x) = 1 \cdot b^{(x+1)} - 2$
B $g(x) = 1 \cdot b^{(x+1)} - 1$
C $h(x) = 1 \cdot b^{(x+1)}0$
D $j(x) = 1 \cdot b^{(x+1)} + 1$
E $k(x) = 1 \cdot b^{(x+1)} + 2$

Use your results to deduce the effect of q.

3. Following the general method of the above activities, choose your own values of a and q to plot 5 different graphs of $y = ab^{(x+p)} + q$ to deduce the effect of p.

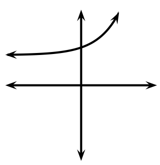
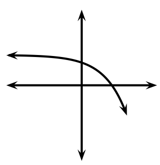
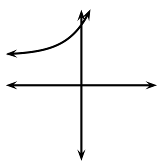
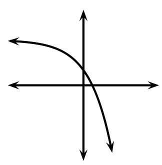
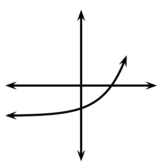
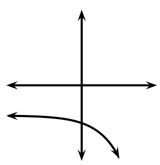
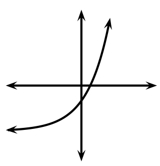
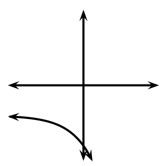
You should have found that the value of a affects whether the graph curves upwards ($a > 0$) or curves downwards ($a < 0$).

You should have also found that the value of p affects the position of the x-intercept.

You should have also found that the value of q affects the position of the y-intercept.

These different properties are summarised in Table 28.1. The axes of symmetry for each graph is shown as a dashed line.

Table 28.1: Table summarising general shapes and positions of functions of the form $y = ab^{(x+p)} + q$.

	$p < 0$		$p > 0$	
	$a > 0$	$a < 0$	$a > 0$	$a < 0$
$q > 0$				
$q < 0$				

28.2.1 Domain and Range

For $y = ab^{(x+p)} + q$, the function is defined for all real values of x. Therefore, the domain is $\{x : x \in \mathbb{R}\}$.

The range of $y = ab^{(x+p)} + q$ is dependent on the sign of a.

If $a > 0$ then:

$$\begin{aligned} b^{(x+p)} &\geq 0 \\ a \cdot b^{(x+p)} &\geq 0 \\ a \cdot b^{(x+p)} + q &\geq q \\ f(x) &\geq q \end{aligned}$$

Therefore, if $a > 0$, then the range is $\{f(x) : f(x) \in [q,\infty)\}$.

If $a < 0$ then:

$$\begin{aligned} b^{(x+p)} &\leq 0 \\ a \cdot b^{(x+p)} &\leq 0 \\ a \cdot b^{(x+p)} + q &\leq q \\ f(x) &\leq q \end{aligned}$$

Therefore, if $a < 0$, then the range is $\{f(x) : f(x) \in (-\infty,q]\}$.

For example, the domain of $g(x) = 3 \cdot 2^{x+1} + 2$ is $\{x : x \in \mathbb{R}\}$. For the range,

$$\begin{aligned} 2^{x+1} &\geq 0 \\ 3 \cdot 2^{x+1} &\geq 0 \\ 3 \cdot 2^{x+1} + 2 &\geq 2 \end{aligned}$$

Therefore the range is $\{g(x) : g(x) \in [2,\infty)\}$.

Exercise: Domain and Range

1. Give the domain of $y = 3^x$.
2. What is the domain and range of $f(x) = 2^x$?
3. Determine the domain and range of $y = (1,5)^{x+3}$.

28.2.2 Intercepts

For functions of the form, $y = ab^{(x+p)} + q$, the intercepts with the x and y axis is calulated by setting $x = 0$ for the y-intercept and by setting $y = 0$ for the x-intercept.

The y-intercept is calculated as follows:

$$\begin{aligned} y &= ab^{(x+p)} + q && (28.1) \\ y_{int} &= ab^{(0+p)} + q && (28.2) \\ &= ab^p + q && (28.3) \end{aligned}$$

For example, the y-intercept of $g(x) = 3 \cdot 2^{x+1} + 2$ is given by setting $x = 0$ to get:

$$\begin{aligned} y &= 3 \cdot 2^{x+1} + 2 \\ y_{int} &= 3 \cdot 2^{0+1} + 2 \\ &= 3 \cdot 2^1 + 2 \\ &= 3 \cdot 2 + 2 \\ &= 8 \end{aligned}$$

The x-intercepts are calculated by setting $y = 0$ as follows:

$$\begin{aligned} y &= ab^{(x+p)} + q && (28.4) \\ 0 &= ab^{(x_{int}+p)} + q && (28.5) \\ ab^{(x_{int}+p)} &= -q && (28.6) \\ b^{(x_{int}+p)} &= -\frac{q}{a} && (28.7) \end{aligned}$$

Which only has a real solution if either $a < 0$ or $Q < 0$. Otherwise, the graph of the function of form $y = ab^{(x+p)} + q$ does not have any x-intercepts.

For example, the x-intercept of $g(x) = 3 \cdot 2^{x+1} + 2$ is given by setting $x = 0$ to get:

$$\begin{aligned} y &= 3 \cdot 2^{x+1} + 2 \\ 0 &= 3 \cdot 2^{x_{int}+1} + 2 \\ -2 &= 3 \cdot 2^{x_{int}+1} \\ 2^{x_{int}+1} &= \frac{-2}{2} \end{aligned}$$

which has no real solution. Therefore, the graph of $g(x) = 3 \cdot 2^{x+1} + 2$ does not have any x-intercepts.

?

Exercise: Intercepts

1. Give the y-intercept of the graph of $y = b^x + 2$.
2. Give the x- and y-intercepts of the graph of $y = \frac{1}{2}(1{,}5)^{x+3} - 0{,}75$.

28.2.3 Asymptotes

There are two asymptotes for functions of the form $y = ab^{(x+p)} + q$. They are determined by examining the domain and range.

We saw that the function was undefined at $x = -p$ and for $y = q$. Therefore the asymptotes are $x = -p$ and $y = q$.

For example, the domain of $g(x) = 3 \cdot 2^{x+1} + 2$ is $\{x : x \in \mathbb{R}, x \neq -1\}$ because $g(x)$ is undefined at $x = -1$. We also see that $g(x)$ is undefined at $y = 2$. Therefore the range is $\{g(x) : g(x) \in (-\infty,2) \cup (2,\infty)\}$.

From this we deduce that the asymptotes are at $x = -1$ and $y = 2$.

?

Exercise: Asymptotes

1. Give the equation of the asymptote of the graph of $y = 3^x - 2$.
2. What is the equation of the horizontal asymptote of the graph of $y = 3(0{,}8)^{x-1} - 3$?

28.2.4 Sketching Graphs of the Form $f(x) = ab^{(x+p)} + q$

In order to sketch graphs of functions of the form, $f(x) = ab^{(x+p)} + q$, we need to calculate determine four characteristics:

1. domain and range
2. y-intercept

3. x-intercept

For example, sketch the graph of $g(x) = 3 \cdot 2^{x+1} + 2$. Mark the intercepts.

We have determined the domain to be $\{x : x \in \mathbb{R}\}$ and the range to be $\{g(x) : g(x) \in [5,\infty)\}$.

The y-intercept is $y_{int} = 8$ and there are no x-intercepts.

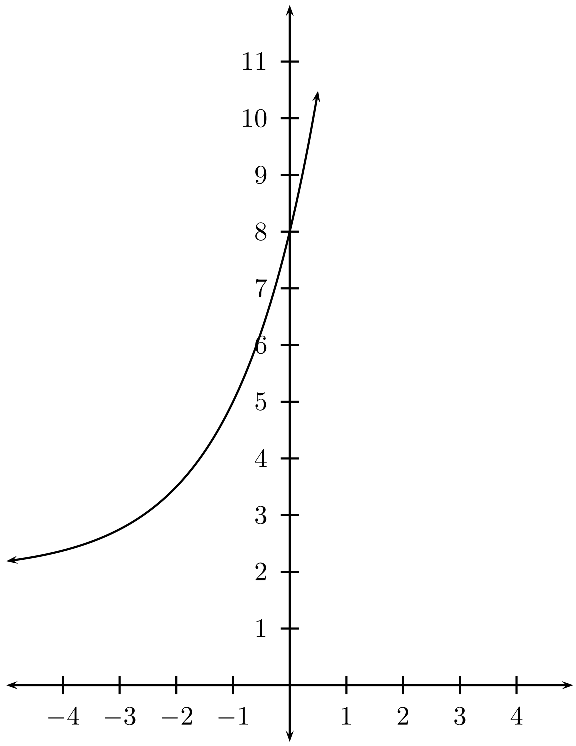


Figure 28.2: Graph of $g(x) = 3 \cdot 2^{x+1} + 2$.

?

Exercise: Sketching Graphs

1. Draw the graphs of the following on the same set of axes. Label the horizontal aymptotes and y-intercepts clearly.
   - A $y = b^x + 2$
   - B $y = b^{x+2}$
   - C $y = 2b^x$
   - D $y = 2b^{x+2} + 2$
   - A Draw the graph of $f(x) = 3^x$.
   - B Explain whre a solution of $3^x = 5$ can be read off the graph.

28.3 End of Chapter Exercises

1. The following table of values has columns giving the y-values for the graph $y = a^x$, $y = a^{x+1}$ and $y = a^x + 1$. Match a graph to a column.

x	A	B	C
-2	7,25	6,25	2,5
-1	3,5	2,5	1
0	2	1	0,4
1	1,4	0,4	0,16
2	1,16	0,16	0,064

2. The graph of $f(x) = 1 + a.2^x$ (a is a constant) passes through the origin.

   A Determine the value of a.

   B Determine the value of $f(-15)$ correct to FIVE decimal places.

   C Determine the value of x, if $P(x; 0{,}5)$ lies on the graph of f.

   D If the graph of f is shifted 2 units to the right to give the function h, write down the equation of h.

3. The graph of $f(x) = a.bx$ $(a \neq 0)$ has the point P(2;144) on f.

   A If $b = 0{,}75$, calculate the value of a.

   B Hence write down the equation of f.

   C Determine, correct to TWO decimal places, the value of $f(13)$.

   D Describe the transformation of the curve of f to h if $h(x) = f(-x)$.

Chapter 29

Gradient at a Point - Grade 11

29.1 Introduction

In Grade 10, we investigated the idea of *average gradient* and saw that the gradient of some functions varied over different intervals. In Grade 11, we further look at the idea of average gradient, and are introduced to the idea of a gradient of a curve at a point.

29.2 Average Gradient

We saw that the average gradient between two points on a curve is the gradient of the straight line passing through the two points.

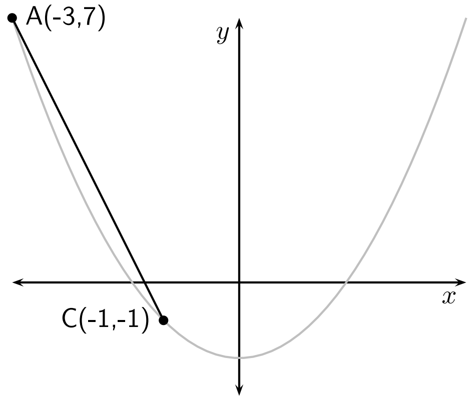
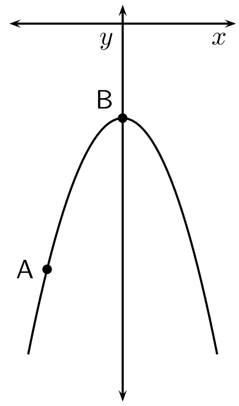


Figure 29.1: The average gradient between two points on a curve is the gradient of the straight line that passes through the points.

What happens to the gradient if we fix the position of one point and move the second point closer to the fixed point?

Activity :: Investigation : Gradient at a Single Point on a Curve
The curve shown is defined by $y = -2x^2 - 5$. Point B is fixed at co-ordinates (0,-5). The position of point A varies. Complete the table below by calculating the y-coordinates of point A for the given x-coordinates and then calculate the average gradient between points A and B.

x_A	y_A	average gradient
-2		
-1.5		
-1		
-0.5		
0		
0.5		
1		
1.5		
2		



What happens to the average gradient as A moves towards B? What happens to the average gradient as A away from B? What is the average gradient when A overlaps with B?

In Figure 29.2, the gradient of the straight line that passes through points A and C changes as A moves closer to C. At the point when A and C overlap, the straight line only passes through one point on the curve. Such a line is known as a tangent to the curve.

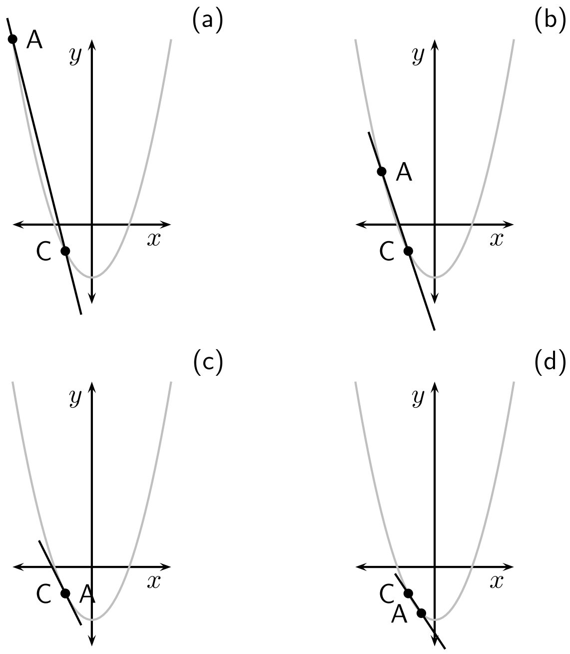


Figure 29.2: The gradient of the straight line between A and C changes as the point A moves along the curve towards C. There comes a point when A and C overlap (as shown in (c)). At this point the line is a tangent to the curve.

We therefore introduce the idea of a gradient at a single point on a curve. The gradient at a point on a curve is simply the gradient of the tangent to the curve at the given point.

Worked Example 122: Average Gradient

Question: Find the average gradient between two points P$(a; g(a))$ and Q$(a + h; g(a+h))$ on a curve $g(x) = x^2$. Then find the average gradient between P$(2; g(2))$

and Q$(4; g(4))$. Finally, explain what happens to the average gradient if P moves closer to Q.

Answer

Step 1 : Label x points

$$x_1 = a$$

$$x_2 = a + h$$

Step 2 : Determine y coordinates

Using the function $g(x) = x^2$, we can determine:

$$y_1 = g(a) = a^2$$

$$\begin{aligned} y_2 &= g(a+h) \\ &= (a+h)^2 \\ &= a^2 + 2ah + h^2 \end{aligned}$$

Step 3 : Calculate average gradient

$$\begin{aligned} \frac{y_2 - y_1}{x_2 - x_1} &= \frac{(a^2 + 2ah + h^2) - (a^2)}{(a+h) - (a)} \\ &= \frac{a^2 + 2ah + h^2 - a^2}{a + h - a} \\ &= \frac{2ah + h^2}{h} \\ &= \frac{h(2a + h)}{h} \\ &= 2a + h \end{aligned} \qquad (29.1)$$

The average gradient between P$(a; g(a))$ and Q$(a+h; g(a+h))$ on the curve $g(x) = x^2$ is $2a + h$.

Step 4 : Calculate the average gradient between P$(2; g(2))$ and Q$(4; g(4))$

We can use the result in (29.1), but we have to determine what is a and h. We do this by looking at the definitions of P and Q. The x coordinate of P is a and the x coordinate of Q is $a + h$ therefore if we assume that $a = 2$ then if $a + h = 4$, which gives $h = 2$.

Then the average gradient is:

$$2a + h = 2(2) + (2) = 6$$

Step 5 : When P moves closer to Q...

When point P moves closer to point Q, h gets smaller. This means that the average gradient also gets smaller. When the point Q overlaps with the point P $h = 0$ and the average gradient is given by $2a$.

We now see that we can write the equation to calculate average gradient in a slightly different manner. If we have a curve defined by $f(x)$ then for two points P and Q with P$(a; f(a))$ and Q$(a + h; f(a + h))$, then the average gradient between P and Q on $f(x)$ is:

$$\begin{aligned} \frac{y_2 - y_1}{x_2 - x_1} &= \frac{f(a+h) - f(a)}{(a+h) - (a)} \\ &= \frac{f(a+h) - f(a)}{h} \end{aligned}$$

This result is important for calculating the gradient at a point on a curve and will be explored in greater detail in Grade 12.

29.3 End of Chapter Exercises

1. A Determine the average gradient of the curve $f(x) = x(x+3)$ between $x = 5$ and $x = 3$.

   B Hence, state what you can deduce about the function f between $x = 5$ and $x = 3$.

2. A(1;3) is a point on $f(x) = 3x^2$.

   A Determine the gradient of the curve at point A.

   B Hence, determine the equation of the tangent line at A.

3. Given: $f(x) = 2x^2$.

   A Determine the average gradient of the curve between $x = -2$ and $x = 1$.

   B Determine the gradient of the curve of f where $x = 2$.

Chapter 30

Linear Programming - Grade 11

30.1 Introduction

In everyday life people are interested in knowing the most efficient way of carrying out a task or achieving a goal. For example, a farmer might want to know how many crops to plant during a season in order to maximise yield (produce) or a stock broker might want to know how much to invest in stocks in order to maximise profit. These are examples of **optimisation** problems, where by optimising we mean finding the maxima or minima of a function.

We have seen optimisation problems of one variable in Chapter 40, where there were no restrictions to the answer. You were then required to find the highest (maximum) or lowest (minimum) possible value of some function. In this chapter we look at optimisation problems with two variables and where the possible solutions are restricted.

30.2 Terminology

There are some basic terms which you need to become familiar with for the linear programming chapters.

30.2.1 Decision Variables

The aim of an optimisation problem is to find the values of the decision variables. These values are unknown at the beginning of the problem. Decision variables usually represent things that can be changed, for example the rate at which water is consumed or the number of birds living in a certain park.

30.2.2 Objective Function

The objective function is a mathematical combination of the decision variables and represents the function that we want to optimise (i.e. maximise or minimise) is called the **objective function**. We will only be looking at objective functions which are functions of two variables. For example, in the case of the farmer, the objective function is the yield and it is dependent on the amount of crops planted. If the farmer has two crops then the objective function $f(x,y)$ is the yield, where x represents the amount of the first crop planted and y represents the amount of the second crop planted. For the stock broker, assuming that there are two stocks to invest in, the objective function $f(x,y)$ is the amount of profit earned by investing x rand in the first stock and y rand in the second.

30.2.3 Constraints

Constraints, or **restrictions**, are often placed on the variables being optimised. For the example of the farmer, he cannot plant a negative number of crops, therefore the constraints would be:

$$x \geq 0$$
$$y \geq 0.$$

Other constraints might be that the farmer cannot plant more of the second crop than the first crop and that no more than 20 units of the first crop can be planted. These constraints can be written as:

$$x \geq y$$
$$x \leq 20$$

Constraints that have the form

$$ax + by \leq c$$

or

$$ax + by = c$$

are called **linear** constraints. Examples of linear constraints are:

$$x + y \leq 0$$
$$-2x = 7$$
$$y \leq \sqrt{2}$$

30.2.4 Feasible Region and Points

Constraints mean that we cannot just take any x and y when looking for the x and y that optimise our objective function. If we think of the variables x and y as a point (x,y) in the xy-plane then we call the set of all points in the xy-plane that satisfy our constraints the **feasible region**. Any point in the feasible region is called a **feasible point**.

For example, the constraints

$$x \geq 0$$
$$y \geq 0.$$

mean that only values of x and y that are positive are allowed. Similarly, the constraint

$$x \geq y$$

means that only values of x that are greater than or equal to the y values are allowed.

$$x \leq 20$$

means that only x values which are less than or equal to 20 are allowed.

Important: The constraints are used to create bounds of the solution.

30.2.5 The Solution

Important: Points that satisfy the constraints are called feasible solutions.

Once we have determined the feasible region the **solution** of our problem will be the feasible point where the objective function is a maximum / minimum. Sometimes there will be more than one feasible point where the objective function is a maximum/minimum — in this case we have more than one solution.

30.3 Example of a Problem

A simple problem that can be solved with linear programming involves Mrs. Nkosi and her farm.

> Mrs Nkosi grows mielies and potatoes on a farm of 100 m^2. She has accepted orders that will need her to grow at least 40 m^2 of mielies and at least 30 m^2 of potatoes. Market research shows that the demand this year will be at least twice as much for mielies as for potatoes and so she wants to use at least twice as much area for mielies as for potatoes. She expects to make a profit of R650 per m^2 for her mielies and R1 500 per m^2 on her sorgum. How should she divide her land so that she can earn the most profit?

Let m represent the area of mielies grown and let p be the area of potatoes grown.

We shall see how we can solve this problem.

30.4 Method of Linear Programming

Method: Linear Programming

1. Identify the decision variables in the problem.
2. Write constraint equations
3. Write objective function as an equation
4. Solve the problem

30.5 Skills you will need

30.5.1 Writing Constraint Equations

You will need to be comfortable with converting a word description to a mathematical description for linear programming. Some of the words that are used is summarised in Table 30.1.

Table 30.1: Phrases and mathematical equivalents.

Words	Mathematical description
x equals a	$x = a$
x is greater than a	$x > a$
x is greater than or equal to a	$x \geq a$
x is less than a	$x < a$
x is less than or equal to a	$x \leq a$
x must be at least a	$x \geq a$
x must be at most b	$x \leq a$

Worked Example 123: Writing constraints as equations

Question: Mrs Nkosi grows mielies and potatoes on a farm of 100 m^2. She has accepted orders that will need her to grow at least 40 m^2 of mielies and at least

30 m^2 of potatoes. Market research shows that the demand this year will be at least twice as much for mielies as for potatoes and so she wants to use at least twice as much area for mielies as for potatoes.

Answer

Step 1 : Identify the decision variables

There are two decision variables: the area used to plant mielies (m) and the area used to plant potatoes (p).

Step 2 : Identify the phrases that constrain the decision variables

- grow at least 40 m^2 of mielies
- grow at least 30 m^2 of potatoes
- area of farm is 100 m^2
- demand is twice as much for mielies as for potatoes

Step 3 : For each phrase, write a constraint

- $m \geq 40$
- $p \geq 30$
- $m + p \leq 100$
- $m \geq 2p$

Exercise: constraints as equation

Write the following constraints as equations:

1. Michael is registering for courses at university. Michael needs to register for at least 4 courses.
2. Joyce is also registering for courses at university. She cannot register for more than 7 courses.
3. In a geography test, Simon is allowed to choose 4 questions from each section.
4. A baker can bake at most 50 chocolate cakes in 1 day.
5. Megan and Katja can carry at most 400 koeksisters.

30.5.2 Writing the Objective Function

If the objective function is not given to you as an equation, you will need to be able to convert a word description to an equation to get the objective function.

You will need to look for words like:

- most profit
- least cost
- largest area

Worked Example 124: Writing the objective function

Question: The cost of hiring a small trailer is R500 per day and the cost of hiring a big trailer is R800 per day. Write down the objective function that can be used to find the cheapest cost for hiring trailers for 1 day.

Answer

Step 1 : Identify the decision variables

There are two decision variables: the number of big trailers (n_b) and the number of small trailers (n_s).

Step 2 : Write the purpose of the objective function

The purpose of the objective function is to minimise cost.

Step 3 : Write the objective function

The cost of hiring n_s small trailers for 1 day is:

$$500 \times n_s$$

The cost of hiring n_b big trailers for 1 day is:

$$800 \times n_b$$

Therefore the objective function, which is the total cost of hiring n_s small trailers and n_b big trailers for 1 day is:

$$500 \times n_s + 800 \times n_b$$

Worked Example 125: Writing the objective function

Question: Mrs Nkosi expects to make a profit of R650 per m^2 for her mielies and R1 500 per m^2 on her potatoes. How should she divide her land so that she can earn the most profit?

Answer

Step 1 : Identify the decision variables

There are two decision variables: the area used to plant mielies (m) and the area used to plant potatoes (p).

Step 2 : Write the purpose of the objective function

The purpose of the objective function is to maximise profit.

Step 3 : Write the objective function

The profit of planting m m^2 of mielies is:

$$650 \times m$$

The profit of planting p m^2 of potatoes is:

$$1500 \times p$$

Therefore the objective function, which is the total profit of planting mielies and potatoes is:

$$650 \times m + 1500 \times p$$

?

Exercise: Writing the objective function

1. The *EduFurn* furniture factory manufactures school chairs and school desks. They make a profit of R50 on each chair sold and of R60 on each desk sold. Write an equation that will show how much profit they will make by selling the chairs and desks?
2. A manufacturer makes small screen GPS's and wide screen GPS's. If the profit on small screen GPS's is R500 and the profit on wide screen GPS's is R250, write an equation that will show the possible maximum profit.

30.5.3 Solving the Problem

The numerical method involves using the points along the boundary of the feasible region, and determining which point has the optimises the objective function.

Activity :: Investigation : Numerical Method
Use the objective function

$$650 \times m + 1500 \times p$$

to calculate Mrs. Nkosi's profit for the following feasible solutions:

m	p	Profit
60	30	
65	30	
70	30	
$66\frac{2}{3}$	$33\frac{1}{3}$	

The question is *How do you find the feasible region?* We will use the graphical method of solving a system of linear equations to determine the feasible. We draw all constraints as graphs and mark the area that satisfies all constraints. This is shown in Figure 30.1 for Mrs. Nkosi's farm.

Now we can use the methods we learnt previously to find the points at the vertices of the feasible region. In Figure 30.1, vertex A is at the intersection of $p = 30$ and $m = 2p$. Therefore, the coordinates of A are (30,60). Similarly vertex B is at the intersection of $p = 30$ and $m = 100-p$. Therefore the coordinates of B are (30,70). Vertex C is at the intersection of $m = 100 - p$ and $m = 2p$, which gives ($33\frac{1}{3}$,$66\frac{2}{3}$) for the coordinates of C.

If we now substitute these points into the objective function, we get the following:

m	p	Profit
60	30	81 000
70	30	87 000
$66\frac{2}{3}$	$33\frac{1}{3}$	89 997

Therefore Mrs. Nkosi makes the most profit if she plants $66\frac{2}{3}$ m^2 of mielies and $66\frac{2}{3}$ m^2 of potatoes. Her profit is R89 997.

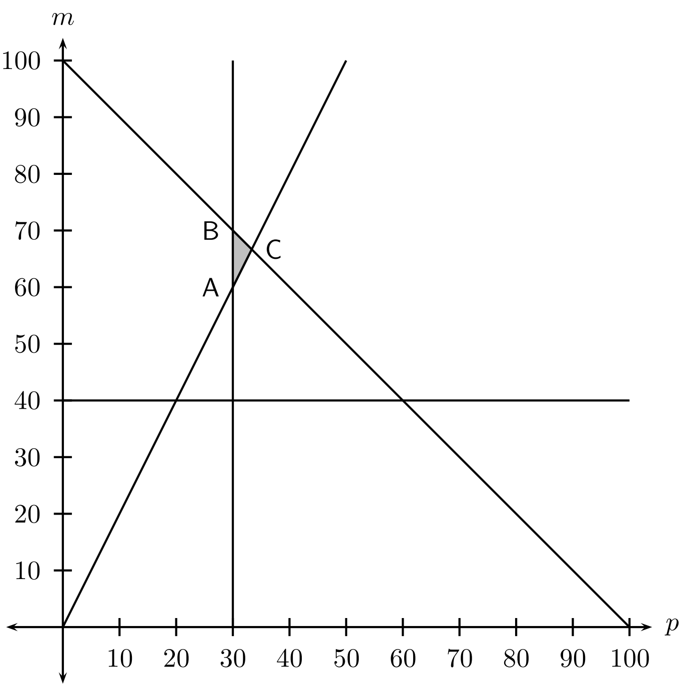


Figure 30.1: Graph of the feasible region

Worked Example 126: Prizes!

Question: As part of their opening specials, a furniture store has promised to give away at least 40 prizes with a total value of at least R2 000. The prizes are kettles and toasters.

1. If the company decides that there will be at least 10 of each prize, write down two more inequalities from these constraints.
2. If the cost of manufacturing a kettle is R60 and a toaster is R50, write down an objective function C which can be used to determine the cost to the company of both kettles and toasters.
3. Sketch the graph of the feasibility region that can be used to determine all the possible combinations of kettles and toasters that honour the promises of the company.
4. How many of each prize will represent the cheapest option for the company?
5. How much will this combination of kettles and toasters cost?

Answer

Step 1 : Identify the decision variables

Let the number of kettles be x_k and the number of toasters be y_t and write down two constraints apart from $x_k \geq 0$ and $y_t \geq 0$ that must be adhered to.

Step 2 : Write constraint equations

Since there will be at least 10 of each prize we can write:

$$x_k \geq 10$$

and

$$y_t \geq 10$$

Also the store has promised to give away at least 40 prizes in total. Therefore:

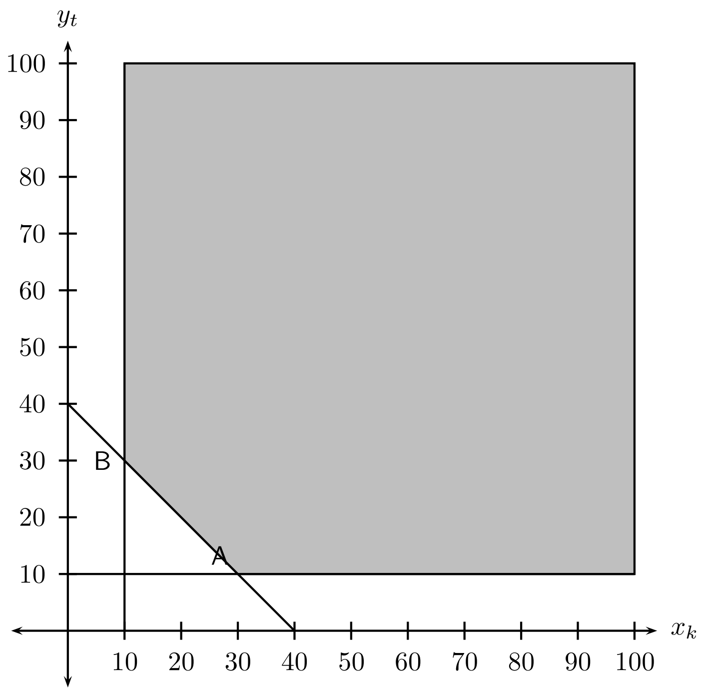
$$x_k + y_t \geq 40$$

Step 3 : Write the objective function

The cost of manufacturing a kettle is R60 and a toaster is R50. Therefore the cost the total cost C is:

$$C = 60x_k + 50y_t$$

Step 4 : Sketch the graph of the feasible region



Step 5 : Determine vertices of feasible region
From the graph, the coordinates of vertex A is (3,1) and the coordinates of vertex B are (1,3).
Step 6 : Calculate cost at each vertex
At vertex A, the cost is:

$$\begin{aligned} C &= 60x_k + 50y_t \\ &= 60(30) + 50(10) \\ &= 1800 + 500 \\ &= 2300 \end{aligned}$$

At vertex B, the cost is:

$$\begin{aligned} C &= 60x_k + 50y_t \\ &= 60(10) + 50(30) \\ &= 600 + 1500 \\ &= 2100 \end{aligned}$$

Step 7 : Write the final answer
The cheapest combination of prizes is 10 kettles and 30 toasters, costing the company R2 100.

30.6 End of Chapter Exercises

1. You are given a test consisting of two sections. The first section is on Algebra and the second section is on Geometry. You are not allowed to answer more than 10 questions from any section, but you have to answer at least 4 Algebra questions. The time allowed

is not more than 30 minutes. An Algebra problem will take 2 minutes and a Geometry problem will take 3 minutes each to solve.

If you answer x_A Algebra questions and y_G Geometry questions,

A Formulate the constraints which satisfy the above constraints.

B The Algebra questions carry 5 marks each and the Geometry questions carry 10 marks each. If T is the total marks, write down an expression for T.

2. A local clinic wants to produce a guide to healthy living. The clinic intends to produce the guide in two formats: a short video and a printed book. The clinic needs to decide how many of each format to produce for sale. Estimates show that no more than 10 000 copies of both items together will be sold. At least 4 000 copies of the video and at least 2 000 copies of the book could be sold, although sales of the book are not expected to exceed 4 000 copies. Let x_v be the number of videos sold, and y_b the number of printed books sold.

   A Write down the constraint inequalities that can be deduced from the given information.

   B Represent these inequalities graphically and indicate the feasible region clearly.

   C The clinic is seeking to maximise the income, I, earned from the sales of the two products. Each video will sell for R50 and each book for R30. Write down the objective function for the income.

   D Determine graphically, by using a search line, the number of videos and books that ought to be sold to maximise the income.

   E What maximum income will be generated by the two guides?

3. A patient in a hospital needs at least 18 grams of protein, 0,006 grams of vitamin C and 0,005 grams of iron per meal, which consists of two types of food, A and B. Type A contains 9 grams of protein, 0,002 grams of vitamin C and no iron per serving. Type B contains 3 grams of protein, 0,002 grams of vitamin C and 0,005 grams of iron per serving. The energy value of A is 800 kilojoules and the of B 400 kilojoules per mass unit. A patient is not allowed to have more than 4 servings of A and 5 servings of B. There are x_A servings of A and y_B servings of B on the patients plate.

   A Write down in terms of x_A and y_B

      i. The mathematical constraints which must be satisfied.

      ii. The kilojoule intake per meal.

   B Represent the constraints graphically on graph paper. Use the scale 1 unit $=$ 20mm on both axes. Shade the feasible region.

   C Deduce from the graphs, the values of x_A and y_B which will give the minimum kilojoule intake per meal for the patient.

4. A certain motorcycle manufacturer produces two basic models, the 'Super X' and the 'Super Y'. These motorcycles are sold to dealers at a profit of R20 000 per 'Super X' and R10 000 per 'Super Y'. A 'Super X' requires 150 hours for assembly, 50 hours for painting and finishing and 10 hours for checking and testing. The 'Super Y' requires 60 hours for assembly, 40 hours for painting and finishing and 20 hours for checking and testing. The total number of hours available per month is: 30 000 in the assembly department, 13 000 in the painting and finishing department and 5 000 in the checking and testing department.

   The above information can be summarised by the following table:

Department	Hours for 'Super X'	Hours for Super 'Y'	Maximum hours available per month
Assembley	150	60	30 000
Painting and Finishing	50	40	13 000
Checking and Testing	10	20	5 000

   Let x be the number of 'Super X' and y be the number of 'Super Y' models manufactured per month.

A Write down the set of constraint inequalities.

B Use the graph paper provided to represent the constraint inequalities.

C Shade the feasible region on the graph paper.

D Write down the profit generated in terms of x and y.

E How many motorcycles of each model must be produced in order to maximise the monthly profit?

F What is the maximum monthly profit?

5. A group of students plan to sell x hamburgers and y chicken burgers at a rugby match. They have meat for at most 300 hamburgers and at most 400 chicken burgers. Each burger of both types is sold in a packet. There are 500 packets available. The demand is likely to be such that the number of chicken burgers sold is at least half the number of hamburgers sold.

   A Write the constraint inequalities.

   B Two constraint inequalities are shown on the graph paper provided. Represent the remaining constraint inequalities on the graph paper.

   C Shade the feasible region on the graph paper.

   D A profit of R3 is made on each hamburger sold and R2 on each chicken burger sold. Write the equation which represents the total profit, P, in terms of x and y.

   E The objective is to maximise profit. How many, of each type of burger, should be sold to maximise profit?

6. Fashion-cards is a small company that makes two types of cards, type X and type Y. With the available labour and material, the company can make not more than 150 cards of type X and not more than 120 cards of type Y per week. Altogether they cannot make more than 200 cards per week.

   There is an order for at least 40 type X cards and 10 type Y cards per week. Fashion-cards makes a profit of R5 for each type X card sold and R10 for each type Y card.

   Let the number of type X cards be x and the number of type Y cards be y, manufactured per week.

   A One of the constraint inequalities which represents the restrictions above is $x \leq 150$. Write the other constraint inequalities.

   B Represent the constraints graphically and shade the feasible region.

   C Write the equation that represents the profit P (the objective function), in terms of x and y.

   D Calculate the maximum weekly profit.

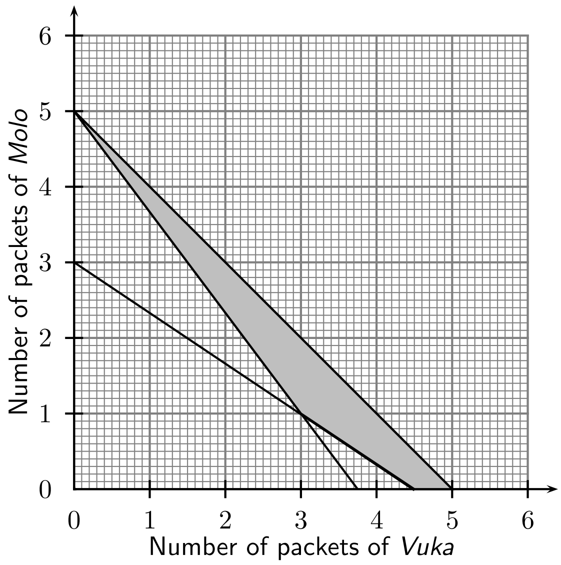
7. To meet the requirements of a specialised diet a meal is prepared by mixing two types of cereal, *Vuka* and *Molo*. The mixture must contain x packets of *Vuka* cereal and y packets of *Molo* cereal. The meal requires at least 15 g of protein and at least 72 g of carbohydrates. Each packet of *Vuka* cereal contains 4 g of protein and 16 g of carbohydrates. Each packet of *Molo* cereal contains 3 g of protein and 24 g of carbohydrates. There are at most 5 packets of cereal available. The feasible region is shaded on the attached graph paper.

   A Write down the constraint inequalities.

   B If *Vuka* cereal costs R6 per packet and *Molo* cereal also costs R6 per packet, use the graph to determine how many packets of each cereal must be used for the mixture to satisfy the above constraints in each of the following cases:

      i. The total cost is a minimum.

      ii. The total cost is a maximum (give all possibilities).



8. A bicycle manufacturer makes two different models of bicycles, namely mountain bikes and speed bikes. The bicycle manufacturer works under the following constraints:
No more than 5 mountain bicycles can be assembled daily.
No more than 3 speed bicycles can be assembled daily.
It takes one man to assemble a mountain bicycle, two men to assemble a speed bicycle and there are 8 men working at the bicycle manufacturer.
Let x represent the number of mountain bicycles and let y represent the number of speed bicycles.

   A Determine algebraically the constraints that apply to this problem.

   B Represent the constraints graphically on the graph paper.

   C By means of shading, clearly indicate the feasible region on the graph.

   D The profit on a mountain bicycle is R200 and the profit on a speed bicycle is R600. Write down an expression to represent the profit on the bicycles.

   E Determine the number of each model bicycle that would maximise the profit to the manufacturer.

Chapter 31

Geometry - Grade 11

31.1 Introduction

Activity :: Extension : History of Geometry
Work in pairs or groups and investigate the history of the development of geometry in the last 1500 years. Describe the various stages of development and how different cultures used geometry to improve their lives.

The works of the following people or cultures can be investigated:

1. Islamic geometry (c. 700 - 1500)
   A Thabit ibn Qurra
   B Omar Khayyam
   C Sharafeddin Tusi
2. Geometry in the 17th - 20th centuries (c. 700 - 1500)

31.2 Right Pyramids, Right Cones and Spheres

A pyramid is a geometric solid that has a polygon base and the base is joined to an apex. Examples of pyramids are shown in Figure 31.1.

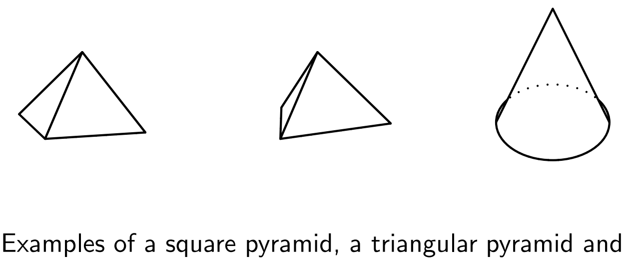


Figure 31.1: Examples of a square pyramid, a triangular pyramid and a cone.

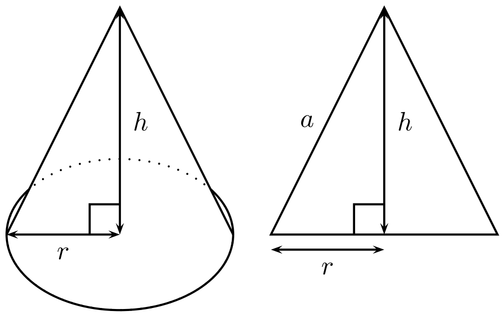
Method: Surface Area of a Pyramid
The surface area of a pyramid is calculated by adding the area of each face together.

Worked Example 127: Surface Area

Question: If a cone has a height of h and a base of radius r, show that the surface area is $\pi r^2 + \pi r\sqrt{r^2+h^2}$.

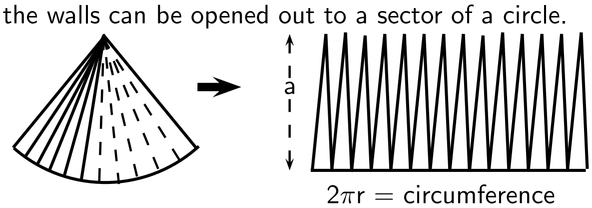
Answer

Step 1 : Draw a picture



Step 2 : Identify the faces that make up the cone

The cone has two faces: the base and the walls. The base is a circle of radius r and the walls can be opened out to a sector of a circle.



This curved surface can be cut into many thin triangles with height close to a (a is called a *slant height*). The area of these triangles will add up to $\frac{1}{2}\times$base$\times$height which is $\frac{1}{2}\times 2\pi r\times a = \pi ra$

Step 3 : Calculate a

a can be calculated by using the Theorem of Pythagoras. Therefore:

$$a = \sqrt{r^2+h^2}$$

Step 4 : Calculate the area of the circular base

$$A_b = \pi r^2$$

Step 5 : Calculate the area of the curved walls

$$\begin{aligned} A_w &= \pi ra \\ &= \pi r\sqrt{r^2+h^2} \end{aligned}$$

Step 6 : Calculate surface area A

$$\begin{aligned} A &= A_b + A_w \\ &= \pi r^2 + \pi r\sqrt{r^2+h^2} \end{aligned}$$

Method:

Volume of a Pyramid

The volume of a pyramid is found by:

$$V = \frac{1}{3}A\cdot h$$

where A is the area of the base and h is the height.

A cone is a pyramid, so the volume of a cone is given by

$$V = \frac{1}{3}\pi r^2 h.$$

A square pyramid has volume

$$V = \frac{1}{3}a^2 h$$

where a is the side length.

Worked Example 128: Volume of a Pyramid

Question: What is the volume of a square pyramid, 3cm high with a side length of 2cm?

Answer

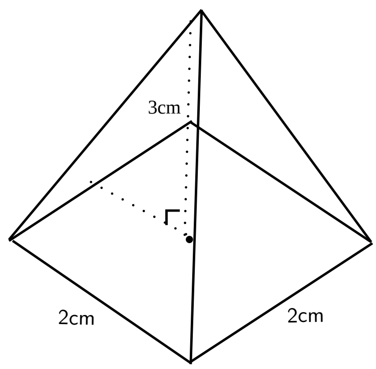
Step 1 : Determine the correct formula

The volume of a pyramid is

$$V = \frac{1}{3}A \cdot h,$$

which for a square base means

$$V = \frac{1}{3}a \cdot a \cdot h.$$



Step 2 : Substitute the given values

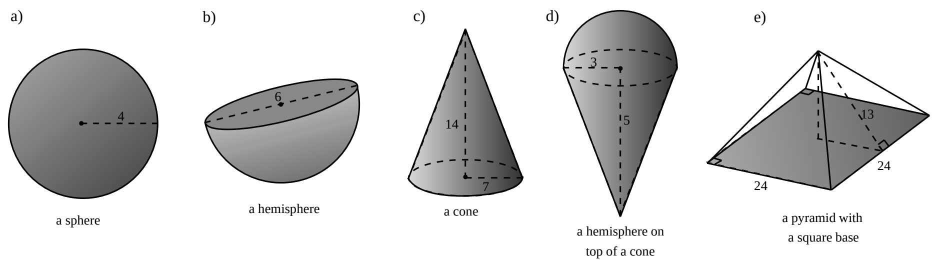
$$\begin{aligned} &= \frac{1}{3} \cdot 2 \cdot 2 \cdot 3 \\ &= \frac{1}{3} \cdot 12 \\ &= 4 \text{ cm}^3 \end{aligned}$$

We accept the following formulae for volume and surface area of a sphere (ball).

$$\begin{aligned} \text{Surface area} &= 4\pi r^2 \\ \text{Volume} &= \frac{4}{3}\pi r^3 \end{aligned}$$

Exercise: Surface Area and Volume

1. Calculate the volumes and surface areas of the following solids: *Hint for (e): find the perpendicular height using Pythagoras.



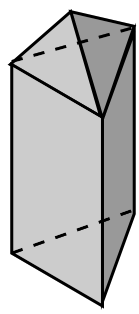
2. Water covers approximately 71% of the Earth's surface. Taking the radius of the Earth to be 6378 km, what is the total area of land (area not covered by water)?

3. A right triangular pyramid is placed on top of a right triangular prism. The prism has an equilateral triangle of side length 20 cm as a base, and has a height of 42 cm. The pyramid has a height of 12 cm.

   A Find the total volume of the object.

   B Find the area of each face of the pyramid.

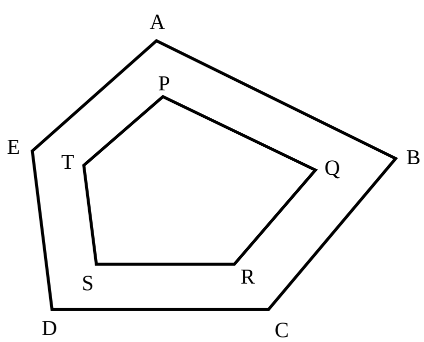
   C Find the total surface area of the object.



31.3 Similarity of Polygons

In order for two polygons to be similar the following must be true:

1. All corresponding angles must be congruent.
2. All corresponding sides must be in the same proportion to each other.



If

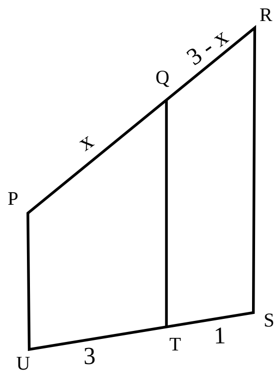
1. $\hat{A} = \hat{P}$; $\hat{B} = \hat{Q}$; $\hat{C} = \hat{R}$; $\hat{D} = \hat{S}$; $\hat{E} = \hat{T}$ and
2. $\frac{AB}{PQ} = \frac{BC}{QR} = \frac{CD}{RS} = \frac{DE}{ST} = \frac{EA}{TP}$

then the polygons ABCDE and PQRST are similar.

Worked Example 129: Similarity of Polygons

Question:

Polygons PQTU and PRSU are similar.
Find the value of x.



Answer

Step 1 : Identify corresponding sides

Since the polygons are similar,

$$\begin{aligned} \frac{PQ}{PR} &= \frac{TU}{SU} \\ \therefore \frac{x}{x+(3-x)} &= \frac{3}{1} \\ \therefore \frac{x}{3} &= 3 \\ \therefore x &= 9 \end{aligned}$$

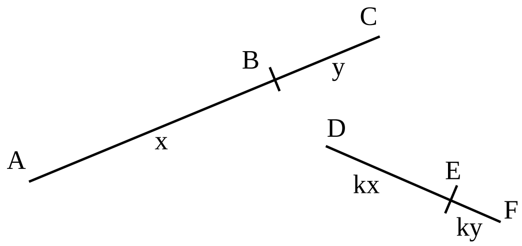
31.4 Triangle Geometry

31.4.1 Proportion

Two line segments are divided in the same proportion if the ratios between their parts are equal.

$$\frac{AB}{BC} = \frac{x}{y} = \frac{kx}{ky} = \frac{DE}{EF}$$

$\therefore$ the line segments are in the same proportion

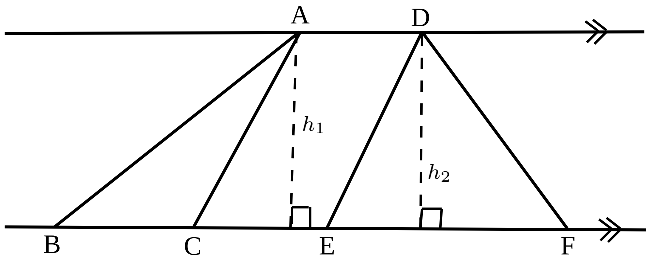


If the line segments are proportional, the following also hold

1. $AC \cdot FE = CB \cdot DF$
2. $\frac{CB}{AC} = \frac{FE}{DF}$
3. $\frac{AB}{BC} = \frac{DE}{FE}$ and $\frac{BC}{AB} = \frac{FE}{DE}$
4. $\frac{AB}{AC} = \frac{DE}{DF}$ and $\frac{AC}{AB} = \frac{DF}{DE}$

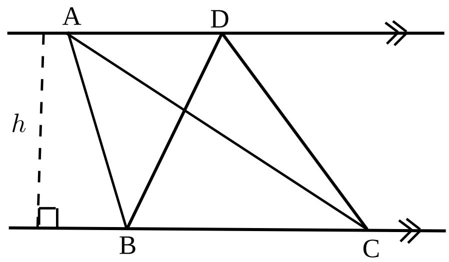
- Triangles with equal heights have areas which are in the same proportion to each other as the bases of the triangles.

$$\begin{aligned} h_1 &= h_2 \\ \therefore \frac{\text{area } \triangle ABC}{\text{area } \triangle DEF} &= \frac{\frac{1}{2}BC \times h_1}{\frac{1}{2}EF \times h_2} = \frac{BC}{EF} \end{aligned}$$



- A special case of this happens when the bases of the triangles are equal:
  Triangles with equal bases between the same parallel lines have the same area.

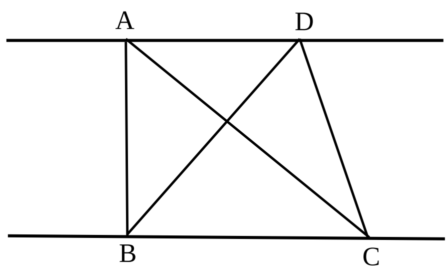
$$\text{area } \triangle ABC = \frac{1}{2} \cdot h \cdot BC = \text{ area } \triangle DBC$$



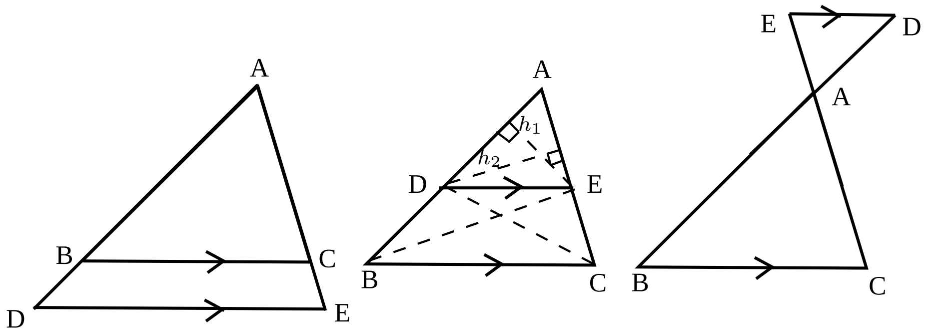
- Triangles on the same side of the same base, with equal areas, lie between parallel lines.

If area △ ABC = area △ BDC,

then AD ∥ BC.



Theorem 1. *Proportion Theorem:A line drawn parallel to one side of a triangle divides the other two sides proportionally.*



Given:△ABC with line DE ∥ BC

R.T.P.:

$$\frac{AD}{DB} = \frac{AE}{EC}$$

Proof:

Draw h_1 from E perpendicular to AD, and h_2 from D perpendicular to AE.
Draw BE and CD.

$$\begin{aligned}
\frac{\text{area } \triangle\text{ADE}}{\text{area } \triangle\text{BDE}} &= \frac{\frac{1}{2}AD \cdot h_1}{\frac{1}{2}DB \cdot h_1} = \frac{AD}{DB} \\
\frac{\text{area } \triangle\text{ADE}}{\text{area } \triangle\text{CED}} &= \frac{\frac{1}{2}AE \cdot h_2}{\frac{1}{2}EC \cdot h_2} = \frac{AE}{EC} \\
\text{but area } \triangle\text{BDE} &= \text{area } \triangle\text{CED (equal base and height)} \\
\therefore \frac{\text{area } \triangle\text{ADE}}{\text{area } \triangle\text{BDE}} &= \frac{\text{area } \triangle\text{ADE}}{\text{area } \triangle\text{CED}} \\
\therefore \frac{AD}{DB} &= \frac{AE}{EC}
\end{aligned}$$

∴ DE divides AB and AC proportionally.

Similarly,

$$\begin{aligned}
\frac{AD}{AB} &= \frac{AE}{AC} \\
\frac{AB}{BD} &= \frac{AC}{CE}
\end{aligned}$$

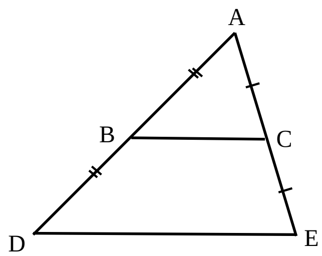
Following from Theorem 1, we can prove the midpoint theorem.:

Theorem 2. *Midpoint Theorem: A line joining the midpoints of two sides of a triangle is parallel to the third side and equal to half the length of the third side.*

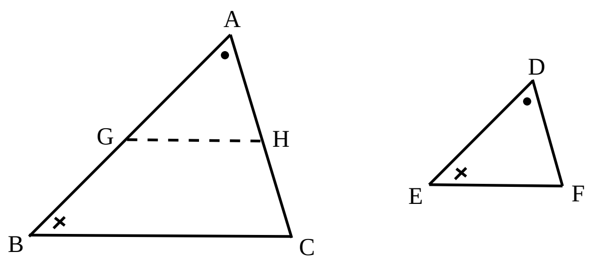
Proof:

This is a special case of the Proportionality Theorem (Theorem 1).

If AB = BD and AC = AE,
then DE ∥ BC and BC = 2DE.



Theorem 3. *Similarity Theorem 1:Equiangular triangles have their sides in proportion and are therefore similar.*



Given:△ABC and △DEF with $\hat{A} = \hat{D}$; $\hat{B} = \hat{E}$; $\hat{C} = \hat{F}$

R.T.P.:

$$\frac{AB}{DE} = \frac{AC}{DF}$$

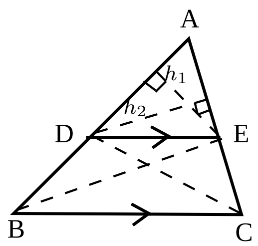
Construct: G on AB, so that AG = DE
H on AC, so that AH = DF

Proof: In △'s AGH and DEF

$$
\begin{array}{lll}
 & AG = DE; AH = DF & \text{(const.)} \\
 & \hat{A} = \hat{D} & \text{(given)} \\
\therefore & \triangle AGH \equiv \triangle DEF & \text{(SAS)} \\
\therefore & A\hat{G}H = \hat{E} = \hat{B} & \\
\therefore & GH \parallel BC & \text{(corres. } \angle\text{'s equal)} \\
\therefore & \frac{AG}{AB} = \frac{AH}{AC} & \text{(proportion theorem)} \\
\therefore & \frac{DE}{AB} = \frac{DF}{AC} & \text{(AG = DE; AH = DF)} \\
\therefore & \triangle ABC \,|||\, \triangle DEF &
\end{array}
$$

Important: ||| means "is similar to"

Theorem 4. *Similarity Theorem 2:Triangles with sides in proportion are equiangular and therefore similar.*



Given:△ABC with line DE such that

$$\frac{AD}{DB} = \frac{AE}{EC}$$

R.T.P.:$DE \parallel BC$; △ADE ||| △ABC

Proof:

Draw h_1 from E perpendicular to AD, and h_2 from D perpendicular to AE.
Draw BE and CD.

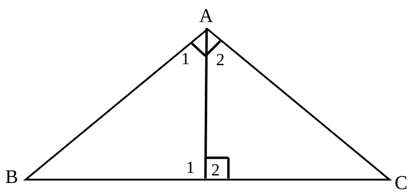
$$\begin{aligned}
\frac{\text{area }\triangle\text{ADE}}{\text{area }\triangle\text{BDE}} &= \frac{\frac{1}{2}AD\cdot h_1}{\frac{1}{2}DB\cdot h_1} = \frac{AD}{DB} \\
\frac{\text{area }\triangle\text{ADE}}{\text{area }\triangle\text{CED}} &= \frac{\frac{1}{2}AE\cdot h_2}{\frac{1}{2}EC\cdot h_2} = \frac{AE}{EC} \\
\text{but } \frac{AD}{DB} &= \frac{AE}{EC} \text{ (given)} \\
\therefore \frac{\text{area }\triangle\text{ADE}}{\text{area }\triangle\text{BDE}} &= \frac{\text{area }\triangle\text{ADE}}{\text{area }\triangle\text{CED}} \\
\therefore \text{area }\triangle\text{BDE} &= \text{area }\triangle\text{CED} \\
\therefore DE \parallel BC & \quad \text{(same side of equal base DE, same area)} \\
\therefore A\hat{D}E &= A\hat{B}C \text{ (corres }\angle\text{'s)} \\
\text{and } A\hat{E}D &= A\hat{C}B
\end{aligned}$$

$$\therefore \triangle\text{ADE and }\triangle\text{ABC are equiangular}$$

$$\therefore \triangle ADE \,|||\, \triangle ABC \text{ (AAA)}$$

Theorem 5. *Pythagoras' Theorem:The square on the hypotenuse of a right angled triangle is equal to the sum of the squares on the other two sides.*

Given:△ ABC with $\hat{A} = 90°$



R.T.P.:$BC^2 = AB^2 + AC^2$

Proof:

$$\begin{aligned}
\text{Let } \hat{C} &= x \\
\therefore \hat{A}_2 &= 90° - x \text{ (}\angle\text{ 's of a }\triangle\text{)} \\
\therefore \hat{A}_1 &= x \\
\hat{B} &= 90° - x \text{ (}\angle\text{ 's of a }\triangle\text{)} \\
\hat{D}_1 &= \hat{D}_2 = \hat{A} = 90°
\end{aligned}$$

$$\therefore \triangle\text{ABD }|||\triangle\text{CBA and }\triangle\text{CAD }|||\triangle\text{ CBA (AAA)}$$

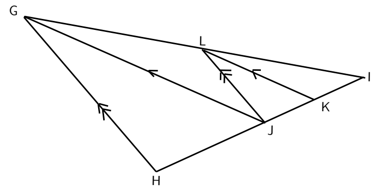
$$\therefore \frac{AB}{CB} = \frac{BD}{BA} = \left(\frac{AD}{CA}\right) \text{ and } \frac{CA}{CB} = \frac{CD}{CA} = \left(\frac{AD}{BA}\right)$$

$$\therefore AB^2 = CB \times BD \text{ and } AC^2 = CB \times CD$$

$$\begin{aligned}
\therefore AB^2 + AC^2 &= CB(BD + CD) \\
&= CB(CB) \\
&= CB^2 \\
\text{i.e. } BC^2 &= AB^2 + AC^2
\end{aligned}$$

Worked Example 130: Triangle Geometry 1

Question: In △ GHI, GH ∥ LJ; GJ ∥ LK and $\frac{JK}{KI} = \frac{5}{3}$. Determine $\frac{HJ}{KI}$.



Answer

Step 1 : Identify similar triangles

$$\begin{aligned} L\hat{I}J &= G\hat{I}H \\ J\hat{L}I &= H\hat{G}I && \text{(Corres. } \angle\text{s)} \\ \therefore \triangle LIJ &\parallel\!| \triangle GIH && \text{(Equiangular } \triangle\text{s)} \end{aligned}$$

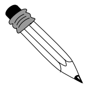
$$\begin{aligned} L\hat{I}K &= G\hat{I}J \\ K\hat{L}I &= J\hat{G}I && \text{(Corres. } \angle\text{s)} \\ \therefore \triangle LIK &\parallel\!| \triangle GIJ && \text{(Equiangular } \triangle\text{s)} \end{aligned}$$

Step 2 : Use proportional sides

$$\begin{aligned} \frac{HJ}{JI} &= \frac{GL}{LI} && (\triangle LIJ \parallel\!| \triangle GIH) \\ \text{and } \frac{GL}{LI} &= \frac{JK}{KI} && (\triangle LIK \parallel\!| \triangle GIJ) \\ &= \frac{5}{3} \\ \therefore \frac{HJ}{JI} &= \frac{5}{3} \end{aligned}$$

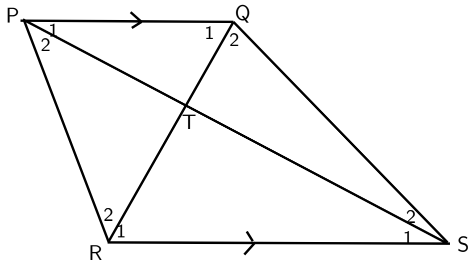
Step 3 : Rearrange to find the required ratio

$$\begin{aligned} \frac{HJ}{KI} &= \frac{HJ}{JI} \times \frac{JI}{KI} \\ &= \frac{5}{3} \times \frac{8}{3} \\ &= \frac{40}{9} \end{aligned}$$



Worked Example 131: Triangle Geometry 2

Question: PQRS is a trapezium, with PQ ∥ RS. Prove that PT · TR = ST · TQ.



Answer

Step 1 : Identify similar triangles

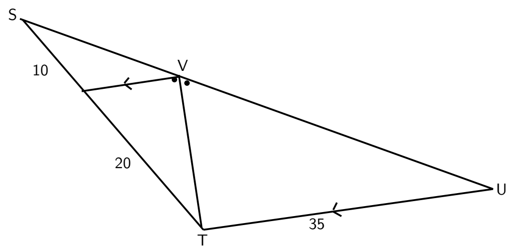
$$\begin{aligned} \hat{P}_1 &= \hat{S}_1 && \text{(Alt. } \angle\text{s)} \\ \hat{Q}_1 &= \hat{R}_1 && \text{(Alt. } \angle\text{s)} \\ \therefore \triangle PTQ &\,|||\, \triangle STR && \text{(Equiangular } \triangle\text{s)} \end{aligned}$$

Step 2 : Use proportional sides

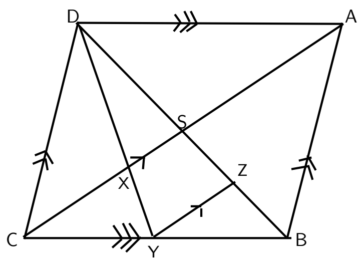
$$\begin{aligned} \frac{PT}{TQ} &= \frac{ST}{TR} && (\triangle\ \text{PTQ}\ |||\ \triangle\ \text{STR}) \\ \therefore PT \cdot TR &= ST \cdot TQ \end{aligned}$$

Exercise: Triangle Geometry

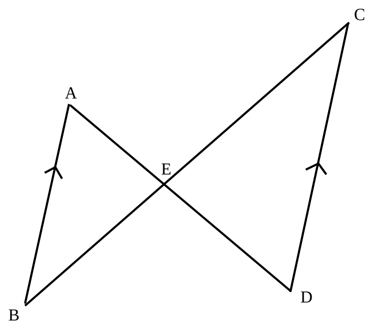
1. Calculate SV



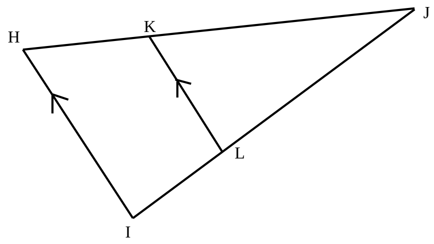
2. $\frac{CB}{YB} = \frac{2}{3}$. Find $\frac{DS}{SB}$.



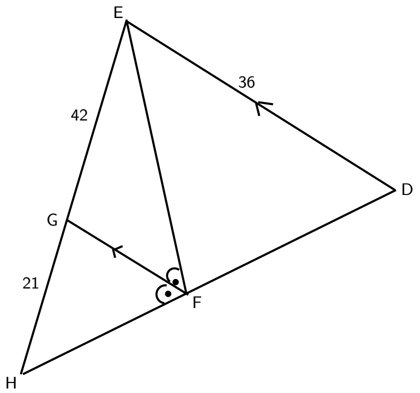
3. Given the following figure with the following lengths, find AE, EC and BE.
   BC = 15 cm, AB = 4 cm, CD = 18 cm, and ED = 9 cm.



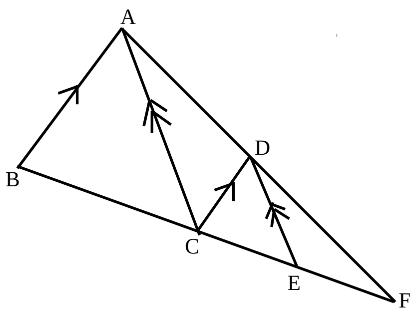
4. Using the following figure and lengths, find IJ and KJ.
   HI = 26 m, KL = 13 m, JL = 9 m and HJ = 32 m.



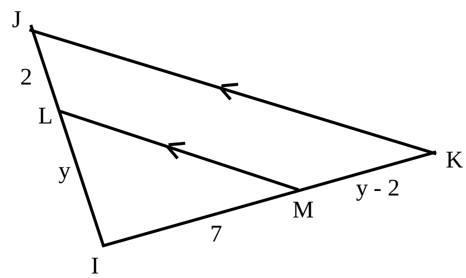
5. Find FH in the following figure.



6. BF = 25 m, AB = 13 m, AD = 9 m, DF = 18m.
   Calculate the lengths of BC, CF, CD, CE and EF, and find the ratio $\frac{DE}{AC}$.



7. If LM ∥ JK, calculate y.



31.5 Co-ordinate Geometry

31.5.1 Equation of a Line between Two Points

There are many different methods of specifying the requirements for determining the equation of a straight line. One option is to find the equation of a straight line, when two points are given.

Assume that the two points are $(x_1; y_1)$ and $(x_2; y_2)$, and we know that the general form of the equation for a straight line is:

$$\boxed{y = mx + c} \qquad (31.1)$$

So, to determine the equation of the line passing through our two points, we need to determine values for m (the gradient of the line) and c (the y-intercept of the line). The resulting equation is

$$\boxed{y - y_1 = m(x - x_1)} \qquad (31.2)$$

where $(x_1; y_1)$ are the co-ordinates of either given point.

Extension: Finding the second equation for a straight line

This is an example of a set of simultaneous equations, because we can write:

$$\begin{aligned} y_1 &= mx_1 + c && (31.3) \\ y_2 &= mx_2 + c && (31.4) \end{aligned}$$

We now have two equations, with two unknowns, m and c.

$$\begin{aligned} \text{Subtract (31.3) from (31.4)} \quad y_2 - y_1 &= mx_2 - mx_1 && (31.5) \\ \therefore \quad m &= \frac{y_2 - y_1}{x_2 - x_1} && (31.6) \\ \text{Re-arrange (31.3) to obtain } c \quad y_1 &= mx_1 + c && (31.7) \\ c &= y_1 - mx_1 && (31.8) \end{aligned}$$

Now, to make things a bit easier to remember, substitute (31.7) into (31.1):

$$\begin{aligned} y &= mx + c && (31.9) \\ &= mx + (y_1 - mx_1) && (31.10) \\ \text{which can be re-arranged to:} \quad y - y_1 &= m(x - x_1) && (31.11) \end{aligned}$$

Important: If you are asked to calculate the equation of a line passing through two points, use:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

to calculate m and then use:

$$y - y_1 = m(x - x_1)$$

to determine the equation.

For example, the equation of the straight line passing through $(-1; 1)$ and $(2; 2)$ is given by first calculating m

$$\begin{aligned} m &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{2 - 1}{2 - (-1)} \\ &= \frac{1}{3} \end{aligned}$$

and then substituting this value into

$$y - y_1 = m(x - x_1)$$

to obtain

$$y - y_1 = \frac{1}{3}(x - x_1).$$

Then substitute $(-1;1)$ to obtain

$$\begin{aligned} y-(1) &= \frac{1}{3}(x-(-1)) \\ y-1 &= \frac{1}{3}x+\frac{1}{3} \\ y &= \frac{1}{3}x+\frac{1}{3}+1 \\ y &= \frac{1}{3}x+\frac{4}{3} \end{aligned}$$

So, $y=\frac{1}{3}x+\frac{4}{3}$ passes through $(-1;1)$ and $(2;2)$.

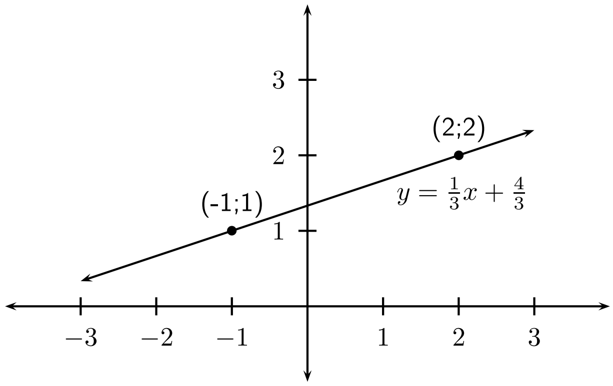


Figure 31.2: The equation of the line passing through $(-1;1)$ and $(2;2)$ is $y=\frac{1}{3}x+\frac{4}{3}$.

Worked Example 132: Equation of Straight Line

Question: Find the equation of the straight line passing through $(-3;2)$ and $(5;8)$.

Answer

Step 1 : Label the points

$$\begin{aligned} (x_1;y_1) &= (-3;2) \\ (x_2;y_2) &= (5;8) \end{aligned}$$

Step 2 : Calculate the gradient

$$\begin{aligned} m &= \frac{y_2-y_1}{x_2-x_1} \\ &= \frac{8-2}{5-(-3)} \\ &= \frac{6}{5+3} \\ &= \frac{6}{8} \\ &= \frac{3}{4} \end{aligned}$$

Step 3 : Determine the equation of the line

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - (2) &= \frac{3}{4}(x - (-3)) \\ y &= \frac{3}{4}(x + 3) + 2 \\ &= \frac{3}{4}x + \frac{3}{4} \cdot 3 + 2 \\ &= \frac{3}{4}x + \frac{9}{4} + \frac{8}{4} \\ &= \frac{3}{4}x + \frac{17}{4} \end{aligned}$$

Step 4 : Write the final answer

The equation of the straight line that passes through $(-3; 2)$ and $(5; 8)$ is $y = \frac{3}{4}x + \frac{17}{4}$.

31.5.2 Equation of a Line through One Point and Parallel or Perpendicular to Another Line

Another method of determining the equation of a straight-line is to be given one point, $(x_1; y_1)$, and to be told that the line is parallel or perpendicular to another line. If the equation of the unknown line is $y = mx + c$ and the equation of the second line is $y = m_0x + c_0$, then we know the following:

$$\text{If the lines are parallel, then} \quad m = m_0 \tag{31.12}$$

$$\text{If the lines are perpendicular, then} \quad m \times m_0 = -1 \tag{31.13}$$

Once we have determined a value for m, we can then use the given point together with:

$$y - y_1 = m(x - x_1)$$

to determine the equation of the line.

For example, find the equation of the line that is parallel to $y = 2x - 1$ and that passes through $(-1; 1)$.

First we determine m. Since the line we are looking for is parallel to $y = 2x - 1$,

$$m = 2$$

The equation is found by substituting m and $(-1; 1)$ into:

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 1 &= 2(x - (-1) \\ y - 1 &= 2(x + 1) \\ y - 1 &= 2x + 2 \\ y &= 2x + 2 + 1 \\ y &= 2x + 3 \end{aligned}$$

31.5.3 Inclination of a Line

In Figure 31.4(a), we see that the line makes an angle θ with the x-axis. This angle is known as the *inclination* of the line and it is sometimes interesting to know what the value of θ is.

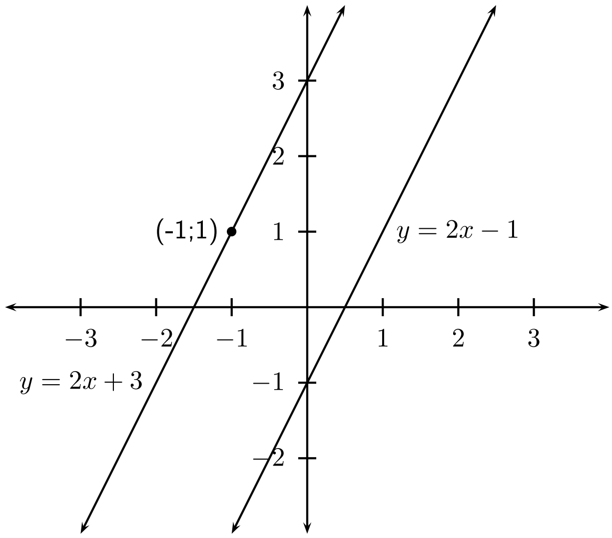


Figure 31.3: The equation of the line passing through $(-1; 1)$ and parallel to $y = 2x - 1$ is $y = 2x + 3$. It can be seen that the lines are parallel to each other. You can test this by using your ruler and measuring the distance between the lines at different points.

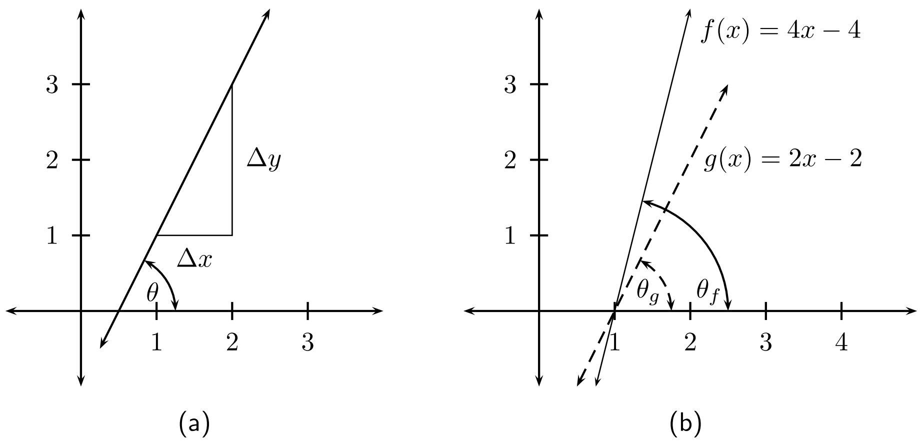


Figure 31.4: (a) A line makes an angle θ with the x-axis. (b) The angle is dependent on the gradient. If the gradient of f is m_f and the gradient of g is m_g then $m_f > m_g$ and $\theta_f > \theta_g$.

Firstly, we note that if the gradient changes, then the value of θ changes (Figure 31.4(b)), so we suspect that the inclination of a line is related to the gradient. We know that the gradient is a ratio of a change in the y-direction to a change in the x-direction.

$$m = \frac{\Delta y}{\Delta x}$$

But, in Figure 31.4(a) we see that

$$\begin{aligned} \tan\theta &= \frac{\Delta y}{\Delta x} \\ \therefore m &= \tan\theta \end{aligned}$$

For example, to find the inclination of the line $y = x$, we know $m = 1$

$$\begin{aligned} \therefore \tan\theta &= 1 \\ \therefore \theta &= 45° \end{aligned}$$

?

Exercise: Co-ordinate Geometry

1. Find the equations of the following lines
   A through points $(-1; 3)$ and $(1; 4)$
   B through points $(7; -3)$ and $(0; 4)$
   C parallel to $y = \frac{1}{2}x + 3$ passing through $(-1; 3)$
   D perpendicular to $y = -\frac{1}{2}x + 3$ passing through $(-1; 2)$
   E perpendicular to $2y + x = 6$ passing through the origin
2. Find the inclination of the following lines
   A $y = 2x - 3$
   B $y = \frac{1}{3}x - 7$
   C $4y = 3x + 8$
   D $y = -\frac{2}{3}x + 3$ (Hint: if m is negative θ must be in the second quadrant)
   E $3y + x - 3 = 0$
3. Show that the line $y = k$ for any constant k is parallel to the x-axis. (Hint: Show that the inclination of this line is $0°$.)
4. Show that the line $x = k$ for any constant k is parallel to the y-axis. (Hint: Show that the inclination of this line is $90°$.)

31.6 Transformations

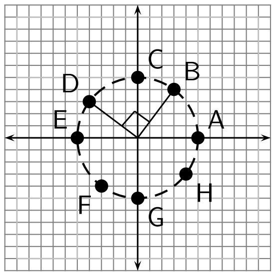
31.6.1 Rotation of a Point

When something is moved around a fixed point, we say that it is *rotated*. What happens to the coordinates of a point that is rotated by $90°$ or $180°$ around the origin?

Activity :: Investigation : Rotation of a Point by $90°$

Complete the table, by filling in the coordinates of the points shown in the figure.

Point	x-coordinate	y-coordinate
A		
B		
C		
D		
E		
F		
G		
H		

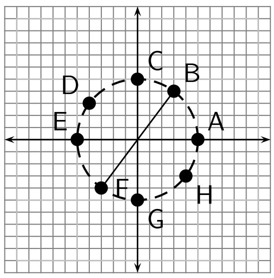


What do you notice about the x-coordinates?
What do you notice about the y-coordinates?
What would happen to the coordinates of point A, if it was rotated to the position of point C? What about point B rotated to the position of D?

Activity :: Investigation : Rotation of a Point by $180°$

Complete the table, by filling in the coordinates of the points shown in the figure.

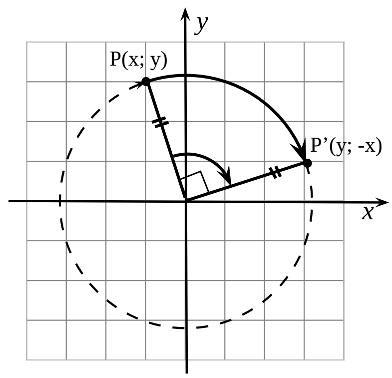
Point	x-coordinate	y-coordinate
A		
B		
C		
D		
E		
F		
G		
H		



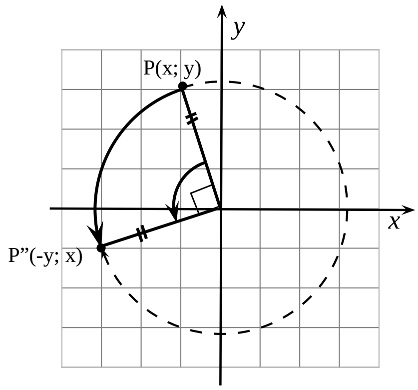
What do you notice about the x-coordinates?
What do you notice about the y-coordinates?
What would happen to the coordinates of point A, if it was rotated to the position of point E? What about point F rotated to the position of B?

From these activities you should have come to the following conclusions:

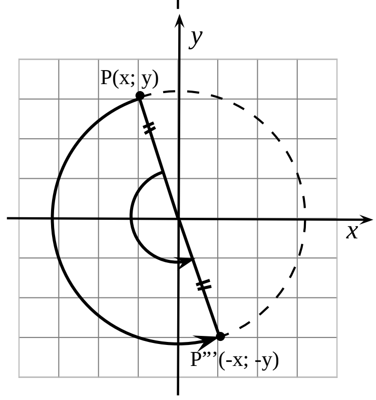
- 90° clockwise rotation:
  The image of a point P$(x; y)$ rotated clockwise through 90° around the origin is P'$(y; -x)$.
  We write the rotation as $(x; y) \rightarrow (y; -x)$.



- 90° anticlockwise rotation:
  The image of a point P$(x; y)$ rotated anticlockwise through 90° around the origin is P'$(-y; x)$.
  We write the rotation as $(x; y) \rightarrow (-y; x)$.



- 180° rotation:
  The image of a point P$(x; y)$ rotated through 180° around the origin is P'$(-x; -y)$.
  We write the rotation as $(x; y) \rightarrow (-x; -y)$.



Exercise: Rotation

1. For each of the following rotations about the origin:
   (i) Write down the rule.
   (ii) Draw a diagram showing the direction of rotation.

   A OA is rotated to OA′ with A(4;2) and A′(-2;4)

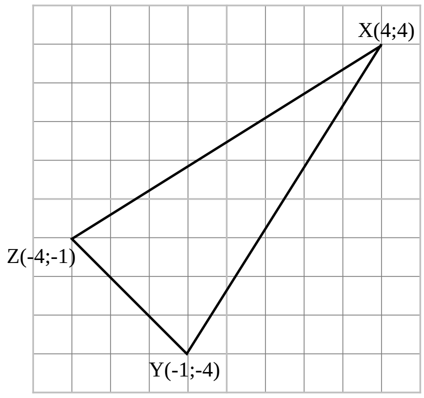
   B OB is rotated to OB′ with B(-2;5) and B′(5;2)

   C OC is rotated to OC′ with C(-1;-4) and C′(1;4)

2. Copy ΔXYZ onto squared paper. The co-ordinates are given on the picture.

   A Rotate ΔXYZ anti-clockwise through an angle of 90° about the origin to give ΔX′Y′Z′. Give the co-ordinates of X′, Y′ and Z′.

   B Rotate ΔXYZ through 180° about the origin to give ΔX′′Y′′Z′′. Give the co-ordinates of X′′, Y′′ and Z′′.

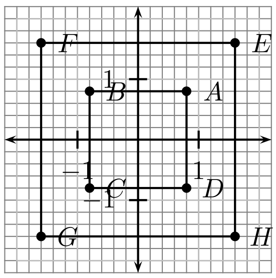


31.6.2 Enlargement of a Polygon 1

When something is made larger, we say that it is *enlarged*. What happens to the coordinates of a polygon that is enlarged by a factor k?

Activity :: Investigation : Enlargement of a Polygon

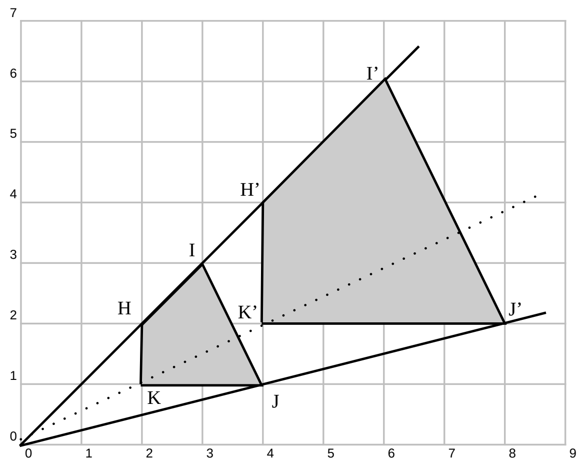
Complete the table, by filling in the coordinates of the points shown in the figure.



Point	x-coordinate	y-coordinate
A		
B		
C		
D		
E		
F		
G		
H		

What do you notice about the x-coordinates?
What do you notice about the y-coordinates?
What would happen to the coordinates of point A, if the square ABCD was enlarged by a factor 2?

Activity :: Investigation : Enlargement of a Polygon 2



In the figure quadrilateral HIJK has been enlarged by a factor of 2 through the origin to become H'I'J'K'. Complete the following table.

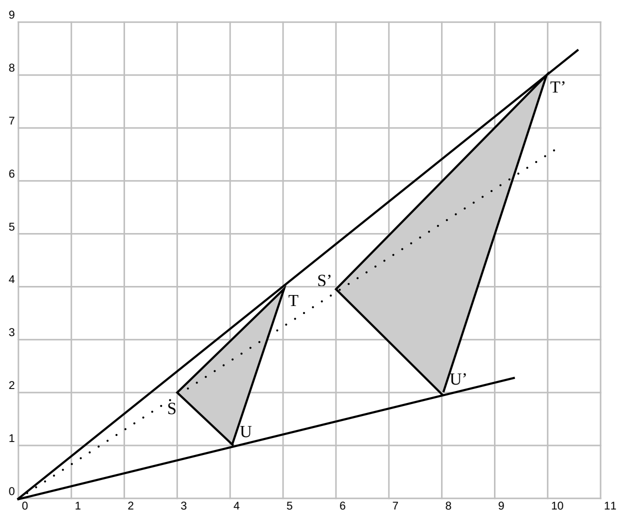
Co-ordinate	Co-ordinate'	Length	Length'
H = (;)	H' = (;)	OH =	OH' =
I = (;)	I' = (;)	OI =	OI' =
J = (;)	J' = (;)	OJ =	OJ' =
K = (;)	K' + (;)	OK =	OK' =

What conclusions can you draw about

1. the co-ordinates
2. the lengths when we enlarge by a factor of 2?

We conclude as follows:
Let the vertices of a triangle have co-ordinates S$(x_1; y_1)$, T$(x_2; y_2)$, U$(x_3; y_3)$. △S'T'U' is an enlargement through the origin of △STU by a factor of c $(c > 0)$.

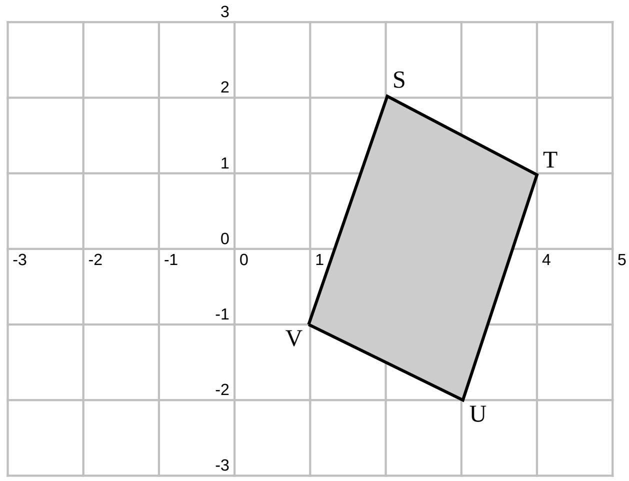
- △STU is a reduction of △S'T'U' by a factor of c.
- △S'T'U' can alternatively be seen as an reduction through the origin of △STU by a factor of $\frac{1}{c}$. (Note that a reduction by $\frac{1}{c}$ is the same as an enlargement by c).
- The vertices of △S'T'U' are S'$(cx_1; cy_1)$, T'(cx_2,cy_2), U'(cx_3,cy_3).
- The distances from the origin are OS' = cOS, OT' = cOT and OU' = cOU.



?

Exercise: Transformations

1. 1) Copy polygon STUV onto squared paper and then answer the following questions.



   A What are the co-ordinates of polygon STUV?

   B Enlarge the polygon through the origin by a constant factor of $c = 2$. Draw this on the same grid. Label it S'T'U'V'.

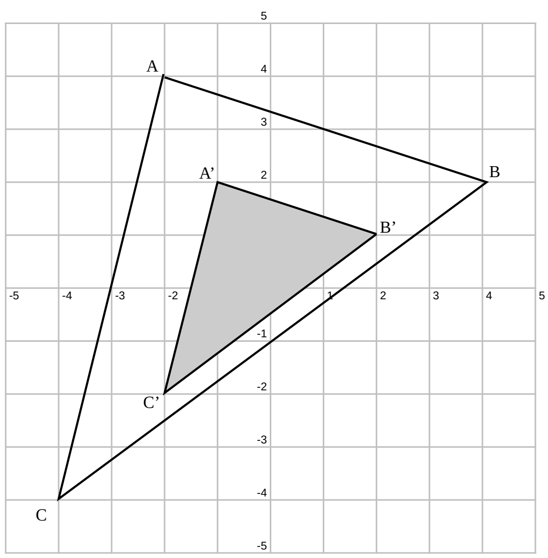
   C What are the co-ordinates of the vertices of S'T'U'V'?

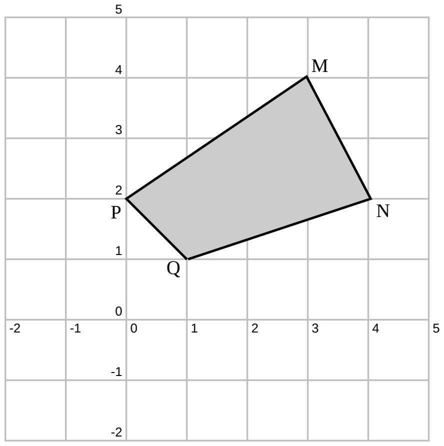
2. $\triangle$ABC is an enlargement of $\triangle$A'B'C' by a constant factor of k through the origin.

   A What are the co-ordinates of the vertices of $\triangle$ABC and $\triangle$A'B'C'?

   B Giving reasons, calculate the value of k.

   C If the area of $\triangle$ABC is m times the area of $\triangle$A'B'C', what is m?



3. 

A What are the co-ordinates of the vertices of polygon MNPQ?

B Enlarge the polygon through the origin by using a constant factor of $c = 3$, obtaining polygon M'N'P'Q'. Draw this on the same set of axes.

C What are the co-ordinates of the new vertices?

D Now draw M"N"P"Q" which is an anticlockwise rotation of MNPQ by 90° around the origin.

E Find the inclination of OM".

Chapter 32

Trigonometry - Grade 11

32.1 History of Trigonometry

Work in pairs or groups and investigate the history of the development of trigonometry. Describe the various stages of development and how different cultures used trigonometry to improve their lives.

The works of the following people or cultures can be investigated:

1. Cultures
   A Ancient Egyptians
   B Mesopotamians
   C Ancient Indians of the Indus Valley
2. People
   A Lagadha (circa 1350-1200 BC)
   B Hipparchus (circa 150 BC)
   C Ptolemy (circa 100)
   D Aryabhata (circa 499)
   E Omar Khayyam (1048-1131)
   F Bhaskara (circa 1150)
   G Nasir al-Din (13th century)
   H al-Kashi and Ulugh Beg (14th century)
   I Bartholemaeus Pitiscus (1595)

32.2 Graphs of Trigonometric Functions

32.2.1 Functions of the form $y = \sin(k\theta)$

In the equation, $y = \sin(k\theta)$, k is a constant and has different effects on the graph of the function. The general shape of the graph of functions of this form is shown in Figure 32.1 for the function $f(\theta) = \sin(2\theta)$.

?

Exercise: Functions of the Form $y = \sin(k\theta)$
On the same set of axes, plot the following graphs:

1. $a(\theta) = \sin 0.5\theta$

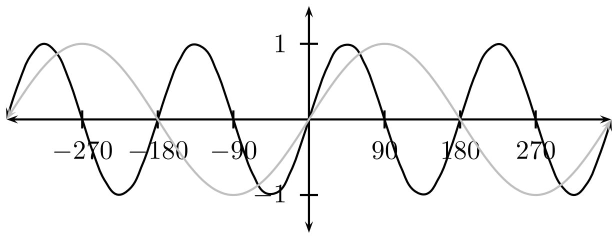


Figure 32.1: Graph of $f(\theta) = \sin(2\theta)$ with the graph of $g(\theta) = \sin(\theta)$ superimposed in gray.

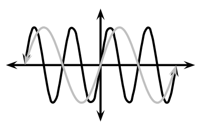
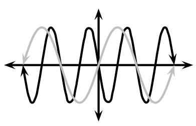
2. $b(\theta) = \sin 1\theta$
3. $c(\theta) = \sin 1.5\theta$
4. $d(\theta) = \sin 2\theta$
5. $e(\theta) = \sin 2.5\theta$

Use your results to deduce the effect of k.

You should have found that the value of k affects the periodicity of the graph. Notice that in the case of the sine graph, the period (length of one wave) is given by $\frac{360°}{k}$.

These different properties are summarised in Table 32.1.

Table 32.1: Table summarising general shapes and positions of graphs of functions of the form $y = \sin(kx)$. The curve $y = \sin(x)$ is shown in gray.

$k > 0$	$k < 0$
	

Domain and Range

For $f(\theta) = \sin(k\theta)$, the domain is $\{\theta : \theta \in \mathbb{R}\}$ because there is no value of $\theta \in \mathbb{R}$ for which $f(\theta)$ is undefined.

The range of $f(\theta) = \sin(k\theta)$ is $\{f(\theta) : f(\theta) \in [-1,1]\}$.

Intercepts

For functions of the form, $y = \sin(k\theta)$, the details of calculating the intercepts with the y axis are given.

There are many x-intercepts.

The y-intercept is calculated by setting $\theta = 0$:

$$\begin{aligned} y &= \sin(k\theta) \\ y_{int} &= \sin(0) \\ &= 0 \end{aligned}$$

32.2.2 Functions of the form $y = \cos(k\theta)$

In the equation, $y = \cos(k\theta)$, k is a constant and has different effects on the graph of the function. The general shape of the graph of functions of this form is shown in Figure 32.2 for the function $f(\theta) = \cos(2\theta)$.

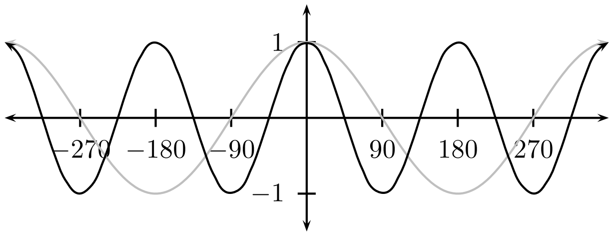


Figure 32.2: Graph of $f(\theta) = \cos(2\theta)$ with the graph of $g(\theta) = cos(\theta)$ superimposed in gray.

Exercise: Functions of the Form $y = \cos(k\theta)$
On the same set of axes, plot the following graphs:

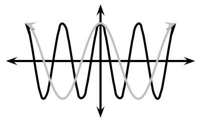
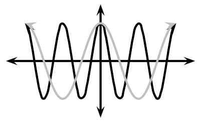
1. $a(\theta) = \cos 0.5\theta$
2. $b(\theta) = \cos 1\theta$
3. $c(\theta) = \cos 1.5\theta$
4. $d(\theta) = \cos 2\theta$
5. $e(\theta) = \cos 2.5\theta$

Use your results to deduce the effect of k.

You should have found that the value of k affects the periodicity of the graph. The period of the cosine graph is given by $\frac{360^\circ}{k}$.

These different properties are summarised in Table 32.2.

Table 32.2: Table summarising general shapes and positions of graphs of functions of the form $y = \cos(kx)$. The curve $y = \cos(x)$ is shown in gray.

$k > 0$	$k < 0$
	

Domain and Range

For $f(\theta) = \cos(k\theta)$, the domain is $\{\theta : \theta \in \mathbb{R}\}$ because there is no value of $\theta \in \mathbb{R}$ for which $f(\theta)$ is undefined.

The range of $f(\theta) = \cos(k\theta)$ is $\{f(\theta) : f(\theta) \in [-1,1]\}$.

Intercepts

For functions of the form, $y = \cos(k\theta)$, the details of calculating the intercepts with the y axis are given.

The y-intercept is calculated as follows:

$$\begin{aligned} y &= \cos(k\theta) \\ y_{int} &= \cos(0) \\ &= 1 \end{aligned}$$

32.2.3 Functions of the form $y = \tan(k\theta)$

In the equation, $y = \tan(k\theta)$, k is a constant and has different effects on the graph of the function. The general shape of the graph of functions of this form is shown in Figure 32.3 for the function $f(\theta) = \tan(2\theta)$.

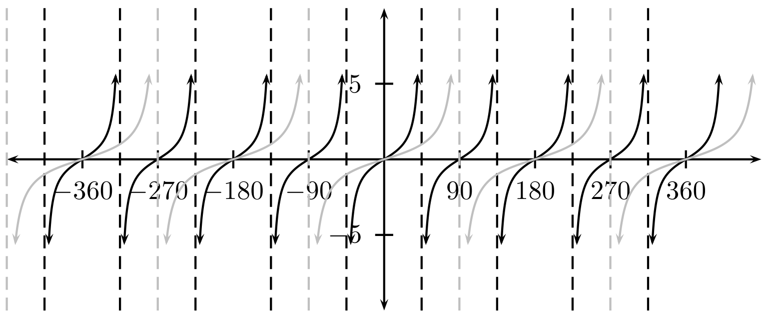


Figure 32.3: The graph of $\tan(2\theta)$ superimposed on the graph of $g(\theta) = \tan(\theta)$ (in gray). The asymptotes are shown as dashed lines.

?

Exercise: Functions of the Form $y = \tan(k\theta)$
On the same set of axes, plot the following graphs:

1. $a(\theta) = \tan 0.5\theta$
2. $b(\theta) = \tan 1\theta$
3. $c(\theta) = \tan 1.5\theta$
4. $d(\theta) = \tan 2\theta$
5. $e(\theta) = \tan 2.5\theta$

Use your results to deduce the effect of k.

You should have found that, once again, the value of k affects the periodicity of the graph. As k increases, the graph is more tightly packed. As k decreases, the graph is more spread out. The period of the tan graph is given by $\frac{180^\circ}{k}$.

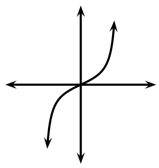
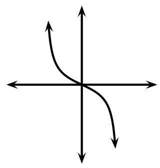
These different properties are summarised in Table 32.3.

Domain and Range

For $f(\theta) = \tan(k\theta)$, the domain of one branch is $\{\theta : \theta \in (-\frac{90^\circ}{k}, \frac{90^\circ}{k})\}$ because the function is undefined for $\theta = \frac{90^\circ}{k}$ and $\theta = \frac{90^\circ}{k}$.

The range of $f(\theta) = \tan(k\theta)$ is $\{f(\theta) : f(\theta) \in (-\infty,\infty)\}$.

Table 32.3: Table summarising general shapes and positions of graphs of functions of the form $y = \tan(k\theta)$.

$k > 0$	$k < 0$
	

Intercepts

For functions of the form, $y = \tan(k\theta)$, the details of calculating the intercepts with the x and y axis are given. There are many x-intercepts; each one is halfway between the asymptotes.

The y-intercept is calculated as follows:

$$\begin{aligned} y &= \tan(k\theta) \\ y_{int} &= \tan(0) \\ &= 0 \end{aligned}$$

Asymptotes

The graph of $\tan k\theta$ has asymptotes because as $k\theta$ approaches $90°$, $\tan k\theta$ approaches infinity. In other words, there is no defined value of the function at the asymptote values.

32.2.4 Functions of the form $y = \sin(\theta + p)$

In the equation, $y = \sin(\theta + p)$, p is a constant and has different effects on the graph of the function. The general shape of the graph of functions of this form is shown in Figure 32.4 for the function $f(\theta) = \sin(\theta + 30°)$.

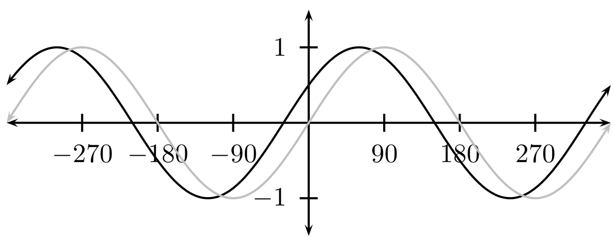


Figure 32.4: Graph of $f(\theta) = \sin(\theta + 30°)$ with the graph of $g(\theta) = \sin(\theta)$ in gray.

Exercise: Functions of the Form $y = \sin(\theta + p)$
On the same set of axes, plot the following graphs:

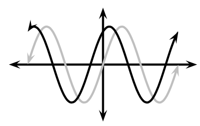
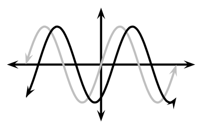
1. $a(\theta) = \sin(\theta - 90°)$
2. $b(\theta) = \sin(\theta - 60°)$
3. $c(\theta) = \sin\theta$
4. $d(\theta) = \sin(\theta + 90°)$
5. $e(\theta) = \sin(\theta + 180°)$

Use your results to deduce the effect of p.

You should have found that the value of p affects the y-intercept and phase shift of the graph. The p value shifts the graph horizontally. If p is positive, the graph shifts left and if p is negative tha graph shifts right.

These different properties are summarised in Table 32.4.

Table 32.4: Table summarising general shapes and positions of graphs of functions of the form $y = \sin(\theta + p)$.

$p > 0$	$p < 0$
	

Domain and Range

For $f(\theta) = \sin(\theta + p)$, the domain is $\{\theta : \theta \in \mathbb{R}\}$ because there is no value of $\theta \in \mathbb{R}$ for which $f(\theta)$ is undefined.

The range of $f(\theta) = \sin(\theta + p)$ is $\{f(\theta) : f(\theta) \in [-1,1]\}$.

Intercepts

For functions of the form, $y = \sin(\theta + p)$, the details of calculating the intercept with the y axis are given.

The y-intercept is calculated as follows: set $\theta = 0°$

$$\begin{aligned} y &= \sin(\theta + p) \\ y_{int} &= \sin(0 + p) \\ &= \sin(p) \end{aligned}$$

32.2.5 Functions of the form $y = \cos(\theta + p)$

In the equation, $y = \cos(\theta + p)$, p is a constant and has different effects on the graph of the function. The general shape of the graph of functions of this form is shown in Figure 32.5 for the function $f(\theta) = \cos(\theta + 30°)$.

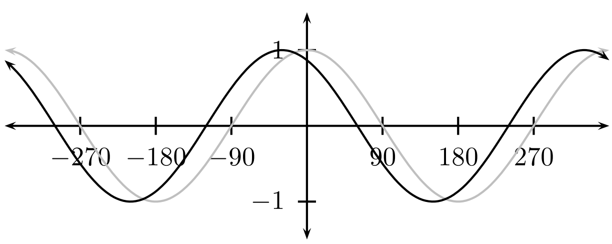


Figure 32.5: Graph of $f(\theta) = \cos(\theta + 30°)$ with the graph of $g(\theta) = \cos(\theta)$ shown in gray.

?

Exercise: Functions of the Form $y = \cos(\theta + p)$
On the same set of axes, plot the following graphs:

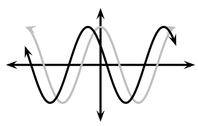
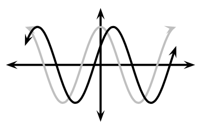
1. $a(\theta) = \cos(\theta - 90°)$
2. $b(\theta) = \cos(\theta - 60°)$
3. $c(\theta) = \cos\theta$
4. $d(\theta) = \cos(\theta + 90°)$
5. $e(\theta) = \cos(\theta + 180°)$

Use your results to deduce the effect of p.

You should have found that the value of p affects the y-intercept and phase shift of the graph. As in the case of the sine graph, positive values of p shift the cosine graph left while negative p values shift the graph right.

These different properties are summarised in Table 32.5.

Table 32.5: Table summarising general shapes and positions of graphs of functions of the form $y = \cos(\theta + p)$. The curve $y = \cos\theta$ is shown in gray.

$p > 0$	$p < 0$
	

Domain and Range

For $f(\theta) = \cos(\theta + p)$, the domain is $\{\theta : \theta \in \mathbb{R}\}$ because there is no value of $\theta \in \mathbb{R}$ for which $f(\theta)$ is undefined.

The range of $f(\theta) = \cos(\theta + p)$ is $\{f(\theta) : f(\theta) \in [-1,1]\}$.

Intercepts

For functions of the form, $y = \cos(\theta + p)$, the details of calculating the intercept with the y axis are given.

The y-intercept is calculated as follows: set $\theta = 0°$

$$\begin{aligned} y &= \cos(\theta + p) \\ y_{int} &= \cos(0 + p) \\ &= \cos(p) \end{aligned}$$

32.2.6 Functions of the form $y = \tan(\theta + p)$

In the equation, $y = \tan(\theta + p)$, p is a constant and has different effects on the graph of the function. The general shape of the graph of functions of this form is shown in Figure 32.6 for the function $f(\theta) = \tan(\theta + 30°)$.

Exercise: Functions of the Form $y = \tan(\theta + p)$
On the same set of axes, plot the following graphs:

1. $a(\theta) = \tan(\theta - 90°)$

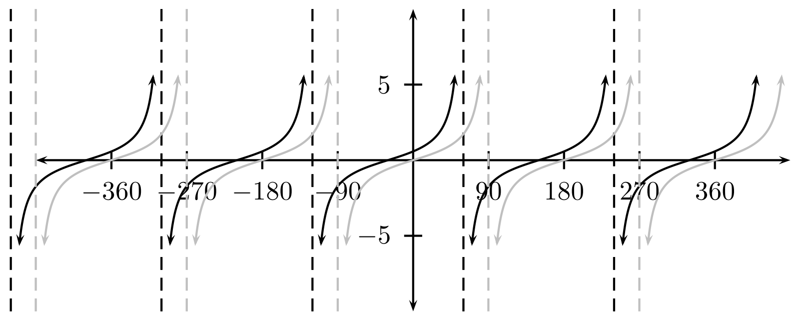


Figure 32.6: The graph of $\tan(\theta + 30°)$ with the graph of $g(\theta) = \tan(\theta)$ shown in gray.

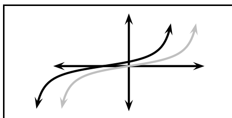
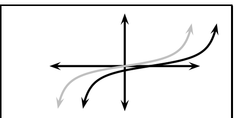
2. $b(\theta) = \tan(\theta - 60°)$
3. $c(\theta) = \tan\theta$
4. $d(\theta) = \tan(\theta + 60°)$
5. $e(\theta) = \tan(\theta + 180°)$

Use your results to deduce the effect of p.

You should have found that the value of p once again affects the y-intercept and phase shift of the graph. There is a horizontal shift to the left if p is positive and to the right if p is negative.

These different properties are summarised in Table 32.6.

Table 32.6: Table summarising general shapes and positions of graphs of functions of the form $y = \tan(\theta + p)$.

$k > 0$	$k < 0$
	

Domain and Range

For $f(\theta) = \tan(\theta + p)$, the domain for one branch is $\{\theta : \theta \in (-90° - p, 90° - p\}$ because the function is undefined for $\theta = -90° - p$ and $\theta = 90° - p$.

The range of $f(\theta) = \tan(\theta + p)$ is $\{f(\theta) : f(\theta) \in (-\infty, \infty)\}$.

Intercepts

For functions of the form, $y = \tan(\theta + p)$, the details of calculating the intercepts with the y axis are given.

The y-intercept is calculated as follows: set $\theta = 0°$

$$\begin{aligned} y &= \tan(\theta + p) \\ y_{int} &= \tan(p) \end{aligned}$$

Asymptotes

The graph of $\tan(\theta + p)$ has asymptotes because as $\theta + p$ approaches $90°$, $\tan(\theta + p)$ approaches infinity. Thus, there is no defined value of the function at the asymptote values.

Exercise: Functions of various form

Using your knowledge of the effects of p and k draw a rough sketch of the following graphs without a table of values.

1. $y = \sin 3x$
2. $y = -\cos 2x$
3. $y = \tan \frac{1}{2}x$
4. $y = \sin(x - 45°)$
5. $y = \cos(x + 45°)$
6. $y = \tan(x - 45°)$
7. $y = 2\sin 2x$
8. $y = \sin(x + 30°) + 1$

32.3 Trigonometric Identities

32.3.1 Deriving Values of Trigonometric Functions for $30°$, $45°$ and $60°$

Keeping in mind that trigonometric functions apply only to right-angled triangles, we can derive values of trigonometric functions for $30°$, $45°$ and $60°$. We shall start with $45°$ as this is the easiest.

Take any right-angled triangle with one angle $45°$. Then, because one angle is $90°$, the third angle is also $45°$. So we have an isosceles right-angled triangle as shown in Figure 32.7.

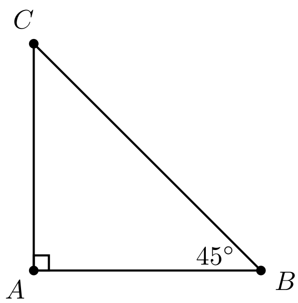


Figure 32.7: An isosceles right angled triangle.

If the two equal sides are of length a, then the hypotenuse, h, can be calculated as:

$$\begin{aligned} h^2 &= a^2 + a^2 \\ &= 2a^2 \\ \therefore \quad h &= \sqrt{2}a \end{aligned}$$

So, we have:

$$\begin{aligned} \sin(45°) &= \frac{\text{opposite}(45°)}{\text{hypotenuse}} \\ &= \frac{a}{\sqrt{2}a} \\ &= \frac{1}{\sqrt{2}} \end{aligned}$$

$$\begin{aligned}\cos(45^\circ) &= \frac{\text{adjacent}(45^\circ)}{\text{hypotenuse}}\\ &= \frac{a}{\sqrt{2}a}\\ &= \frac{1}{\sqrt{2}}\end{aligned}$$

$$\begin{aligned}\tan(45^\circ) &= \frac{\text{opposite}(45^\circ)}{\text{adjacent}(45^\circ)}\\ &= \frac{a}{a}\\ &= 1\end{aligned}$$

We can try something similar for 30° and 60°. We start with an equilateral triangle and we bisect one angle as shown in Figure 32.8. This gives us the right-angled triangle that we need, with one angle of 30° and one angle of 60°.

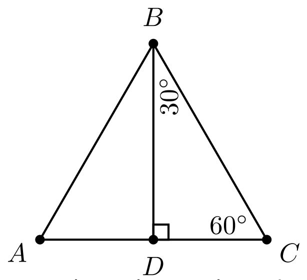


Figure 32.8: An equilateral triangle with one angle bisected.

If the equal sides are of length a, then the base is $\frac{1}{2}a$ and the length of the vertical side, v, can be calculated as:

$$\begin{aligned}v^2 &= a^2 - (\frac{1}{2}a)^2\\ &= a^2 - \frac{1}{4}a^2\\ &= \frac{3}{4}a^2\\ \therefore \quad v &= \frac{\sqrt{3}}{2}a\end{aligned}$$

So, we have:

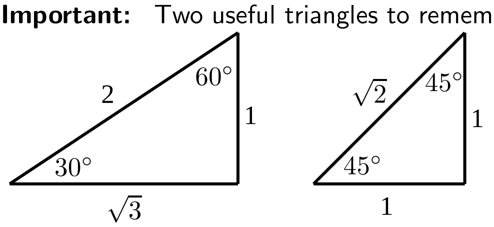
$$\begin{aligned}\sin(30^\circ) &= \frac{\text{opposite}(30^\circ)}{\text{hypotenuse}} \\ &= \frac{\frac{a}{2}}{a} \\ &= \frac{1}{2}\end{aligned} \qquad \begin{aligned}\sin(60^\circ) &= \frac{\text{opposite}(60^\circ)}{\text{hypotenuse}} \\ &= \frac{\frac{\sqrt{3}}{2}a}{a} \\ &= \frac{\sqrt{3}}{2}\end{aligned}$$

$$\begin{aligned}\cos(30^\circ) &= \frac{\text{adjacent}(30^\circ)}{\text{hypotenuse}} \\ &= \frac{\frac{\sqrt{3}}{2}a}{a} \\ &= \frac{\sqrt{3}}{2}\end{aligned} \qquad \begin{aligned}\cos(60^\circ) &= \frac{\text{adjacent}(60^\circ)}{\text{hypotenuse}} \\ &= \frac{\frac{a}{2}}{a} \\ &= \frac{1}{2}\end{aligned}$$

$$\begin{aligned}\tan(30^\circ) &= \frac{\text{opposite}(30^\circ)}{\text{adjacent}(30^\circ)} \\ &= \frac{\frac{a}{2}}{\frac{\sqrt{3}}{2}a} \\ &= \frac{1}{\sqrt{3}}\end{aligned} \qquad \begin{aligned}\tan(60^\circ) &= \frac{\text{opposite}(60^\circ)}{\text{adjacent}(60^\circ)} \\ &= \frac{\frac{\sqrt{3}}{2}a}{\frac{a}{2}} \\ &= \sqrt{3}\end{aligned}$$

You do not have to memorise these identities if you know how to work them out.

Important: Two useful triangles to remember



32.3.2 Alternate Definition for $\tan\theta$

We know that $\tan\theta$ is defined as:

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

This can be written as:

$$\begin{aligned}\tan\theta &= \frac{\text{opposite}}{\text{adjacent}} \times \frac{\text{hypotenuse}}{\text{hypotenuse}} \\ &= \frac{\text{opposite}}{\text{hypotenuse}} \times \frac{\text{hypotenuse}}{\text{adjacent}}\end{aligned}$$

But, we also know that $\sin\theta$ is defined as:

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$$

and that $\cos\theta$ is defined as:

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$$

Therefore, we can write

$$\begin{aligned}\tan\theta &= \frac{\text{opposite}}{\text{hypotenuse}} \times \frac{\text{hypotenuse}}{\text{adjacent}} \\ &= \sin\theta \times \frac{1}{\cos\theta} \\ &= \frac{\sin\theta}{\cos\theta}\end{aligned}$$

Important: $\tan\theta$ can also be defined as:

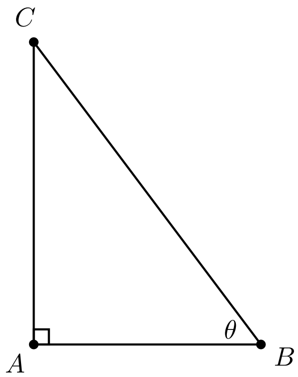
$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

32.3.3 A Trigonometric Identity

One of the most useful results of the trigonometric functions is that they are related to each other. We have seen that $\tan\theta$ can be written in terms of $\sin\theta$ and $\cos\theta$. Similarly, we shall show that:

$$\sin^2\theta + \cos^2\theta = 1$$

We shall start by considering $\triangle ABC$,



We see that:

$$\sin\theta = \frac{AC}{BC}$$

and

$$\cos\theta = \frac{AB}{BC}.$$

We also know from the Theorem of Pythagoras that:

$$AB^2 + AC^2 = BC^2.$$

So we can write:

$$\begin{aligned}\sin^2\theta + \cos^2\theta &= \left(\frac{AC}{BC}\right)^2 + \left(\frac{AB}{BC}\right)^2 \\ &= \frac{AC^2}{BC^2} + \frac{AB^2}{BC^2} \\ &= \frac{AC^2 + AB^2}{BC^2} \\ &= \frac{BC^2}{BC^2} \qquad \text{(from Pythagoras)} \\ &= 1\end{aligned}$$

Worked Example 133: Trigonometric Identities A

Question: Simplify using identities:

1. $\tan^2\theta \cdot \cos^2\theta$
2. $\frac{1}{\cos^2\theta} - \tan^2\theta$

Answer

Step 1 : Use known identities to replace $\tan\theta$

$$\begin{aligned} &= \tan^2\theta \cdot \cos^2\theta \\ &= \frac{\sin^2\theta}{\cos^2\theta} \cdot \cos^2\theta \\ &= \sin^2\theta \end{aligned}$$

Step 2 : Use known identities to replace $\tan\theta$

$$\begin{aligned} &= \frac{1}{\cos^2\theta} - \tan^2\theta \\ &= \frac{1}{\cos^2\theta} - \frac{\sin^2\theta}{\cos^2\theta} \\ &= \frac{1-\sin^2\theta}{\cos^2\theta} \\ &= \frac{\cos^2\theta}{\cos^2\theta} = 1 \end{aligned}$$

Worked Example 134: Trigonometric Identities B

Question: Prove: $\frac{1-\sin x}{\cos x} = \frac{\cos x}{1+\sin x}$

Answer

$$\begin{aligned} \text{LHS} &= \frac{1-\sin x}{\cos x} \\ &= \frac{1-\sin x}{\cos x} \times \frac{1+\sin x}{1+\sin x} \\ &= \frac{1-\sin^2 x}{\cos x(1+\sin x)} \\ &= \frac{\cos^2 x}{\cos x(1+\sin x)} \\ &= \frac{\cos x}{1+\sin x} = \text{RHS} \end{aligned}$$

Exercise: Trigonometric identities

1. Simplify the following using the fundamental trigonometric identities:
   A $\frac{\cos\theta}{\tan\theta}$
   B $\cos^2\theta.\tan^2\theta\ +\ \tan^2\theta.\sin^2\theta$
   C $1-\tan^2\theta.\sin^2\theta$
   D $1-\sin\theta.\cos\theta.\tan\theta$
   E $1-\sin^2\theta$
   F $\left(\frac{1-\cos^2\theta}{\cos^2\theta}\right)-\cos^2\theta$
2. Prove the following:
   A $\frac{1+\sin\theta}{\cos\theta}=\frac{\cos\theta}{1-\sin\theta}$
   B $\sin^2\theta+(\cos\theta-\tan\theta)(\cos\theta+\tan\theta)=1-\tan^2\theta$
   C $\frac{(2\cos^2\theta-1)}{1}+\frac{1}{(1+\tan^2\theta)}=\frac{1-\tan^2\theta}{1+\tan^2\theta}$
   D $\frac{1}{\cos\theta}-\frac{\cos\theta\tan^2\theta}{1}=1$
   E $\frac{2\sin\theta\cos\theta}{\sin\theta+\cos\theta}=\sin\theta+\cos\theta-\frac{1}{\sin\theta+\cos\theta}$
   F $\left(\frac{\cos\theta}{\sin\theta}+\tan\theta\right)\cdot\cos\theta=\frac{1}{\sin\theta}$

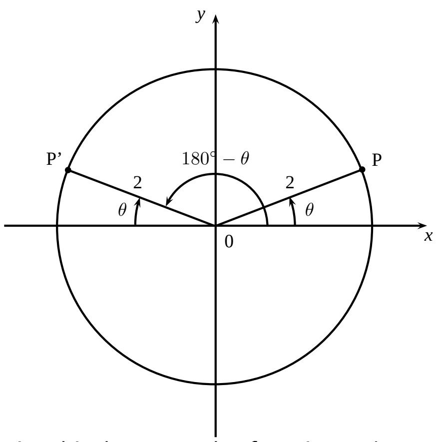
32.3.4 Reduction Formula

Any trigonometric function whose argument is $90°\pm\theta$, $180°\pm\theta$, $270°\pm\theta$ and $360°\pm\theta$ (hence $-\theta$) can be written simply in terms of θ. For example, you may have noticed that the cosine graph is identical to the sine graph except for a phase shift of $90°$. From this we may expect that $\sin(90°+\theta)=\cos\theta$.

Function Values of $180°\pm\theta$

Activity :: Investigation : Reduction Formulae for Function Values of $180°\pm\theta$

1. **Function Values of** $(180°-\theta)$
   A In the figure P and P' lie on the circle with radius 2. OP makes an angle $\theta=30°$ with the x-axis. P thus has co-ordinates $(\sqrt{3};1)$. If P' is the reflection of P about the y-axis (or the line $x=0$), use symmetry to write down the co-ordinates of P'.
   B Write down values for $\sin\theta$, $\cos\theta$ and $\tan\theta$.
   C Using the co-ordinates for P' determine $\sin(180°-\theta)$, $\cos(180°-\theta)$ and $\tan(180°-\theta)$.
   (d) From your results try and determine a relationship between the function values of $(180°-\theta)$ and θ.

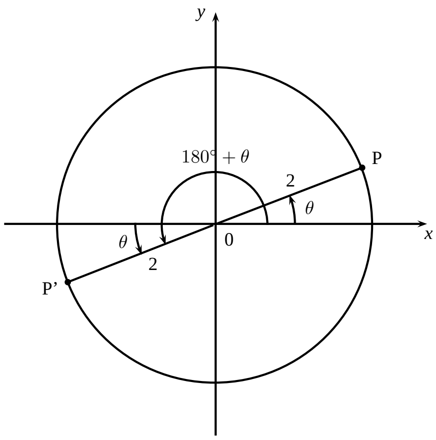


2. **Function values of** $(180°+\theta)$

A In the figure P and P' lie on the circle with radius 2. OP makes an angle $\theta = 30°$ with the x-axis. P thus has co-ordinates $(\sqrt{3}; 1)$. P' is the inversion of P through the origin (reflection about both the x- and y-axes) and lies at an angle of $180° + \theta$ with the x-axis. Write down the co-ordinates of P'.

B Using the co-ordinates for P' determine $\sin(180° + \theta)$, $\cos(180° + \theta)$ and $\tan(180° + \theta)$.

C From your results try and determine a relationship between the function values of $(180° + \theta)$ and θ.



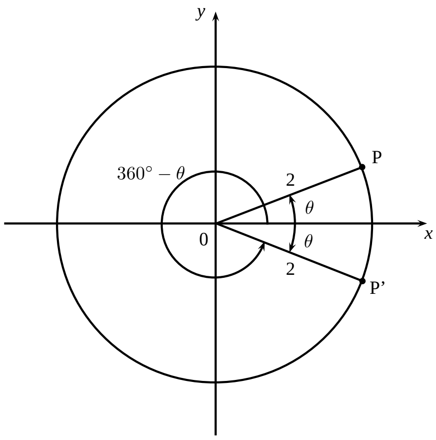
Activity :: Investigation : Reduction Formulae for Function Values of $360° \pm \theta$

1. **Function values of** $(360° - \theta)$

A In the figure P and P' lie on the circle with radius 2. OP makes an angle $\theta = 30°$ with the x-axis. P thus has co-ordinates $(\sqrt{3}; 1)$. P' is the reflection of P about the x-axis or the line $y = 0$. Using symmetry, write down the co-ordinates of P'.

B Using the co-ordinates for P' determine $\sin(360° - \theta)$, $\cos(360° - \theta)$ and $\tan(360° - \theta)$.

C From your results try and determine a relationship between the function values of $(360° - \theta)$ and θ.



It is possible to have an angle which is larger than $360°$. The angle completes one revolution to give $360°$ and then continues to give the required angle. We get the following results:

$$\begin{aligned} \sin(360° + \theta) &= \sin\theta \\ \cos(360° + \theta) &= \cos\theta \\ \tan(360° + \theta) &= \tan\theta \end{aligned}$$

Note also, that if k is any integer, then

$$\begin{aligned} \sin(360° \cdot k + \theta) &= \sin\theta \\ \cos(360° \cdot k + \theta) &= \cos\theta \\ \tan(360° \cdot k + \theta) &= \tan\theta \end{aligned}$$

Worked Example 135: Basic use of a reduction formula

Question: Write $\sin 293°$ as the function of an acute angle.

Answer

We note that $293° = 360° - 67°$ thus

$$\begin{aligned}\sin 293° &= \sin(360° - 67°)\\ &= -\sin 67°\end{aligned}$$

where we used the fact that $\sin(360° - \theta) = -\sin\theta$. Check, using your calculator, that these values are in fact equal:

$$\begin{aligned}\sin 293° &= -0{,}92\cdots\\ -\sin 67° &= -0{,}92\cdots\end{aligned}$$

Worked Example 136: More complicated...

Question: Evaluate without using a calculator:

$$\tan^2 210° - (1 + \cos 120°)\sin^2 225°$$

Answer

$$\begin{aligned}&\tan^2 210° - (1 + \cos 120°)\sin^2 225°\\ =\ &[\tan(180° + 30°)]^2 - [1 + \cos(180° - 60°)] \cdot [\sin(180° + 45°)]^2\\ =\ &(\tan 30°)^2 - [1 + (-\cos 60°)] \cdot (-\sin 45°)^2\\ =\ &\left(\frac{1}{\sqrt{3}}\right)^2 - \left(1 - \frac{1}{2}\right) \cdot \left(-\frac{1}{\sqrt{2}}\right)^2\\ =\ &\frac{1}{3} - \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\\ =\ &\frac{1}{3} - \frac{1}{4} = \frac{1}{12}\end{aligned}$$

Exercise: Reduction Formulae

1. Write these equations as a function of θ only:
   - A $\sin(180° - \theta)$
   - B $\cos(180° - \theta)$
   - C $\cos(360° - \theta)$
   - D $\cos(360° + \theta)$
   - E $\tan(180° - \theta)$
   - F $\cos(360° + \theta)$
2. Write the following trig functions as a function of an acute angle, then find the actual angle with your calculator:
   - A $\sin 163°$

B $\cos 327°$
C $\tan 248°$
D $\cos 213°$

3. Determine the following without the use of a calculator:
   A $\tan\ 150°.\sin\ 30°\ +\ \cos\ 330°$
   B $\tan\ 300°.\cos\ 120°$
   C $(1-\cos\ 30°)(1-\sin\ 210°)$
   D $\cos\ 780°\ +\ \sin\ 315°.\tan\ 420°$

4. Determine the following by reducing to an acute angle and using special angles. Do not use a calculator:
   A $\cos\ 300°$
   B $\sin\ 135°$
   C $\cos\ 150°$
   D $\tan\ 330°$
   E $\sin\ 120°$
   F $\tan^2 225°$
   G $\cos\ 315°$
   H $\sin^2 420°$
   I $\tan\ 405°$
   J $\cos\ 1020°$
   K $\tan^2 135°$
   L $1-\sin^2 210°$

Function Values of $(-\theta)$

When the argument of a trigonometric function is $(-\theta)$ we can add $360°$ without changing the result. Thus for sine and cosine

$$\sin(-\theta) = \sin(360° - \theta) = -\sin\theta$$

$$\cos(-\theta) = \cos(360° - \theta) = \cos\theta$$

Function Values of $90° \pm \theta$

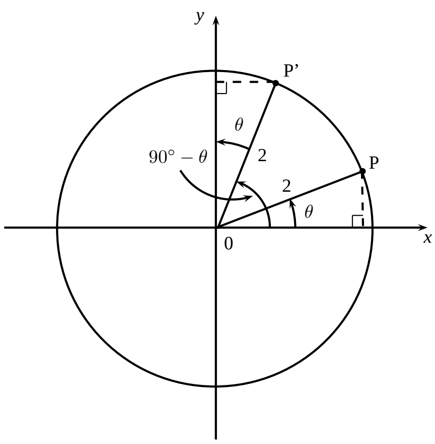
Activity :: Investigation : Reduction Formulae for Function Values of $90° \pm \theta$

1. **Function values of** $(90° - \theta)$

   A In the figure P and P' lie on the circle with radius 2. OP makes an angle $\theta = 30°$ with the x-axis. P thus has co-ordinates $(\sqrt{3}; 1)$. P' is the reflection of P about the line $y = x$. Using symmetry, write down the co-ordinates of P'.

   B Using the co-ordinates for P' determine $\sin(90°-\theta)$, $\cos(90°-\theta)$ and $\tan(90°-\theta)$.

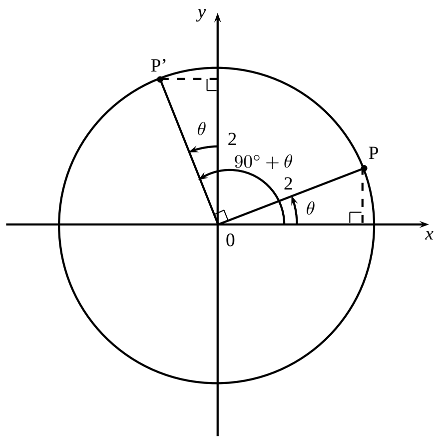
   C From your results try and determine a relationship between the function values of $(90° - \theta)$ and θ.



2. **Function values of** $(90° + \theta)$
   A In the figure P and P' lie on the circle with radius 2. OP makes an angle $\theta = 30°$ with the x-axis. P thus has co-ordinates $(\sqrt{3}; 1)$. P' is the rotation of P through $90°$. Using symmetry, write down the co-ordinates of P'. (Hint: consider P' as the reflection of P about the line $y = x$ followed by a reflection about the y-axis)

   B Using the co-ordinates for P' determine $\sin(90°+\theta)$, $\cos(90°+\theta)$ and $\tan(90°+\theta)$.

   C From your results try and determine a relationship between the function values of $(90° + \theta)$ and θ.



Complementary angles are positive acute angles that add up to $90°$. e.g. $20°$ and $70°$ are complementary angles.

Sine and cosine are known as *co-functions*. The other co-functions are secant and cosecant, and tangent and cotangent.

The function value of an angle is equal to the co-function of its complement (the co-co rule).

Thus for sine and cosine we have

$$\begin{aligned} \sin(90° - \theta) &= \cos\theta \\ \cos(90° - \theta) &= \sin\theta \end{aligned}$$

Worked Example 137: Co-co rule

Question: Write each of the following in terms of $40°$ using $\sin(90° - \theta) = \cos\theta$ and $\cos(90° - \theta) = \sin\theta$.

1. $\cos 50°$
2. $\sin 320°$
3. $\cos 230°$

Answer

1. $\cos 50° = \cos(90° - 40°) = \sin 40°$
2. $\sin 320° = \sin(360° - 40°) = -\sin 40°$
3. $\cos 230° = \cos(180° + 50°) = -\cos 50° = -\cos(90° - 40°) = -\sin 40°$

Function Values of $(\theta - 90°)$

$\sin(\theta - 90°) = -\cos\theta$ and $\cos(\theta - 90°) = \sin\theta$.

These results may be proved as follows:

$$\begin{aligned} \sin(\theta - 90°) &= \sin[-(90° - \theta)] \\ &= -\sin(90° - \theta) \\ &= -\cos\theta \end{aligned}$$

and likewise for $\cos(\theta - 90°) = \sin\theta$

Summary

The following summary may be made

second quadrant $(180° - \theta)$ **or** $(90° + \theta)$ $\sin(180° - \theta) = +\sin\theta$ $\cos(180° - \theta) = -\cos\theta$ $\tan(180° - \theta) = -\tan\theta$ $\sin(90° + \theta) = +\cos\theta$ $\cos(90° + \theta) = -\sin\theta$	**first quadrant** (θ) **or** $(90° - \theta)$ all trig functions are positive $\sin(360° + \theta) = \sin\theta$ $\cos(360° + \theta) = \cos\theta$ $\tan(360° + \theta) = \tan\theta$ $\sin(90° - \theta) = \sin\theta$ $\cos(90° - \theta) = \cos\theta$
third quadrant $(180° + \theta)$ $\sin(180° + \theta) = -\sin\theta$ $\cos(180° + \theta) = -\cos\theta$ $\tan(180° + \theta) = +\tan\theta$	**fourth quadrant** $(360° - \theta)$ $\sin(360° - \theta) = -\sin\theta$ $\cos(360° - \theta) = +\cos\theta$ $\tan(360° - \theta) = -\tan\theta$

Important:

1. These reduction formulae hold for any angle θ. For convenience, we usually work with θ as if it is acute, i.e. $0° < \theta < 90°$.
2. When determining function values of $180° \pm \theta$, $360° \pm \theta$ and $-\theta$ the functions never change.
3. When determining function values of $90° \pm \theta$ and $\theta - 90°$ the functions changes to its co-function (co-co rule).

Extension: Function Values of $(270° \pm \theta)$
Angles in the third and fourth quadrants may be written as $270° \pm \theta$ with θ an acute angle. Similar rules to the above apply. We get

third quadrant $(270° - \theta)$	**fourth quadrant** $(270° + \theta)$
$\sin(270° - \theta) = -\cos\theta$	$\sin(270° + \theta) = -\cos\theta$
$\cos(270° - \theta) = -\sin\theta$	$\cos(270° + \theta) = +\sin\theta$

32.4 Solving Trigonometric Equations

Chapters **??** and **??** focussed on the solution of algebraic equations and excluded equations that dealt with trigonometric functions (i.e. $\sin$ and $\cos$). In this section, the solution of trigonometric equations will be discussed.

The methods described in Chapters **??** and **??** also apply here. In most cases, trigonometric identities will be used to simplify equations, before finding the final solution. The final solution can be found either graphically or using inverse trigonometric functions.

32.4.1 Graphical Solution

As an example, to introduce the methods of solving trigonometric equations, consider

$$\sin\theta = 0{,}5 \tag{32.1}$$

As explained in Chapters **??** and **??**, the solution of Equation 32.1 is obtained by examining the intersecting points of the graphs of:

$$\begin{aligned} y &= \sin\theta \\ y &= 0{,}5 \end{aligned}$$

Both graphs, for $-720° < \theta < 720°$, are shown in Figure 32.9 and the intersection points of the graphs are shown by the dots.

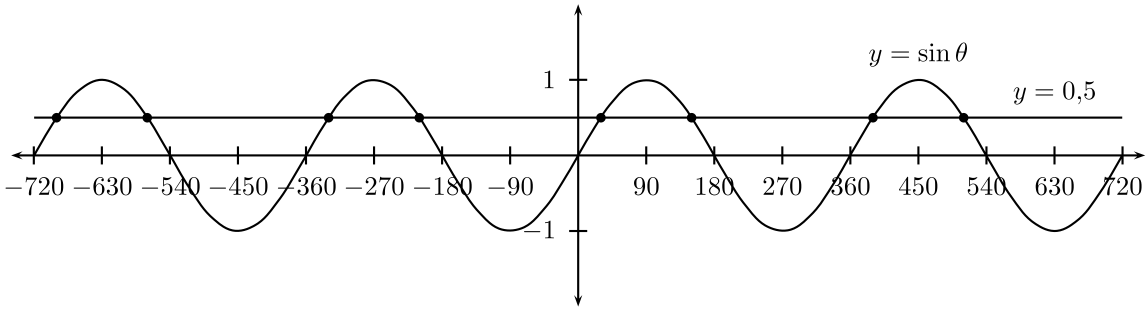


Figure 32.9: Plot of $y = \sin\theta$ and $y = 0{,}5$ showing the points of intersection, hence the solutions to the equation $\sin\theta = 0{,}5$.

In the domain for θ of $-720° < \theta < 720°$, there are 8 possible solutions for the equation $\sin\theta = 0{,}5$. These are $\theta = [-690°, -570°, -330°, -210°, 30°, 150°, 390°, 510°]$

Worked Example 138:

Question: Find θ, if $\tan\theta + 0{,}5 = 1{,}5$, with $0° < \theta < 90°$. Determine the solution graphically.

Answer

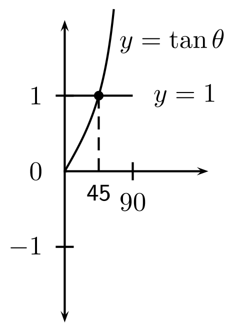
Step 1 : Write the equation so that all the terms with the unknown quantity (i.e. θ) are on one side of the equation.

$$\begin{aligned} \tan\theta + 0{,}5 &= 1{,}5 \\ \tan\theta &= 1 \end{aligned}$$

Step 2 : Identify the two functions which are intersecting.

$$\begin{aligned} y &= \tan\theta \\ y &= 1 \end{aligned}$$

Step 3 : Draw graphs of both functions, over the required domain and identify the intersection point.



The graphs intersect at $\theta = 45°$.

32.4.2 Algebraic Solution

The inverse trigonometric functions $\arcsin$, $\arccos$ and $\arctan$ can also be used to solve trigonometric equations. These are shown as second functions on most calculators: $\sin^{-1}$, $\cos^{-1}$ and $\tan^{-1}$.

Using inverse trigonometric functions, the equation

$$\sin\theta = 0{,}5$$

is solved as

$$\begin{aligned} \sin\theta &= 0{,}5 \\ \therefore \quad \theta &= \arcsin 0{,}5 \\ &= 30^\circ \end{aligned}$$

Worked Example 139:

Question: Find θ, if $\tan\theta + 0{,}5 = 1{,}5$, with $0^\circ < \theta < 90^\circ$. Determine the solution using inverse trigonometric functions.

Answer

Step 1 : Write the equation so that all the terms with the unknown quantity (i.e. θ) are on one side of the equation. Then solve for the angle using the inverse function.

$$\begin{aligned} \tan\theta + 0{,}5 &= 1{,}5 \\ \tan\theta &= 1 \\ \therefore \quad \theta &= \arctan 1 \\ &= 45^\circ \end{aligned}$$

Trigonometric equations often look very simple. Consider solving the equation $\sin\theta = 0{,}7$. We can take the inverse sine of both sides to find that $\theta = \sin^{-1}(0{,}7)$. If we put this into a calculator we find that $\sin^{-1}(0{,}7) = 44{,}42^\circ$. This is true, however, it does not tell the whole story.

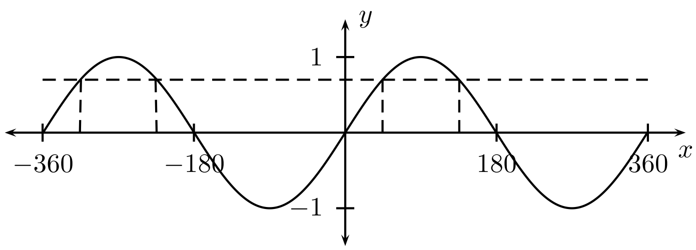


Figure 32.10: The sine graph. The dotted line represents $y = 0{,}7$. There are four points of intersection on this interval, thus four solutions to $\sin\theta = 0{,}7$.

As you can see from figure 32.10, there are *four* possible angles with a sine of 0.7 between -360° and 360°. If we were to extend the range of the sine graph to infinity we would in fact see that there are an infinite number of solutions to this equation! This difficulty (which is caused by the periodicity of the sine function) makes solving trigonometric equations much harder than they may seem to be.

Any problem on trigonometric equations will require two pieces of information to solve. The first is the equation itself and the second is the *range* in which your answers must lie. The hard part is making sure you find all of the possible answers within the range. Your calculator will always give you the *smallest* answer (*i.e.* the one that lies between $-90°$ and $90°$ for tangent and sine and one between $0°$ and $180°$ for cosine). Bearing this in mind we can already solve trigonometric equations within these ranges.

Worked Example 140:

Question: Find the values of x for which $\sin\left(\frac{x}{2}\right) = 0{,}5$ if it is given that $x < 90°$.
Answer
Because we are told that x is an acute angle, we can simply apply an inverse trigonometric function to both sides.

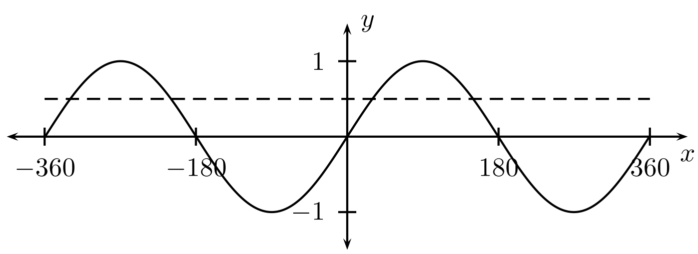
$$\sin\left(\frac{x}{2}\right) = 0.5 \quad (32.2)$$
$$\Rightarrow \frac{x}{2} = \arcsin 0.5 \quad (32.3)$$
$$\Rightarrow \frac{x}{2} = 30° \quad (32.4)$$
$$\therefore x = 60° \quad (32.5)$$

We can, of course, solve trigonometric equations in any range by drawing the graph.

Worked Example 141:

Question: For what values of x does $\sin x = 0{,}5$, when $-360° \leq x \leq 360°$?

Answer
Step 1 : Draw the graph
We take a look at the graph of $\sin x = 0{,}5$ on the interval $[-360°, 360°]$. We want to know when the y value of the graph is $0{,}5$, so we draw in a line at $y = 0{,}5$.



Step 2 :
Notice that this line touches the graph four times. This means that there are four solutions to the equation.
Step 3 :
Read off the x values of those intercepts from the graph as $x = -330°$, $-210°$, $30°$ and $150°$.

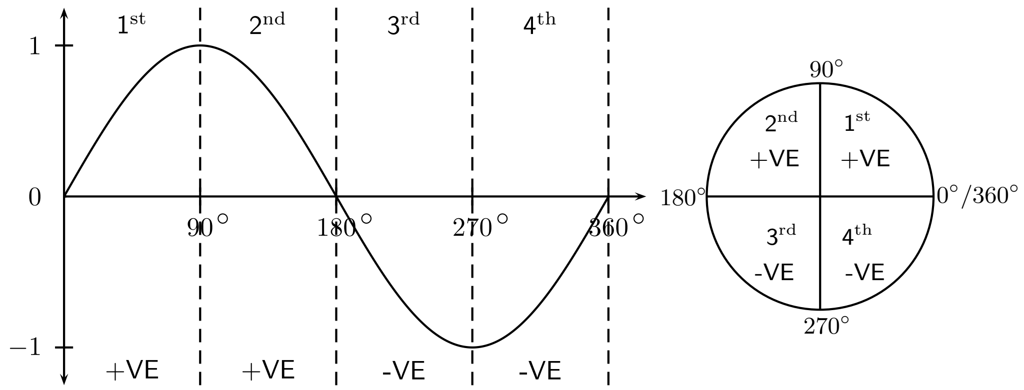
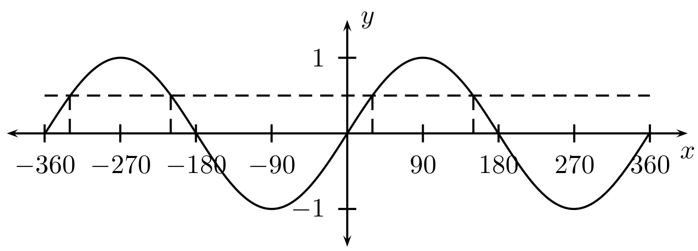


Figure 32.11: The graph and unit circle showing the sign of the sine function.



This method can be time consuming and inexact. We shall now look at how to solve these problems algebraically.

32.4.3 Solution using CAST diagrams

The Sign of the Trigonometric Function

The first step to finding the trigonometry of any angle is to determine the *sign* of the ratio for a given angle. We shall do this for the sine function first and do the same for the cosine and tangent.

In figure 32.11 we have split the sine graph into four *quadrants*, each $90°$ wide. We call them quadrants because they correspond to the four quadrants of the unit circle. We notice from figure 32.11 that the sine graph is positive in the 1^{st} and 2^{nd} quadrants and negative in the 3^{rd} and 4^{th}. Figure 32.12 shows similar graphs for cosine and tangent.

All of this can be summed up in two ways. Table 32.7 shows which trigonometric functions are positive and which are negative in each quadrant.

	1^{st}	2^{nd}	3^{rd}	4^{th}
sin	+VE	+VE	-VE	-VE
cos	+VE	-VE	-VE	+VE
tan	+VE	-VE	+VE	-VE

Table 32.7: The signs of the three basic trigonometric functions in each quadrant.

A more convenient way of writing this is to note that all functions are positive in the 1^{st} quadrant, only sine is positive in the 2^{nd}, only tangent in the 3^{rd} and only cosine in the 4^{th}. We express

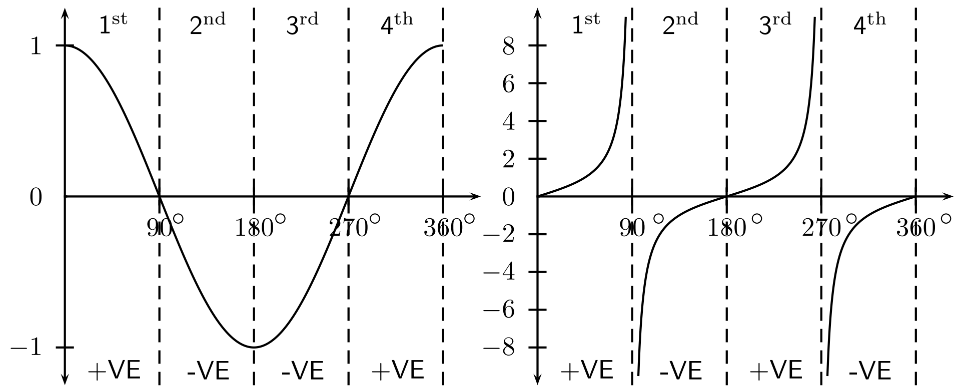


Figure 32.12: Graphs showing the sign of the cosine and tangent functions.

this using the CAST diagram (figure 32.13). This diagram is known as a CAST diagram as the letters, taken anticlockwise from the bottom right, read C-A-S-T. The letter in each quadrant tells us which trigonometric functions are *positive* in that quadrant. The 'A' in the 1^{st} quadrant stands for all (meaning sine, cosine and tangent are all positive in this quadrant). 'S', 'C' and 'T' ,of course, stand for sine, cosine and tangent. The diagram is shown in two forms. The version on the left shows the CAST diagram including the unit circle. This version is useful for equations which lie in large or negative ranges. The simpler version on the right is useful for ranges between $0°$ and $360°$. Another useful diagram shown in figure 32.13 gives the formulae to use in each quadrant when solving a trigonometric equation.

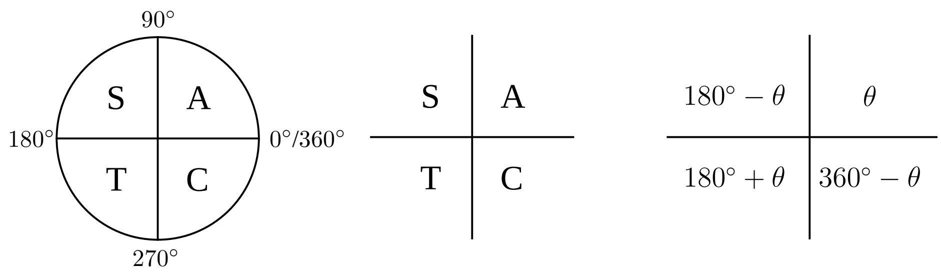


Figure 32.13: The two forms of the CAST diagram and the formulae in each quadrant.

Magnitude of the trigonometric functions

Now that we know in which quadrants our solutions lie, we need to know which angles in these quadrants satisfy our equation.
Calculators give us the smallest possible answer (sometimes negative) which satisfies the equation. For example, if we wish to solve $\sin\theta = 0{,}3$ we can apply the inverse sine function to both sides of the equation to find–

$$\begin{aligned}\theta &= \arcsin 0{,}3 \\ &= 17{,}46°\end{aligned}$$

However, we know that this is just one of infinitely many possible answers. We get the rest of the answers by finding relationships between this small angle, θ, and answers in other quadrants. To do this we use our small angle θ as a *reference angle*. We then look at the sign of the trigonometric function in order to decide in which quadrants we need to work (using the CAST diagram) and add multiples of the period to each, remembering that sine, cosine and tangent are periodic (repeating) functions. To add multiples of the period we use $360° \cdot n$ (where n is an integer) for sine and cosine and $180° \cdot n$, $n \in \mathbb{Z}$, for the tangent.

Worked Example 142:

Question: Solve for θ:

$$\sin\theta = 0{,}3$$

Answer

Step 1 : Determine in which quadrants the solution lies

We look at the sign of the trigonometric function. $\sin\theta$ is given as a positive amount (0,3). Reference to the CAST diagram shows that sine is positive in the first and second quadrants.

S	A
T	C

Step 2 : Determine the reference angle

The small angle θ is the angle returned by the calculator:

$$\begin{aligned} \sin\theta &= 0{,}3 \\ \Rightarrow \theta &= \arcsin 0{,}3 \\ \Rightarrow \theta &= 17{,}46° \end{aligned}$$

Step 3 : Determine the general solution

Our solution lies in quadrants I and II. We therefore use θ and $180° - \theta$, and add the $360° \cdot n$ for the periodicity of sine.

$180° - \theta$	θ
$180° + \theta$	$360° - \theta$

$$\begin{aligned} \text{I}: \quad \theta &= 17{,}46° + 360° \cdot n, n \in \mathbb{Z} \\ \text{II}: \quad \theta &= 180° - 17{,}46° + 360° \cdot n, n \in \mathbb{Z} \\ &= 162{,}54° + 360° \cdot n, n \in \mathbb{Z} \end{aligned}$$

This is called the *general solution*.

Step 4 : Find the specific solutions

We can then find *all* the values of θ by substituting $n = \ldots, -1{,}0, 1, 2, \ldots$etc.
For example,
If $n = 0, \quad \theta = 17{,}46°; 162{,}54°$
If $n = 1, \quad \theta = 377{,}46°; 522{,}54°$
If $n = -1, \quad \theta = -342{,}54°; -197{,}46°$
We can find as many as we like or find specific solutions in a given interval by choosing more values for n.

32.4.4 General Solution Using Periodicity

Up until now we have only solved trigonometric equations where the argument (the bit after the function, e.g. the θ in $\cos\theta$ or the $(2x - 7)$ in $\tan(2x - 7)$), has been θ. If there is anything more complicated than this we need to be a little more careful.

Let us try to solve $\tan(2x - 10°) = 2{,}5$ in the range $-360° \leq x \leq 360°$. We want solutions for positive tangent so using our CAST diagram we know to look in the 1$^{\text{st}}$ and 3$^{\text{rd}}$ quadrants. Our calculator tells us that $\arctan(2{,}5) = 68{,}2°$. This is our reference angle. So to find the general solution we proceed as follows:

$$\begin{aligned} \tan(2x - 10°) &= 2{,}5 \ [68{,}2°] \\ \text{I}: \quad 2x - 10° &= 68{,}2° + 180° \cdot n \\ 2x &= 78{,}2° + 180° \cdot n \\ x &= 39{,}1° + 90° \cdot n, n \in \mathbb{Z} \end{aligned}$$

This is the general solution. Notice that we added the $10°$ and divided by 2 only at the end. Notice that we added $180° \cdot n$ because the tangent has a period of $180°$. This is **also** divided by 2 in the last step to keep the equation balanced. We chose quadrants I and III because tan was positive and we used the formulae θ in quadrant I and $(180° + \theta)$ in quadrant III. To find solutions where $-360° < x < 360°$ we substitue integers for n:

- $n = 0$; $x = 39{,}1°$; $129{,}1°$
- $n = 1$; $x = 129{,}1°$; $219{,}1°$
- $n = 2$; $x = 219{,}1°$; $309{,}1°$
- $n = 3$; $x = 309{,}1°$; $399{,}1°$ (too big!)
- $n = -1$; $x = -50{,}9°$; $39{,}1°$
- $n = -2$; $x = -140{,}1°$; $-50{,}9°$
- $n = -3$; $x = -230{,}9°$; $-140{,}9°$
- $n = -4$; $x = -320{,}9°$; $-230{,}9°$

Solution: $x = -320{,}9°; -230°; -140{,}9°; -50{,}9°; 39{,}1°; 129{,}1°; 219{,}1°$ and $309{,}1°$

32.4.5 Linear Trigonometric Equations

Just like with regular equations without trigonometric functions, solving trigonometric equations can become a lot more complicated. You should solve these just like normal equations to isolate a single trigonometric ratio. Then you follow the strategy outlined in the previous section.

Worked Example 143:

Question: Write down the general solution isf $3\cos(\theta - 15°) - 1 = -2{,}583$
Answer

$$\begin{aligned}
3\cos(\theta - 15°) - 1 &= -2{,}583 \\
3\cos(\theta - 15°) &= -1{,}583 \\
\cos(\theta - 15°) &= -0{,}5276...\ [58{,}2°] \\
\text{II}: \quad \theta - 15° &= 180° - 58{,}2° + 360° \cdot n,\ n \in \mathbb{Z} \\
\theta &= 136{,}8° + 360° \cdot n,\ n \in \mathbb{Z} \\
\text{III}: \quad \theta - 15° &= 180° + 58{,}2° + 360° \cdot n,\ n \in \mathbb{Z} \\
\theta &= 253{,}2° + 360° \cdot n,\ n \in \mathbb{Z}
\end{aligned}$$

32.4.6 Quadratic and Higher Order Trigonometric Equations

The simplest quadratic trigonometric equation is of the form

$$\sin^2 x - 2 = -1.5$$

This type of equation can be easily solved by rearranging to get a more familiar linear equation

$$\begin{aligned}
\sin^2 x &= 0.5 \\
\Rightarrow \sin x &= \pm\sqrt{0.5}
\end{aligned}$$

This gives two linear trigonometric equations. The solutions to either of these equations will satisfy the original quadratic.

The next level of complexity comes when we need to solve a trinomial which contains trigonometric functions. It is much easier in this case to use *temporary variables*. Consider solving

$$\tan^2(2x+1) + 3\tan(2x+1) + 2 = 0$$

Here we notice that $\tan(2x+1)$ occurs twice in the equation, hence we let $y = \tan(2x+1)$ and rewrite:

$$y^2 + 3y + 2 = 0$$

That should look rather more familiar. We can immediately write down the factorised form and the solutions:

$$\begin{aligned} (y+1)(y+2) &= 0 \\ \Rightarrow y = -1 \quad \text{OR} \quad y &= -2 \end{aligned}$$

Next we just substitute back for the temporary variable:

$$\tan(2x+1) = -1 \quad \text{or} \quad \tan(2x+1) = -2$$

And then we are left with two linear trigonometric equations. Be careful: sometimes one of the two solutions will be outside the *range* of the trigonometric function. In that case you need to discard that solution. For example consider the same equation with cosines instead of tangents

$$\cos^2(2x+1) + 3\cos(2x+1) + 2 = 0$$

Using the same method we find that

$$\cos(2x+1) = -1 \quad \text{or} \quad \cos(2x+1) = -2$$

The second solution cannot be valid as cosine must lie between -1 and 1. We must, therefore, reject the second equation. Only solutions to the first equation will be valid.

32.4.7 More Complex Trigonometric Equations

Here are two examples on the level of the hardest trigonometric equations you are likely to encounter. They require using everything that you have learnt in this chapter. If you can solve these, you should be able to solve anything!

Worked Example 144:

Question: Solve $2\cos^2 x - \cos x - 1 = 0$ for $x \in [-180°; 360°]$

Answer

Step 1 : Use a temporary variable

We note that $\cos x$ occurs twice in the equation. So, let $y = \cos x$. Then we have $2y^2 - y - 1 = 0$ Note that with practice you may be able to leave out this step.

Step 2 : Solve the quadratic equation

Factorising yields

$$\begin{aligned} (2y+1)(y-1) &= 0 \\ \therefore \quad y = -0{,}5 \quad \text{or} \quad y &= 1 \end{aligned}$$

Step 1 : Substitute back and solve the two resulting equations

We thus get

$$\cos x = -0{,}5 \quad \text{or} \quad \cos x = 1$$

Both equations are valid (*i.e.* lie in the range of cosine).

General solution:

$$\begin{aligned} \cos x &= -0{,}5 \quad [60^\circ] \\ \text{II}: \quad x &= 180^\circ - 60^\circ + 360^\circ \cdot n, n \in \mathbb{Z} \\ &= 120^\circ + 360^\circ \cdot n, n \in \mathbb{Z} \\ \text{III}: \quad x &= 180^\circ + 60^\circ + 360^\circ \cdot n, n \in \mathbb{Z} \\ &= 240^\circ + 360^\circ \cdot n, n \in \mathbb{Z} \end{aligned}$$

$$\begin{aligned} \cos x &= 1 \quad [90^\circ] \\ \text{I; IV}: \quad x &= 0^\circ + 360^\circ \cdot n, n \in \mathbb{Z} \\ &= 360^\circ \cdot n, n \in \mathbb{Z} \end{aligned}$$

Now we find the specific solutions in the interval $[-180^\circ; 360^\circ]$. Appropriate values of n yield

$$x = -120^\circ; 0^\circ; 120^\circ; 240^\circ; 360^\circ$$

Worked Example 145:

Question: Solve for x in the interval $[-360^\circ; 360^\circ]$:

$$2\sin^2 - \sin x \cos x = 0$$

Answer

Step 2 : Factorise

Factorising yields

$$\sin x(2\sin x - \cos x) = 0$$

which gives two equations

$$\sin x = 0$$

$$\begin{aligned} 2\sin x &= \cos x \\ \frac{2\sin x}{\cos x} &= \frac{\cos x}{\cos x} \\ 2\tan x &= 1 \\ \tan x &= \tfrac{1}{2} \end{aligned}$$

Step 3 : Solve the two trigonometric equations

General solution:

$$\begin{aligned} \sin x &= 0 \quad [0^\circ] \\ \therefore \quad x &= 180^\circ \cdot n, n \in \mathbb{Z} \end{aligned}$$

$$\begin{aligned} \tan x &= \tfrac{1}{2} \quad [26{,}57^\circ] \\ \text{I; III}: \quad x &= 26{,}57^\circ + 180^\circ \cdot n, n \in \mathbb{Z} \end{aligned}$$

Specific solution in the interval $[-360^\circ; 360^\circ]$:

$$x = -360^\circ; -206{,}57^\circ; -180^\circ; -26{,}57^\circ; 0^\circ; 26{,}57^\circ; 180^\circ; 206{,}25^\circ; 360^\circ$$

Exercise: Solving Trigonometric Equations

1. A Find the general solution of each of the following equations. Give answers to one decimal place.

   B Find all solutions in the interval $\theta \in [-180^\circ; 360^\circ]$.

   i. $\sin\theta = -0{,}327$
   ii. $\cos\theta = 0{,}231$
   iii. $\tan\theta = -1{,}375$
   iv. $\sin\theta = 2{,}439$

2. A Find the general solution of each of the following equations. Give answers to one decimal place.
   B Find all solutions in the interval $\theta \in [0°; 360°]$.
   i. $\cos\theta = 0$
   ii. $\sin\theta = \frac{\sqrt{3}}{2}$
   iii. $2\cos\theta - \sqrt{3} = 0$
   iv. $\tan\theta = -1$
   v. $5\cos\theta = -2$
   vi. $3\sin\theta = -1{,}5$
   vii. $2\cos\theta + 1{,}3 = 0$
   viii. $0{,}5\tan\theta + 2{,}5 = 1{,}7$
3. A Write down the general solution for x if $\tan x = -1{,}12$.
   B Hence determine values of $x \in [-180°; 180°]$.
4. A Write down the general solution for θ if $\sin\theta = -0{,}61$.
   B Hence determine values of $\theta \in [0°; 720°]$.
5. A Solve for A if $\sin(\mathrm{A} + 20°) = 0{,}53$
   B Write down the values of A$\in [0°; 360°]$
6. A Solve for x if $\cos(x + 30°) = 0{,}32$
   B Write down the values of $x \in [-180°; 360°]$
7. A Solve for θ if $\sin^2(\theta) + 0{,}5\sin\theta = 0$
   B Write down the values of $\theta \in [0°; 360°]$

32.5 Sine and Cosine Identities

There are a few identities relating to the trigonometric functions that make working with triangles easier. These are:

1. the sine rule
2. the cosine rule
3. the area rule

and will be described and applied in this section.

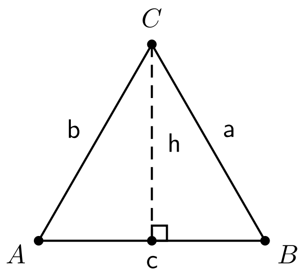
32.5.1 The Sine Rule

Definition: The Sine Rule
The sine rule applies to any triangle:

$$\frac{\sin\hat{A}}{a} = \frac{\sin\hat{B}}{b} = \frac{\sin\hat{C}}{c}$$

where a is the side opposite $\hat{A}$, b is the side opposite $\hat{B}$ and c is the side opposite $\hat{C}$.

Consider $\triangle ABC$.



The area of $\triangle ABC$ can be written as:

$$\text{area} \quad \triangle \text{ABC} = \frac{1}{2}\text{c} \cdot \text{h}.$$

However, h can be calculated in terms of $\hat{A}$ or $\hat{B}$ as:

$$\begin{aligned} \sin \hat{A} &= \frac{h}{b} \\ \therefore \quad h &= b \cdot \sin \hat{A} \end{aligned}$$

and

$$\begin{aligned} \sin \hat{B} &= \frac{h}{a} \\ \therefore \quad h &= a \cdot \sin \hat{B} \end{aligned}$$

Therefore the area of $\triangle ABC$ is:

$$\frac{1}{2}c \cdot h = \frac{1}{2}c \cdot b \cdot \sin \hat{A} = \frac{1}{2}c \cdot a \cdot \sin \hat{B}$$

Similarly, by drawing the perpendicular between point B and line AC we can show that:

$$\frac{1}{2}c \cdot b \cdot \sin \hat{A} = \frac{1}{2}a \cdot b \cdot \sin \hat{C}$$

Therefore the area of $\triangle ABC$ is:

$$\frac{1}{2}c \cdot b \cdot \sin \hat{A} = \frac{1}{2}c \cdot a \cdot \sin \hat{B} = \frac{1}{2}a \cdot b \cdot \sin \hat{C}$$

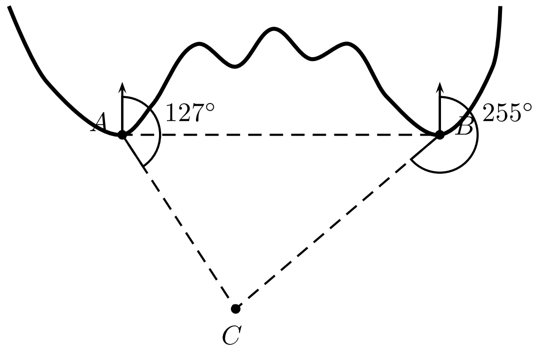
If we divide through by $\frac{1}{2}a \cdot b \cdot c$, we get:

$$\frac{\sin \hat{A}}{a} = \frac{\sin \hat{B}}{b} = \frac{\sin \hat{C}}{c}$$

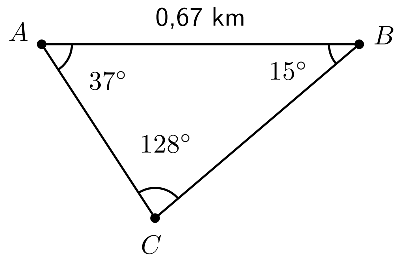
This is known as the sine rule and applies to *any* triangle.

Worked Example 146: Lighthouses

Question: There is a coastline with two lighthouses, one on either side of a beach. The two lighthouses are $0{,}67$ km apart and one is exactly due east of the other. The lighthouses tell how close a boat is by taking bearings to the boat (remember – a bearing is an angle measured clockwise from north). These bearings are shown. Use the sine rule to calculate how far the boat is from each lighthouse.



Answer
We can see that the two lighthouses and the boat form a triangle. Since we know the distance between the lighthouses and we have two angles we can use trigonometry to find the remaining two sides of the triangle, the distance of the boat from the two lighthouses.



We need to know the lengths of the two sides AC and BC. We can use the sine rule to find our missing lengths.

$$\begin{aligned} \frac{BC}{\sin \hat{A}} &= \frac{AB}{\sin \hat{C}} \\ BC &= \frac{AB \cdot \sin \hat{A}}{\sin \hat{C}} \\ &= \frac{(0{,}67\text{km})\sin(37°)}{\sin(128°)} \\ &= 0{,}51 \text{ km} \end{aligned}$$

$$\begin{aligned} \frac{AC}{\sin \hat{B}} &= \frac{AB}{\sin \hat{C}} \\ AC &= \frac{AB \cdot \sin \hat{B}}{\sin \hat{C}} \\ &= \frac{(0{,}67\text{km})\sin(15°)}{\sin(128°)} \\ &= 0{,}22 \text{ km} \end{aligned}$$

?

Exercise: Sine Rule

1. Show that

$$\frac{\sin \hat{A}}{a} = \frac{\sin \hat{B}}{b} = \frac{\sin \hat{C}}{c}$$

is equivalent to:

$$\frac{a}{\sin \hat{A}} = \frac{b}{\sin \hat{B}} = \frac{c}{\sin \hat{C}}.$$

Note: either of these two forms can be used.

2. Find all the unknown sides and angles of the following triangles:
   A $\triangle$PQR in which $\hat{\mathrm{Q}} = 64°$; $\hat{\mathrm{R}} = 24°$ and $r = 3$.
   B $\triangle$KLM in which $\hat{\mathrm{K}} = 43°$; $\hat{\mathrm{M}} = 50°$ and $m = 1$
   C $\triangle$ABC in which $\hat{\mathrm{A}} = 32{,}7°$; $\hat{\mathrm{C}} = 70{,}5°$ and $a = 52{,}3$
   D $\triangle$XYZ in which $\hat{\mathrm{X}} = 56°$; $\hat{\mathrm{Z}} = 40°$ and $x = 50$
3. In $\triangle$ABC, $\hat{\mathrm{A}} = 116°$; $\hat{\mathrm{C}} = 32°$ and $\mathrm{AC} = 23$ m. Find the length of the side AB.
4. In $\triangle$RST, $\hat{\mathrm{R}} = 19°$; $\hat{\mathrm{S}} = 30°$ and $\mathrm{RT} = 120$ km. Find the length of the side ST.
5. In $\triangle$KMS, $\hat{\mathrm{K}} = 20°$; $\hat{\mathrm{M}} = 100°$ and $s = 23$ cm. Find the length of the side m.

32.5.2 The Cosine Rule

Definition: The Cosine Rule
The cosine rule applies to any triangle and states that:

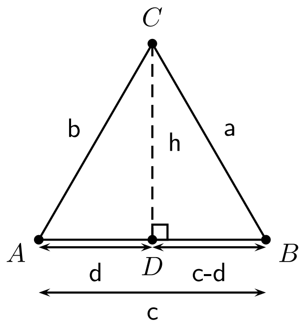
$$\begin{aligned} a^2 &= b^2 + c^2 - 2bc\cos\hat{A} \\ b^2 &= c^2 + a^2 - 2ca\cos\hat{B} \\ c^2 &= a^2 + b^2 - 2ab\cos\hat{C} \end{aligned}$$

where a is the side opposite $\hat{A}$, b is the side opposite $\hat{B}$ and c is the side opposite $\hat{C}$.

The cosine rule relates the length of a side of a triangle to the angle opposite it and the lengths of the other two sides.

Consider $\triangle ABC$ which we will use to show that:

$$a^2 = b^2 + c^2 - 2bc\cos\hat{A}.$$



In $\triangle DCB$:

$$a^2 = (c-d)^2 + h^2 \tag{32.6}$$

from the theorem of Pythagoras.

In $\triangle ACD$:

$$b^2 = d^2 + h^2 \tag{32.7}$$

from the theorem of Pythagoras.

We can eliminate h^2 from (32.6) and (32.7) to get:

$$\begin{aligned} b^2 - d^2 &= a^2 - (c-d)^2 \\ a^2 &= b^2 + (c^2 - 2cd + d^2) - d^2 \\ &= b^2 + c^2 - 2cd + d^2 - d^2 \\ &= b^2 + c^2 - 2cd \end{aligned} \tag{32.8}$$

In order to eliminate d we look at $\triangle ACD$, where we have:

$$\cos \hat{A} = \frac{d}{b}.$$

So,

$$d = b \cos \hat{A}.$$

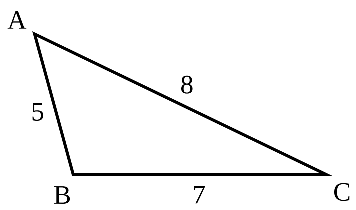
Substituting this into (32.8), we get:

$$a^2 = b^2 + c^2 - 2bc \cos \hat{A} \tag{32.9}$$

The other cases can be proved in an identical manner.

Worked Example 147:

Question: Find $\hat{\text{A}}$:



Answer
Applying the cosine rule:

$$\begin{aligned} a^2 &= b^2 + c^2 - 2bd \cos \hat{\text{A}} \\ \therefore \quad \cos \hat{\text{A}} &= \frac{b^2 + c^2 - a^2}{2bc} \\ &= \frac{8^2 + 5^2 - 7^2}{2 \cdot 8 \cdot 5} \\ &= 0{,}5 \\ \therefore \quad \hat{\text{A}} &= \arccos 0{,}5 = 60^\circ \end{aligned}$$

Exercise: The Cosine Rule

1. Solve the following triangles *i.e.* find all unknown sides and angles

   A $\triangle$ABC in which $\hat{\text{A}} = 70^\circ$; $b = 4$ and $c = 9$

   B $\triangle$XYZ in which $\hat{\text{Y}} = 112^\circ$; $x = 2$ and $y = 3$

   C $\triangle$RST in which RS$= 2$; ST$= 3$ and RT$= 5$

   D $\triangle$KLM in which KL$= 5$; LM$= 10$ and KM$= 7$

E △JHK in which $\hat{\text{H}} = 130°$; JH$= 13$ and HK$= 8$
F △DEF in which $d = 4$; $e = 5$ and $f = 7$

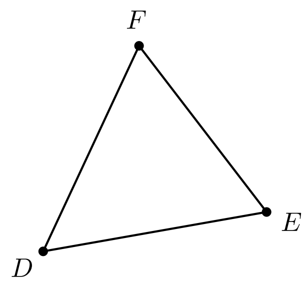
2. Find the length of the third side of the △XYZ where:
   A $\hat{\text{X}} = 71{,}4°$; $y = 3{,}42$ km and $z = 4{,}03$ km
   B ; $x = 103{,}2$ cm; $\hat{\text{Y}} = 20{,}8°$ and $z = 44{,}59$ cm
3. Determine the largest angle in:
   A △JHK in which JH$= 6$; HK$= 4$ and JK$= 3$
   B △PQR where $p = 50$; $q = 70$ and $r = 60$

32.5.3 The Area Rule

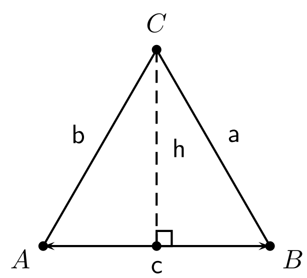
Definition: The Area Rule
The area rule applies to any triangle and states that the area of a triangle is given by half the product of any two sides with the sine of the angle between them.

That means that in the $\triangle DEF$, the area is given by:

$$A = \frac{1}{2}DE \cdot EF \sin \hat{E} = \frac{1}{2}EF \cdot FD \sin \hat{F} = \frac{1}{2}FD \cdot DE \sin \hat{D}$$



In order show that this is true for all triangles, consider $\triangle ABC$.



The area of any triangle is half the product of the base and the perpendicular height. For $\triangle ABC$, this is:

$$A = \frac{1}{2}c \cdot h.$$

However, h can be written in terms of $\hat{A}$ as:

$$h = b \sin \hat{A}$$

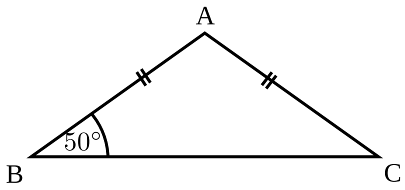
So, the area of $\triangle ABC$ is:

$$A = \frac{1}{2}c \cdot b \sin \hat{A}.$$

Using an identical method, the area rule can be shown for the other two angles.

Worked Example 148: The Area Rule

Question: Find the area of △ABC:



Answer

△ABC is isosceles, therefore AB=AC= 7 and $\hat{C} = \hat{B} = 50°$. Hence $\hat{A} = 180° - 50° - 50° = 80°$. Now we can use the area rule to find the area:

$$\begin{aligned} A &= \frac{1}{2} cb \sin \hat{A} \\ &= \frac{1}{2} \cdot 7 \cdot 7 \cdot \sin 80° \\ &= 24{,}13 \end{aligned}$$

Exercise: The Area Rule

Draw sketches of the figures you use in this exercise.

1. Find the area of △PQR in which:
   A $\hat{P} = 40°$; $q = 9$ and $r = 25$
   B $\hat{Q} = 30°$; $r = 10$ and $p = 7$
   C $\hat{R} = 110°$; $p = 8$ and $q = 9$
2. Find the area of:
   A △XYZ with XY= 6 cm; XZ= 7 cm and $\hat{Z} = 28°$
   B △PQR with PR= 52 cm; PQ= 29 cm and $\hat{P} = 58{,}9°$
   C △EFG with FG= 2,5 cm; EG= 7,9 cm and $\hat{G} = 125°$
3. Determine the area of a parallelogram in which two adjacent sides are 10 cm and 13 cm and the angle between them is 55°.
4. If the area of △ABC is 5000 m^2 with $a = 150$ m and $b = 70$ m, what are the two possible sizes of $\hat{C}$?

Summary of the Trigonometric Rules and Identities

Pythagorean Identity	Ratio Identity
$\cos^2\theta + \sin^2\theta = 1$	$\tan\theta = \frac{\sin\theta}{\cos\theta}$

Odd/Even Identities	Periodicity Identities	Cofunction Identities
$\sin(-\theta) = -\sin\theta$	$\sin(\theta \pm 360°) = \sin\theta$	$\sin(90° - \theta) = \cos\theta$
$\cos(-\theta) = \cos\theta$	$\cos(\theta \pm 360°) = \cos\theta$	$\cos(90° - \theta) = \sin\theta$

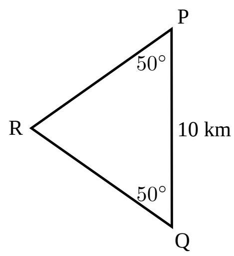
Sine Rule	Area Rule	Cosine Rule
	$\text{Area} = \frac{1}{2}bc\cos A$	$a^2 = b^2 + c^2 - 2bc\cos A$
$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$	$\text{Area} = \frac{1}{2}ac\cos B$	$b^2 = a^2 + c^2 - 2ac\cos B$
	$\text{Area} = \frac{1}{2}ab\cos C$	$c^2 = a^2 + b^2 - 2ab\cos C$

32.6 Exercises

1. Q is a ship at a point 10 km due South of another ship P. R is a lighthouse on the coast such that $\hat{\text{P}} = \hat{\text{Q}} = 50°$.
   Determine:

   A the distance QR

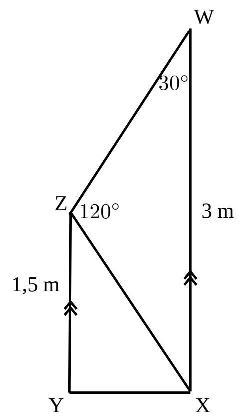
   B the shortest distance from the lighthouse to the line joining the two ships (PQ).



2. WXYZ is a trapezium (WX‖XZ) with WX= 3 m; YZ= 1,5 m; $\hat{\text{Z}} = 120°$ and $\hat{\text{W}} = 30°$

   A Determine the distances XZ and XY

   B Find the angle $\hat{\text{C}}$

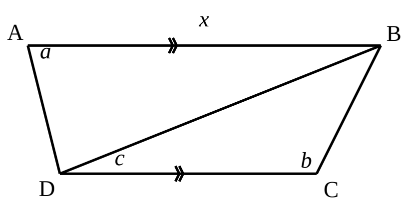


3. On a flight from Johannesburg to Cape Town, the pilot discovers that he has been flying 3° off course. At this point the plane is 500 km from Johannesburg. The direct distance between Cape Town and Johannesburg airports is 1 552 km. Determine, to the nearest km:

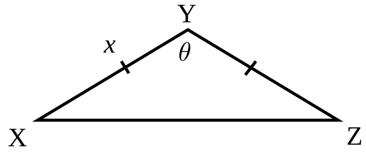
   A The distance the plane has to travel to get to Cape Town and hence the extra distance that the plane has had to travel due to the pilot's error.

   B The correction, to one hundredth of a degree, to the plane's heading (or direction).

4. ABCD is a trapezium (*i.e.* AB‖CD). AB= x; $\text{B}\hat{\text{A}}\text{D} = a$; $\text{B}\hat{\text{C}}\text{D} = b$ and $\text{B}\hat{\text{D}}\text{C} = c$.
   Find an expression for the length of CD in terms of x, a, b and c.



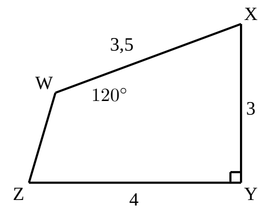
5. A surveyor is trying to determine the distance between points X and Z. However the distance cannot be determined directly as a ridge lies between the two points. From a point Y which is equidistant from X and Z, he measures the angle $\mathrm{X\hat{Y}Z}$



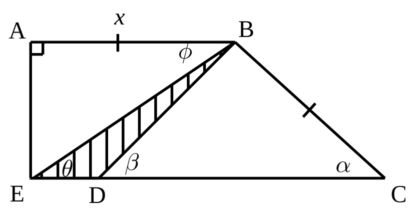
   A If XY= x and $\mathrm{X\hat{Y}Z} = \theta$, show that XZ= $x\sqrt{2(1-\cos\theta)}$

   B Calculate XZ (to the nearest kilometre) if $x =$ 240 km and $\theta = 132°$

6. Find the area of WXYZ (to two decimal places):



7. Find the area of the shaded triangle in terms of x, α, β, θ and ϕ:



Chapter 33

Statistics - Grade 11

33.1 Introduction

This chapter gives you an opportunity to build on what you have learned in previous Grades about data handling and probility. The work done will be mostly of a practical nature. Through problem solving and activities, you will end up mastering further methods of collecting, organising, displaying and analysing data. You will also learn how to interpret data, and not always to accept the data at face value, because data are sometimes unscrupulously misused and abused in order to try to prove or support a viewpoint. Measures of central tendency (mean, median and mode) and dispersion (range, percentiles, quartiles, inter-quartile, semi-inter-quartile range, variance and standard deviation) will be investigated. Of course, the activities involving probability will be familiar to most of you - for example, you have played dice games or card games even before you came to school. Your basic understanding of probability and chance gained so far will be deepened to enable you to come to a better understanding of how chance and uncertainty can be measured and understood.

33.2 Standard Deviation and Variance

The measures of central tendency (mean, median and mode) and measures of dispersion (quartiles, percentiles, ranges) provide information on the data values at the centre of the data set and provide information on the spread of the data. The information on the spread of the data is however based on data values at specific points in the data set, e.g. the end points for range and data points that divide the data set into 4 equal groups for the quartiles. The behaviour of the entire data set is therefore not examined.

A method of determining the spread of data is by calculating a measure of the possible distances between the data and the mean. The two important measures that are used are called the *variance* and the *standard deviation* of the data set.

33.2.1 Variance

The variance of a data set is the average squared distance between the mean of the data set and each data value. An example of what this means is shown in Figure 33.1. The graph represents the results of 100 tosses of a fair coin, which resulted in 45 heads and 55 tails. The mean of the results is 50. The squared distance between the heads value and the mean is $(45-50)^2 = 25$ and the squared distance between the tails value and the mean is $(55-50)^2 = 25$. The average of these two squared distances gives the variance, which is $\frac{1}{2}(25+25) = 25$.

Population Variance

Let the population consist of n elements $\{x_1,x_2,\ldots,x_n\}$. with mean $\bar{x}$ (read as "x bar"). The variance of the population, denoted by σ^2, is the average of the square of the distance of each data value from the mean value.

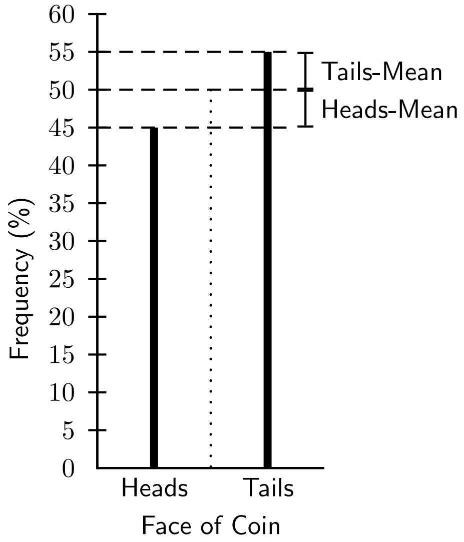


Figure 33.1: The graph shows the results of 100 tosses of a fair coin, with 45 heads and 55 tails. The mean value of the tosses is shown as a vertical dotted line. The difference between the mean value and each data value is shown.

$$\sigma^2 = \frac{(\sum(x - \bar{x}))^2}{n}. \tag{33.1}$$

Since the population variance is squared, it is not directly comparable with the mean and the data themselves.

Sample Variance

Let the sample consist of the n elements $\{x_1, x_2, \ldots, x_n\}$, taken from the population, with mean $\bar{x}$. The variance of the sample, denoted by s^2, is the average of the squared deviations from the sample mean:

$$s^2 = \frac{\sum(x - \bar{x})^2}{n - 1}. \tag{33.2}$$

Since the sample variance is squared, it is also not directly comparable with the mean and the data themselves.

A common question at this point is "Why is the numerator squared?" One answer is: to get rid of the negative signs. Numbers are going to fall above and below the mean and, since the variance is looking for distance, it would be counterproductive if those distances factored each other out.

Difference between Population Variance and Sample Variance

As seen a distinction is made between the variance, σ^2, of a whole population and the variance, s^2 of a sample extracted from the population.

When dealing with the complete population the (population) variance is a constant, a parameter which helps to describe the population. When dealing with a sample from the population the (sample) variance varies from sample to sample. Its value is only of interest as an estimate for the population variance.

Properties of Variance

If the variance is defined, we can conclude that it is never negative because the squares are positive or zero. The unit of variance is the square of the unit of observation. For example, the variance of a set of heights measured in centimeters will be given in square centimeters. This fact is inconvenient and has motivated many statisticians to instead use the square root of the variance, known as the standard deviation, as a summary of dispersion.

33.2.2 Standard Deviation

Since the variance is a squared quantity, it cannot be directly compared to the data values or the mean value of a data set. It is therefore more useful to have a quantity which is the square root of the variance. This quantity is known as the standard deviation.

In statistics, the standard deviation is the most common measure of statistical dispersion. Standard deviation measures how spread out the values in a data set are. More precisely, it is a measure of the average distance between the values of the data in the set. If the data values are all similar, then the standard deviation will be low (closer to zero). If the data values are highly variable, then the standard variation is high (further from zero).

The standard deviation is always a positive number and is always measured in the same units as the original data. For example, if the data are distance measurements in metres, the standard deviation will also be measured in metres.

Population Standard Deviation

Let the population consist of n elements $\{x_1, x_2, \ldots, x_n\}$. with mean $\bar{x}$. The standard deviation of the population, denoted by σ, is the square root of the average of the square of the distance of each data value from the mean value.

$$\sigma = \sqrt{\frac{\sum(x - \bar{x})^2}{n}} \tag{33.3}$$

Sample Standard Deviation

Let the sample consist of n elements $\{x_1, x_2, \ldots, x_n\}$, taken from the population, with mean $\bar{x}$. The standard deviation of the sample, denoted by s, is the square root of the average of the squared deviations from the sample mean:

$$s = \sqrt{\frac{\sum(x - \bar{x})^2}{n-1}} \tag{33.4}$$

It is often useful to set your data out in a table so that you can apply the formulae easily. For example to calculate the standard deviation of 57; 53; 58; 65; 48; 50; 66; 51, you could set it out in the following way:

$$\begin{aligned} mean &= \frac{\text{sum of items}}{\text{number of items}} \\ &= \frac{\sum x}{n} \\ &= \frac{448}{6} \\ &= 56 \end{aligned}$$

Note: To get the deviations, subtract each number from the mean.

$\bar{X}$	**Deviation** $(X - \bar{X})$	**Deviation squared** $(X - \bar{X})^2$
57	1	1
53	-3	9
58	2	4
65	9	81
48	-8	64
50	-6	36
66	10	100
51	-5	25
$\sum X = 448$	$\sum x = 0$	$\sum (X - X)^2 = 320$

Note: The sum of the deviations of scores about their mean is zero. This always happens; that is $(X - \bar{X}) = 0$, for any set of data. Why is this? Find out.
Calculate the variance (add the squared results together and divide this total by the number of items).

$$\begin{aligned} \textbf{Variance} &= \frac{\sum (X - \bar{X})^2}{n} \\ &= \frac{320}{8} \\ &= 40 \end{aligned}$$

$$\begin{aligned} \textbf{Standard deviation} &= \sqrt{variance} \\ &= \sqrt{\frac{\sum (X - \bar{X})^2}{n}} \\ &= \sqrt{\frac{320}{8}} \\ &= \sqrt{40} \\ &= 6.32 \end{aligned}$$

Difference between Population Variance and Sample Variance

As with variance, there is a distinction between the standard deviation,σ, of a whole population and the standard deviation, s, of sample extracted from the population.

When dealing with the complete population the (population) standard deviation is a constant, a parameter which helps to describe the population. When dealing with a sample from the population the (sample) standard deviation varies from sample to sample.

In other words, the standard deviation can be calculated as follows:

1. Calculate the mean value $\bar{x}$.
2. For each data value x_i calculate the difference $x_i - \bar{x}$ between x_i and the mean value $\bar{x}$.
3. Calculate the squares of these differences.
4. Find the average of the squared differences. This quantity is the variance, σ^2.
5. Take the square root of the variance to obtain the standard deviation, σ.

Worked Example 149: Variance and Standard Deviation

Question: What is the variance and standard deviation of the population of possibilities associated with rolling a fair die?

Answer

Step 1 : Determine how many outcomes make up the population

When rolling a fair die, the population consists of 6 possible outcomes. The data set is therefore $x = \{1,2,3,4,5,6\}$. and n=6.

Step 2 : Calculate the population mean

The population mean is calculated by:

$$\begin{aligned} \bar{x} &= \frac{1}{6}(1+2+3+4+5+6) \\ &= 3{,}5 \end{aligned}$$

Step 3 : Calculate the population variance

The population variance is calculated by:

$$\begin{aligned} \sigma^2 &= \frac{\sum(x-\bar{x})^2}{n} \\ &= \frac{1}{6}(6{,}25+2{,}25+0{,}25+0{,}25+2{,}25+6{,}25) \\ &= 2{,}917 \end{aligned}$$

Step 4 : Alternately the population variance is calculated by:

X	$(X-\bar{X})$	$(X-\bar{X})^2$
1	-2.5	6.25
2	-1.5	2.25
3	-0.5	0.25
4	0.5	0.25
5	1.5	2.25
6	2.5	6.25
$\sum X = 21$	$\sum x = 0$	$\sum(X-X)^2 = 17.5$

Step 5 : Calculate the standard deviation

The (population) standard deviation is calculated by:

$$\begin{aligned} \sigma &= \sqrt{2{,}917} \\ &= 1{,}708. \end{aligned}$$

Notice how this standard deviation is somewhere in between the possible deviations.

33.2.3 Interpretation and Application

A large standard deviation indicates that the data values are far from the mean and a small standard deviation indicates that they are clustered closely around the mean.

For example, each of the three samples (0, 0, 14, 14), (0, 6, 8, 14), and (6, 6, 8, 8) has a mean of 7. Their standard deviations are 7, 5 and 1, respectively. The third set has a much smaller standard deviation than the other two because its values are all close to 7. The value of the standard deviation can be considered 'large' or 'small' only in relation to the sample that is being measured. In this case, a standard deviation of 7 may be considered large. Given a different sample, a standard deviation of 7 might be considered small.

Standard deviation may be thought of as a measure of uncertainty. In physical science for example, the reported standard deviation of a group of repeated measurements should give the precision of those measurements. When deciding whether measurements agree with a theoretical prediction, the standard deviation of those measurements is of crucial importance: if the mean of the measurements is too far away from the prediction (with the distance measured in standard deviations), then we consider the measurements as contradicting the prediction. This makes sense since they fall outside the range of values that could reasonably be expected to occur if the prediction were correct and the standard deviation appropriately quantified. See prediction interval.

33.2.4 Relationship between Standard Deviation and the Mean

The mean and the standard deviation of a set of data are usually reported together. In a certain sense, the standard deviation is a "natural" measure of statistical dispersion if the center of the data is measured about the mean. This is because the standard deviation from the mean is smaller than from any other point.

?

Exercise: Means and standard deviations

1. Bridget surveyed the price of petrol at petrol stations in Cape Town and Durban. The raw data, in rands per litre, are given below:

Cape Town	3.96	3.76	4.00	3.91	3.69	3.72
Durban	3.97	3.81	3.52	4.08	3.88	3.68

   A Find the mean price in each city and then state which city has the lowest mean.
   B Assuming that the data is a population find the standard deviation of each city's prices.
   C Assuming the data is a sample find the standard deviation of each city's prices.
   D Giving reasons which city has the more consistently priced petrol?
2. The following data represents the pocket money of a sample of teenagers.
   150; 300; 250; 270; 130; 80; 700; 500; 200; 220; 110; 320; 420; 140.
   What is the standard deviation?
3. Consider a set of data that gives the weights of 50 cats at a cat show.
   A When is the data seen as a population?
   B When is the data seen as a sample?
4. Consider a set of data that gives the results of 20 pupils in a class.
   A When is the data seen as a population?
   B When is the data seen as a sample?

33.3 Graphical Representation of Measures of Central Tendency and Dispersion

The measures of central tendency (mean, median, mode) and the measures of dispersion (range, semi-inter-quartile range, quartiles, percentiles, inter-quartile range) are numerical methods of summarising data. This section presents methods of representing the summarised data using graphs.

33.3.1 Five Number Summary

One method of summarising a data set is to present a *five number summary*. The five numbers are: minimum, first quartile, median, third quartile and maximum.

33.3.2 Box and Whisker Diagrams

A *box and whisker* diagram is a method of depicting the five number summary, graphically.

The main features of the box and whisker diagram are shown in Figure 33.2. The box can lie horizontally (as shown) or vertically. For a horizonatal diagram, the left edge of the box is placed at the first quartile and the right edge of the box is placed at the third quartile. The height of the box is arbitrary, as there is no y-axis. Inside the box there is some representation of central tendency, with the median shown with a vertical line dividing the box into two. Additionally, a star or asterix is placed at the mean value, centered in the box in the vertical direction. The whiskers which extend to the sides reach the minimum and maximum values.

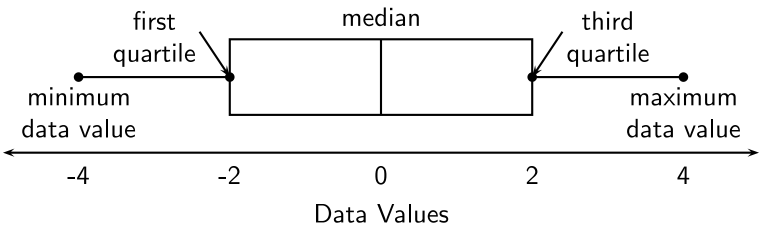


Figure 33.2: Main features of a box and whisker diagram

Worked Example 150: Box and Whisker Diagram

Question: Draw a box and whisker diagram for the data set
$x = \{1{,}25; 1{,}5; 2{,}5; 2{,}5; 3{,}1; 3{,}2; 4{,}1; 4{,}25; 4{,}75; 4{,}8; 4{,}95; 5{,}1\}$.

Answer

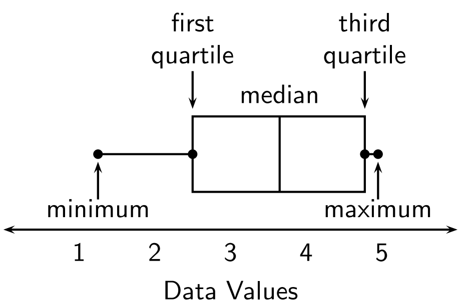
Step 1 : Determine the five number summary

Minimum = 1,25
Maximum = 4,95
Position of first quartile = between 3 and 4
Position of second quartile = between 6 and 7
Position of third quartile = between 9 and 10

Data value between 3 and 4 = $\frac{1}{2}(2{,}5 + 2{,}5) = 2{,}5$
Data value between 6 and 7 = $\frac{1}{2}(3{,}2 + 4{,}1) = 3{,}65$
Data value between 9 and 10 = $\frac{1}{2}(4{,}75 + 4{,}8) = 4{,}775$

The five number summary is therefore: 1,25; 2,5; 3,65; 4,775; 4,95.

Step 2 : Draw a box and whisker diagram and mark the positions of the minimum, maximum and quartiles.



?

Exercise: Box and whisker plots

1. Lisa works as a telesales person. She keeps a record of the number of sales she makes each month. The data below show how much she sells each month.
   49; 12; 22; 35; 2; 45; 60; 48; 19; 1; 43; 12
   Give a five number summary and a box and whisker plot of her sales.
2. Jason is working in a computer store. He sells the following number of computers each month:
   27; 39; 3; 15; 43; 27; 19; 54; 65; 23; 45; 16
   Give a five number summary and a box and whisker plot of his sales,
3. The number of rugby matches attended by 36 season ticket holders is as follows:
   15; 11; 7; 34; 24; 22; 31; 12; 9
   12; 9; 1; 3; 15; 5; 8; 11; 2
   25; 2; 6; 18; 16; 17; 20; 13; 17
   14; 13; 11; 5; 3; 2; 23; 26; 40

   A Sum the data.
   B Using an appropriate graphical method (give reasons) represent the data.
   C Find the median, mode and mean.
   D Calculate the five number summary and make a box and whisker plot.
   E What is the variance and standard deviation?
   F Comment on the data's spread.
   G Where are 95% of the results expected to lie?
4. Rose has worked in a florists shop for nine months. She sold the following number of wedding bouquets:
   16; 14; 8; 12; 6; 5; 3; 5; 7

   A What is the five-number summary of the data?
   B Since there is an odd number of data points what do you observe when calculating the five-numbers?

33.3.3 Cumulative Histograms

Cumulative histograms, also known as ogives, are a plot of cumulative frequency and are used to determine how many data values lie above or below a particular value in a data set. The cumulative frequency is calculated from a frequency table, by adding each frequency to the total of the frequencies of all data values before it in the data set. The last value for the cumulative frequency will always be equal to the total number of data values, since all frequencies will already have been added to the previous total. The cumulative frequency is plotted at the upper limit of the interval.

For example, the cumulative frequencies for Data Set 2 are shown in Table 33.2 and is drawn in Figure 33.3.

Notice the frequencies plotted at the upper limit of the intervals, so the points (30;1) (62;2) (97;3), etc have been plotted. This is different from the frequency polygon where we plot frequencies at the midpoints of the intervals.

?

Exercise: Intervals

Intervals	$0 < n \leq 1$	$1 < n \leq 2$	$2 < n \leq 3$	$3 < n \leq 4$	$4 < n \leq 5$	$5 < n \leq 6$
Frequency	30	32	35	34	37	32
Cumulative Frequency	30	30 + 32	30 + 32 + 35	30 + 32 + 35 + 34	30 + 32 + 35 + 34 + 37	30 + 32 + 35 + 34 + 37 + 32
	30	62	97	131	168	200

Table 33.1: Cumulative Frequencies for Data Set 2.

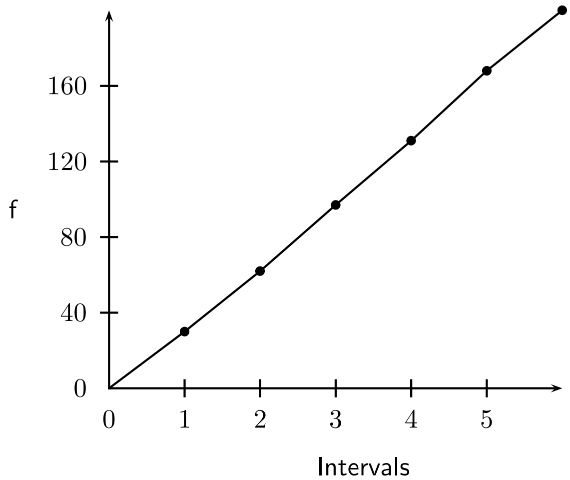


Figure 33.3: Example of a cumulative histogram for Data Set 2.

1. Use the following data of peoples ages to answer the questions.
   2; 5; 1; 76; 34; 23; 65; 22; 63; 45; 53; 38
   4; 28; 5; 73; 80; 17; 15; 5; 34; 37; 45; 56
   A Using an interval width of 8 construct a cumulative frequency distribution
   B How many are below 30?
   C How many are below 60?
   D Giving an explanation state below what value the bottom 50% of the ages fall
   E Below what value do the bottom 40% fall?
   F Construct a frequency polygon and an ogive.
   G Compare these two plots
2. The weights of bags of sand in grams is given below (rounded to the nearest tenth):
   50.1; 40.4; 48.5; 29.4; 50.2; 55.3; 58.1; 35.3; 54.2; 43.5
   60.1; 43.9; 45.3; 49.2; 36.6; 31.5; 63.1; 49.3; 43.4; 54.1
   A Decide on an interval width and state what you observe about your choice.
   B Give your lowest interval.
   C Give your highest interval.
   D Construct a cumultative frequency graph and a frequency polygon.
   E Compare the cumulative frequency graph and frequency polygon.
   F Below what value do 53% of the cases fall?
   G Below what value fo 60% of the cases fall?

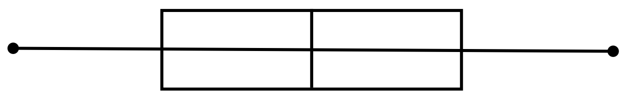
33.4 Distribution of Data

33.4.1 Symmetric and Skewed Data

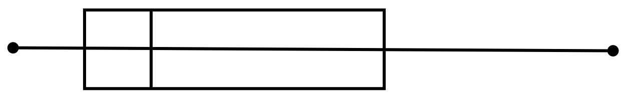
The shape of a data set is important to know.

Definition: Shape of a data set
This describes how the data is distributed relative to the mean and median.

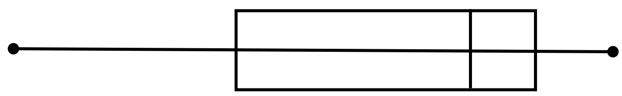
- Symmetrical data sets are balanced on either side of the median. It does not have to be exactly equal to be symmetric



- Skewed data is spread out on one side more than on the other. It can be skewed right or skewed left.



skewed right



skewed left

33.4.2 Relationship of the Mean, Median, and Mode

The relationship of the mean, median, and mode to each other can provide some information about the relative shape of the data distribution. If the mean, median, and mode are approximately equal to each other, the distribution can be assumed to be approximately symmetrical. With both the mean and median known the following can be concluded:

- (mean - median) ≈ 0 then the data is symmetrical
- (mean - median) > 0 then the data is positively skewed (skewed to the right). This means that the median is close to the start of the data set.
- (mean - median) < 0 then the data is negatively skewed (skewed to the left). This means that the median is close to the end of the data set.

Exercise: Distribution of Data

1. Three sets of 12 pupils each had test score recorded. The test was out of 50. Use the given data to answer the following questions.
   A Make a stem and leaf plot for each set.
   B For each of the sets calculate the mean and the five number summary.
   C For each of the classes find the difference between the mean and the median and then use that to make box and whisker plots on the same set of axes.

Set 1	Set 2	Set 3
25	32	43
47	34	47
15	35	16
17	32	43
16	25	38
26	16	44
24	38	42
27	47	50
22	43	50
24	29	44
12	18	43
31	25	42

Table 33.2: Cumulative Frequencies for Data Set 2.

D State which of the three are skewed (either right or left).
E Is set A skewed or symmetrical?
F Is set C symmetrical? Why or why not?

2. Two data sets have the same range and interquartile range, but one is skewed right and the other is skewed left. Sketch the box and whisker plots and then invent data (6 points in each set) that meets the requirements.

33.5 Scatter Plots

A scatter-plot is a graph that shows the relationship between two variables. We say this is bivariate data and we plot the data from two different sets using ordered pairs. For example, we could have mass on the horizontal axis (first variable) and height on the second axis (second variable), or we could have current on the horizontal axis and voltage on the vertical axis.

Ohm's Law is an important relationship in physics. Ohm's law describes the relationship between current and voltage in a conductor, like a piece of wire. When we measure the voltage (dependent variable) that results from a certain current (independent variable) in a wire, we get the data points as shown in Table 33.3.

Table 33.3: Values of current and voltage measured in a wire.

Current	**Voltage**	**Current**	**Voltage**
0	0.4	2.4	1.4
0.2	0.3	2.6	1.6
0.4	0.6	2.8	1.9
0.6	0.6	3	1.9
0.8	0.4	3.2	2
1	1	3.4	1.9
1.2	0.9	3.6	2.1
1.4	0.7	3.8	2.1
1.6	1	4	2.4
1.8	1.1	4.2	2.4
2	1.3	4.4	2.5
2.2	1.1	4.6	2.5

When we plot this data as points, we get the scatter plot shown in Figure 33.4.

If we are to come up with a function that best describes the data, we would have to say that a straight line best describes this data.

htb

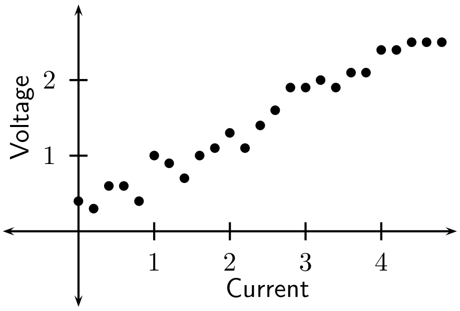


Figure 33.4: Example of a scatter plot

Extension: Ohm's Law
Ohm's Law describes the relationship between current and voltage in a conductor. The gradient of the graph of voltage vs. current is known as the *resistance* of the conductor.

Activity :: Research Project : Scatter Plot
The function that best describes a set of data can take any form. We will restrict ourselves to the forms already studied, that is, linear, quadratic or exponential. Plot the following sets of data as scatter plots and deduce the type of function that best describes the data. The type of function can either be quadratic or exponential.

1.

x	y	x	y	x	y	x	y
-5	9.8	0	14.2	-2.5	11.9	2.5	49.3
-4.5	4.4	0.5	22.5	-2	6.9	3	68.9
-4	7.6	1	21.5	-1.5	8.2	3.5	88.4
-3.5	7.9	1.5	27.5	-1	7.8	4	117.2
-3	7.5	2	41.9	-0.5	14.4	4.5	151.4

2.

x	y	x	y	x	y	x	y
-5	75	0	5	-2.5	27.5	2.5	7.5
-4.5	63.5	0.5	3.5	-2	21	3	11
-4	53	1	3	-1.5	15.5	3.5	15.5
-3.5	43.5	1.5	3.5	-1	11	4	21
-3	35	2	5	-0.5	7.5	4.5	27.5

3.

Height (cm)	147 168	150 170	152 173	155 175	157 178	160 180	163 183	165
Weight (kg)	52 63	53 64	54 66	56 68	57 70	59 72	60 74	61

Definition: outlier
A point on a scatter plot which is widely separated from the other points or a result differing greatly from others in the same sample is called an outlier.

?

Exercise: Scatter Plots

1. A class's results for a test were recorded along with the amount of time spent studying for it. The results are given below.

Score (percent)	Time spent studying (minutes)
67	100
55	85
70	150
90	180
45	70
75	160
50	80
60	90
84	110
30	60
66	96
96	200

   A Draw a diagram labelling horizontal and vertical axes.
   B State with reasons, the cause or independent variable and the effect or dependent variable.
   C Plot the data pairs
   D What do you observe about the plot?
   E Is there any pattern emerging?

2. The rankings of eight tennis players is given along with the time they spend practising.

Practice time (min)	Ranking
154	5
390	1
130	6
70	8
240	3
280	2
175	4
103	7

   A Construct a scatter plot and explain how you chose the dependent (cause) and independent (effect) variables.
   B What pattern or trend do you observe?

3. Eight childrens sweet consumption and sleep habits were recorded. The data is given in the following table.

Number of sweets (per week)	Average sleeping time (per day)
15	4
12	4.5
5	8
3	8.5
18	3
23	2
11	5
4	8

   A What is the dependent (cause) variable?
   B What is the independent (effect) variable?
   C Construct a scatter plot of the data.
   D What trend do you observe?

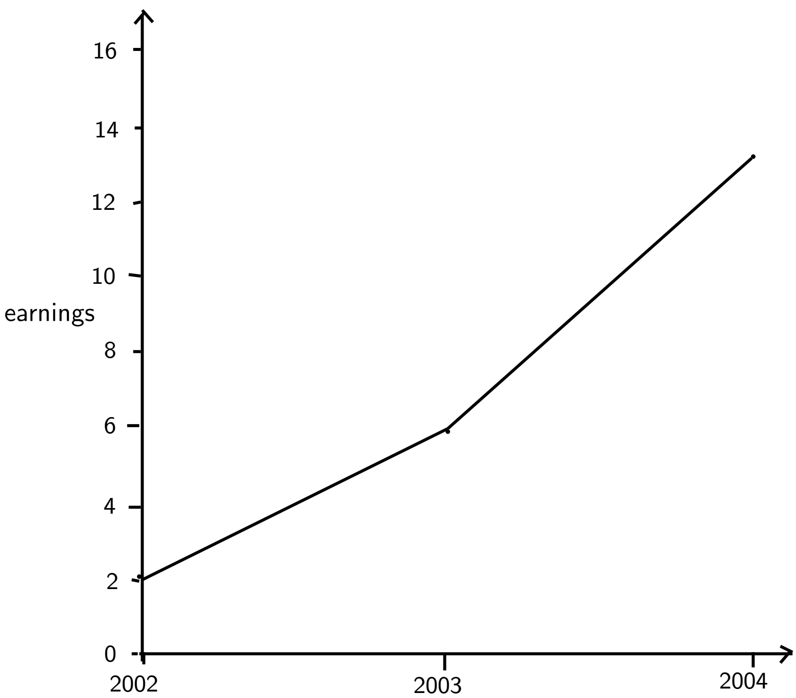
33.6 Misuse of Statistics

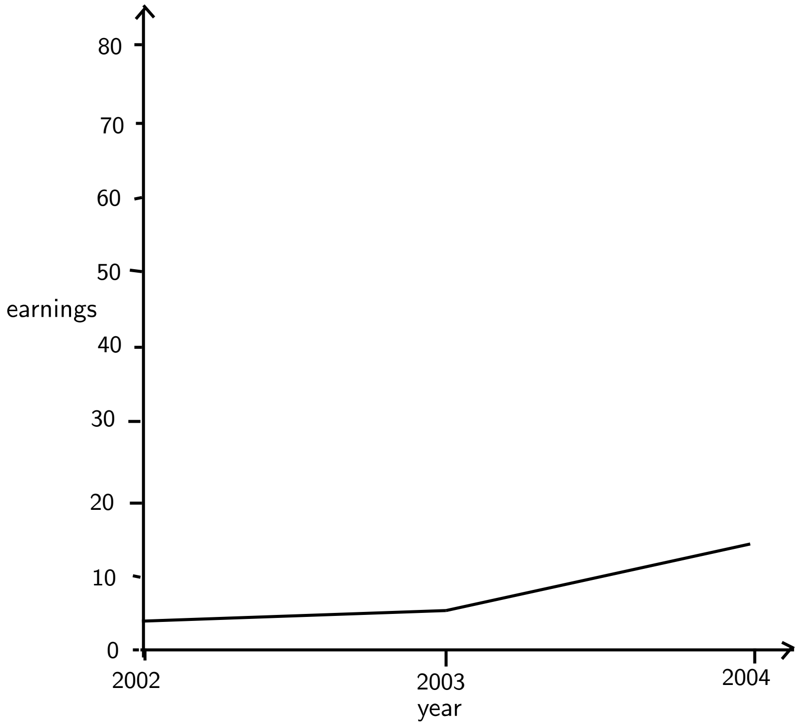
Statistics can be manipulated in many ways that can be misleading. Graphs need to be carefully analysed and questions must always be asked about 'the story behind the figures.' Common manipulations are:

1. Changing the scale to change the appearence of a graph
2. Omissions and biased selection of data
3. Focus on particular research questions
4. Selection of groups

Activity :: Investigation : Misuse of statistics

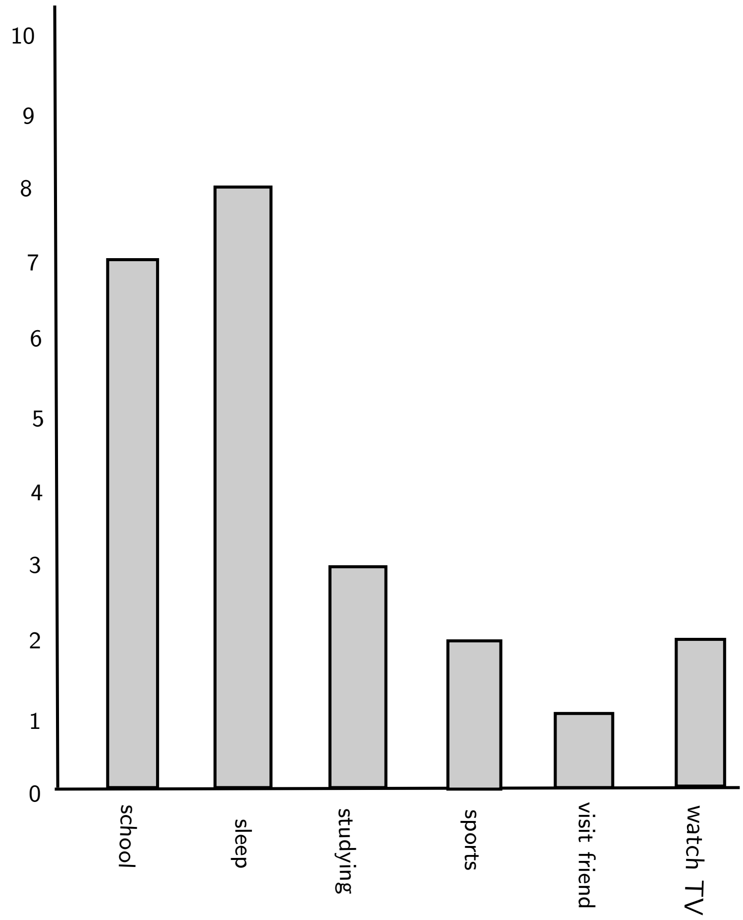
1. Examine the following graphs and comment on the effects of changing scale.

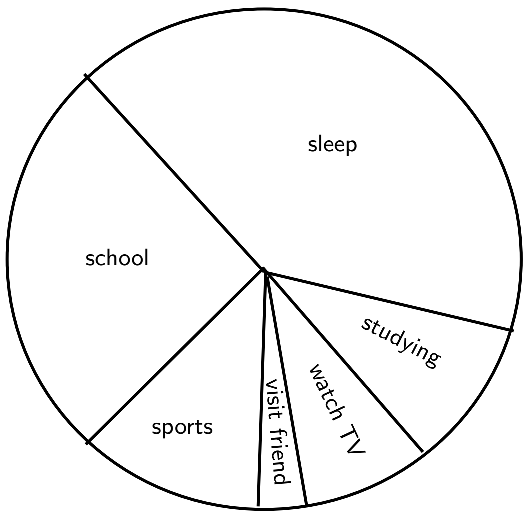




2. Examine the following three plots and comment on omission, selection and bias. Hint: What is wrong with the data and what is missing from the bar and pie charts?

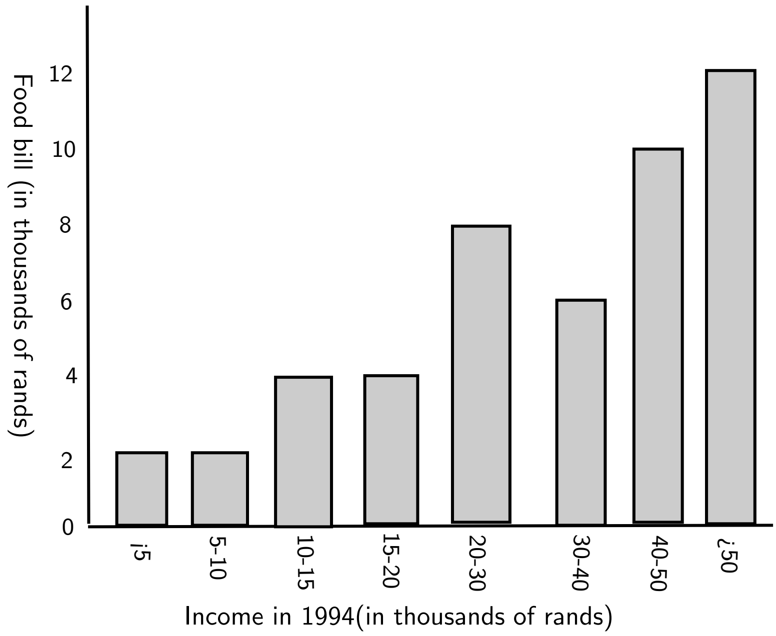
Activity	Hours
Sleep	8
Sports	2
School	7
Visit friend	1
Watch TV	2
Studying	3





Exercise: Misuse of Statistics

The bar graph below shows the results of a study that looked at the cost of food compared to the income of a household in 1994.



Income (thousands of rands)	Food bill (thousands of rands)
<5	2
5-10	2
10-15	4
15-20	4
20-30	8
30-40	6
40-50	10
>50	12

1. What is the dependent variable? Why?
2. What conclusion can you make about this variable? Why? Does this make sense?
3. What would happen if the graph was changed from food bill in thousands of rands to percentage of income?
4. Construct this bar graph using a table. What conclusions can be drawn?
5. Why do the two graphs differ despite showing the same information?
6. What else is observed? Does this affect the fairness of the results?

33.7 End of Chapter Exercises

1. Many accidents occur during the holidays between Durban and Johannesburg. A study was done to see if speeding was a factor in the high accident rate. Use the results given to answer the following questions.

Speed (km/h)	Frequency
$60 < x \leq 70$	3
$70 < x \leq 80$	2
$80 < x \leq 90$	6
$90 < x \leq 100$	40
$100 < x \leq 110$	50
$110 < x \leq 120$	30
$120 < x \leq 130$	15
$130 < x \leq 140$	12
$140 < x \leq 150$	3
$150 < x \leq 160$	2

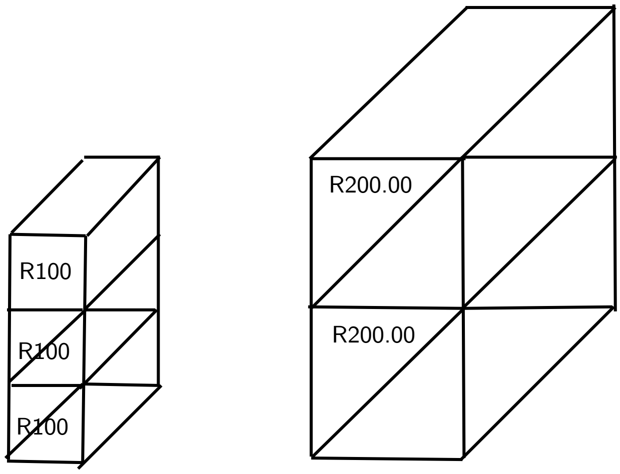
A Draw a graph to illustrate this information.

B Use your graph to find the median speed and the interquartile range.

C What percent of cars travel more than 120km/h on this road?

D Do cars generally exceed the speed limit?

2. The following two diagrams (showing two schools contribution to charity) have been exaggerated. Explain how they are misleading and redraw them so that they are not misleading.



3. The monthly income of eight teachers are given as follows:
R10 050; R14 300; R9 800; R15 000; R12 140; R13 800; R11 990; R12 900.

A What is the mean income and the standard deviation?

B How many of the salaries are within one standard deviation of the mean?

C If each teacher gets a bonus of R500 added to their pay what is the new mean and standard deviation?

D If each teacher gets a bonus of 10% on their salary what is the new mean and standard deviation?

E Determine for both of the above, how many salaries are within one standard deviation of the mean.

F Using the above information work out which bonus is more beneficial for the teachers.

Chapter 34

Independent and Dependent Events - Grade 11

34.1 Introduction

In probability theory an event is either independent or dependent. This chapter describes the differences and how each type of event is worked with.

34.2 Definitions

Two events are independent if knowing something about the value of one event does not give any information about the value of the second event. For example, the event of getting a "1" when a die is rolled and the event of getting a "1" the second time it is thrown are independent.

Definition: Independent Events
Two events A and B are independent if when one of them happens, it doesn't affect the other one happening or not.

The probability of two independent events occurring, $P(A \cap B)$, is given by:

$$P(A \cap B) = P(A) \times P(B) \tag{34.1}$$

Worked Example 151: Independent Events

Question: What is the probability of rolling a 1 and then rolling a 6 on a fair die?
Answer
Step 1 : Identify the two events and determine whether the events are independent or not
Event A is rolling a 1 and event B is rolling a 6. Since the outcome of the first event does not affect the outcome of the second event, the events are independent.
Step 2 : Determine the probability of the specific outcomes occurring, for each event
The probability of rolling a 1 is $\frac{1}{6}$ and the probability of rolling a 6 is $\frac{1}{6}$.
Therefore, $P(A) = \frac{1}{6}$ and $P(B) = \frac{1}{6}$.
Step 3 : Use equation 34.1 to determine the probability of the two events occurring together.

$$\begin{aligned} P(A \cap B) &= P(A) \times P(B) \\ &= \frac{1}{6} \times \frac{1}{6} \\ &= \frac{1}{36} \end{aligned}$$

The probability of rolling a 1 and then rolling a 6 on a fair die is $\frac{1}{36}$.

Consequently, two events are dependent if the outcome of the first event affects the outcome of the second event.

Worked Example 152: Dependent Events

Question: A cloth bag has 4 coins, 1 R1 coin, 2 R2 coins and 1 R5 coin. What is the probability of first selecting a R1 coin followed by selecting a R2 coin?

Answer

Step 1 : Identify the two events and determine whether the events are independent or not

Event A is selecting a R1 coin and event B is next selecting a R2. Since the outcome of the first event affects the outcome of the second event (because there are less coins to choose from after the first coin has been selected), the events are dependent.

Step 2 : Determine the probability of the specific outcomes occurring, for each event

The probability of first selecting a R1 coin is $\frac{1}{4}$ and the probability of next selecting a R2 coin is $\frac{2}{3}$ (because after the R1 coin has been selected, there are only three coins to choose from).

Therefore, $P(A) = \frac{1}{4}$ and $P(B) = \frac{2}{3}$.

Step 3 : Use equation 34.1 to determine the probability of the two events occurring together.

The same equation as for independent events are used, but the probabilities are calculated differently.

$$\begin{aligned} P(A \cap B) &= P(A) \times P(B) \\ &= \frac{1}{4} \times \frac{2}{3} \\ &= \frac{2}{12} \\ &= \frac{1}{6} \end{aligned}$$

The probability of first selecting a R1 coin followed by selecting a R2 coin is $\frac{1}{6}$.

34.2.1 Identification of Independent and Dependent Events

Use of a Contingency Table

A two-way contingency table (studied in an earlier grade) can be used to determine whether events are independent or dependent.

Definition: two-way contingency table
A two-way contingency table is used to represent possible outcomes when two events are combined in a statistical analysis.

For example we can draw and analyse a two-way contingency table to solve the following problem.

Worked Example 153: Contingency Tables

Question: A medical trial into the effectiveness of a new medication was carried out. 120 males and 90 females responded. Out of these 50 males and 40 females responded positively to the medication.

1. Was the medication's succes independent of gender? Explain.
2. Give a table for the independent of gender results.

Answer
Step 1 : Draw a contingency table

	Male	Female	Totals
Positive result	50	40	90
No Positive result	70	50	120
Totals	120	90	210

Step 2 : Work out probabilities
P(male).P(positive result)=$\frac{120}{210} = 0.57$
P(female).P(positive result)=$\frac{90}{210} = 0.43$
P(male and positive result)=$\frac{50}{210} = 0.24$
Step 3 : Draw conclusion
P(male and positive result) is the observed probability and P(male).P(positive result) is the expected probability. These two are quite different. So there is no evidence that the medications success is independent of gender.
Step 4 : Gender-independent results
To get gender independence we need the positve results in the same ratio as the gender. The gender ratio is: 120:90, or 4:3, so the number in the male and positive column would have to be $\frac{4}{7}$ of the total number of patients responding positively which gives 22. This leads to the following table:

	Male	Female	Totals
Positive result	22	68	90
No Positive result	98	22	120
Totals	120	90	210

Use of a Venn Diagram

We can also use Venn diagrams to check whether events are dependent or independent.

Definition: Independent events
Events are said to be independent if the result or outcome of the event does not affect the result or outcome of another event. So P(A/C)=P(A), where P(A/C) represents the probability of event A after event C has occured.

Definition: Dependent events
If the outcome of one event is affected by the outcome of another event such that $P(A/C) \neq P(A)$

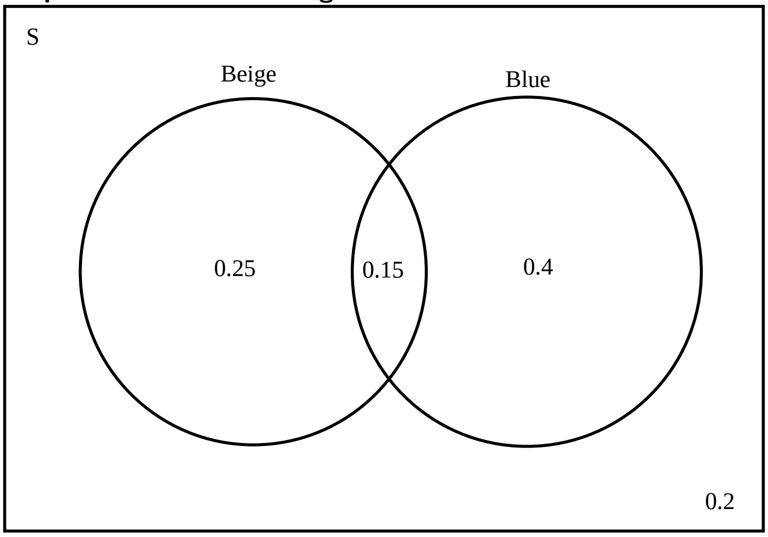
Also note that $P(A/C) = \frac{P(A \cap C)}{P(C)}$ For example, we can draw a Venn diagram and a contingency table to illustrate and analyse the following example.

Worked Example 154: Venn diagrams and events

Question: A school decided that it's uniform needed upgrading. The colours on offer were beige or blue or beige and blue. 40% of the school wanted beige, 55% wanted blue and 15% said a combination would be fine. Are the two events independent?

Answer

Step 1 : Draw a Venn diagram



Step 2 : Draw up a contingency table

	Beige	Not Beige	Totals
Blue	0.15	0.4	0.55
Not Blue	0.25	0.2	0.35
Totals	0.40	0.6	1

Step 3 : Work out the probabilities

P(Blue)=0.4, P(Beige)=0.55, P(Both)=0.15, P(Neither)=0.20
Probability of choosing beige after blue is:

$$\begin{aligned} P(Beige/Blue) &= \frac{P(Beige \cap Blue)}{P(Blue)} \\ &= \frac{0.15}{0.55} \\ &= 0.27 \end{aligned}$$

Step 4 : Solve the problem

Since $P(Beige/Blue) \neq P(Beige)$ the events are statistically independent.

Extension: Applications of Probability Theory

Two major applications of probability theory in everyday life are in risk assessment and in trade on commodity markets. Governments typically apply probability methods in environmental regulation where it is called "pathway analysis", and are often measuring well-being using methods that are stochastic in nature, and choosing projects to undertake based on statistical analyses of their probable effect on the population as a whole. It is not correct to say that statistics are involved in the modelling itself, as typically the assessments of risk are one-time and thus require more fundamental probability models, e.g. "the probability of another 9/11". A law of small numbers tends to apply to all such choices and perception of the effect of such choices, which makes probability measures a political matter.

A good example is the effect of the perceived probability of any widespread Middle East conflict on oil prices - which have ripple effects in the economy as a whole. An assessment by a commodity trade that a war is more likely vs. less likely sends prices up or down, and signals other traders of that opinion. Accordingly, the probabilities are not assessed independently nor necessarily very rationally. The theory of behavioral finance emerged to describe the effect of such groupthink on pricing, on policy, and on peace and conflict.

It can reasonably be said that the discovery of rigorous methods to assess and combine probability assessments has had a profound effect on modern society. A good example is the application of game theory, itself based strictly on probability, to the Cold War and the mutual assured destruction doctrine. Accordingly, it may be of some importance to most citizens to understand how odds and probability assessments are made, and how they contribute to reputations and to decisions, especially in a democracy.

Another significant application of probability theory in everyday life is reliability. Many consumer products, such as automobiles and consumer electronics, utilize reliability theory in the design of the product in order to reduce the probability of failure. The probability of failure is also closely associated with the product's warranty.

34.3 End of Chapter Exercises

1. In each of the following contingency tables give the expected numbers for the events to be perfectly independent and decide if the events are independent or dependent.

A

	Brown eyes	Not Brown eyes	Totals
Black hair	50	30	80
Red hair	70	80	150
Totals	120	110	230

B

	Point A	Point B	Totals
Busses left late	15	40	55
Busses left on time	25	20	35
Totals	40	60	100

C

	Durban	Bloemfontein	Totals
Liked living there	130	30	160
Did not like living there	140	200	340
Totals	270	230	500

D

	Multivitamin A	Multivitamin B	Totals
Improvement in health	400	300	700
No improvement in health	140	120	260
Totals	540	420	960

2. A company has a probability of 0.4 of meeting their quota on time and a probability of 0.25 of meeting their quota late. Also there is a 0.10 chance of not meeting their quota on time. Use a Venn diagram and a contingency table to show the information and decide if the events are independent.

3. A study was undertaken to see how many people in Port Elizabeth owned either a Volkswagen or a Toyota. 3% owned both, 25% owned a Toyota and 60% owned a Volkswagen. Draw a contingency table to show all events and decide if car ownership is independent.

4. Jane invested in the stock market. The probability that she will not lose all her money is 1.32. What is the probability that she will lose all her money? Explain.

5. If D and F are mutually exclusive events, with P(D')=0.3 and P(D or F)=0.94, find P(F).

6. A car sales person has pink, lime-green and purple models of car A and purple, orange and multicolour models of car B. One dark night a thief steals a car.

   A What is the experiment and sample space?

   B Draw a Venn diagram to show this.

   C What is the probability of stealing either model A or model B?

   D What is the probability of stealing both model A and model B?

7. Event X's probability is 0.43, Event Y's probability is 0.24. The probability of both occuring together is 0.10. What is the probability that X or Y will occur (this inculdes X and Y occuring simultaneously)?

8. P(H)=0.62, P(J)=0.39 and P(H and J)=0.31. Calculate:

   A P(H')

   B P(H or J)

   C P(H' or J')

   D P(H' or J)

   E P(H' and J')

9. The last ten letters of the alphabet were placed in a hat and people were asked to pick one of them. Event D is picking a vowel, Event E is picking a consonant and Evetn F is picking the last four letters. Calculate the following probabilities:

   A P(F')

   B P(F or D)

   C P(neither E nor F)

   D P(D and E)

   E P(E and F)

   F P(E and D')

10. At Dawnview High there are 400 Grade 12's. 270 do Computer Science, 300 do English and 50 do Typing. All those doing Computer Science do English, 20 take Computer Science and Typing and 35 take English and Typing. Using a Venn diagram calculate the probability that a pupil drawn at random will take:

    A English, but not Typing or Computer Science

    B English but not Typing

    C English and Typing but not Computer Science

    D English or Typing

Part IV

Grade 12

Chapter 35

Logarithms - Grade 12

In mathematics many ideas are related. We saw that addition and subtraction are related and that multiplication and division are related. Similarly, exponentials and logarithms are related.

Logarithms, commonly referred to as *logs*, are the inverse of exponentials. The logarithm of a number x in the base a is defined as the number n such that $a^n = x$.

So, if $a^n = x$, then:

$$\log_a(x) = n \tag{35.1}$$

Extension: Inverse Function

When we say "inverse function" we mean that the answer becomes the question and the question becomes the answer. For example, in the equation $a^b = x$ the "question" is "what is a raised to the power b." The answer is "x." The inverse function would be $log_a x = b$ or "by what power must we raise a to obtain x." The answer is "b."

The mathematical symbol for logarithm is $\log_a(x)$ and it is read "log to the base a of x". For example, $\log_{10}(100)$ is "log to the base 10 of 100".

Activity :: Logarithm Symbols : Write the following out in words. The first one is done for you.

1. $\log_2(4)$ **is log to the base 2 of 4**
2. $\log_{10}(14)$
3. $\log_{16}(4)$
4. $\log_x(8)$
5. $\log_y(x)$

35.1 Definition of Logarithms

The logarithm of a number is the *index* to which the base must be raised to give that number. From the first example of the activity $\log_2(4)$ (read log to the base 2 of 4) means the power of 2 that will give 4. Therefore,

$$\log_2(4) = 2 \tag{35.2}$$

The *index-form* is then $2^2 = 4$ and the *logarithmic-form* is $\log_2 4 = 2$.

Definition: Logarithms
If $a^n = x$, then: $\log_a(x) = n$, where $a > 0$; $a \neq 1$ and $x > 0$.

Activity :: Applying the definition : Find the value of:

1. $\log_7 343$

$$\begin{aligned} & Reasoning: \\ & 7^3 = 343 \\ therefore, \ & \log_7 343 = 3 \end{aligned}$$

2. $\log_2 8$
3. $\log_4 \frac{1}{64}$
4. $\log_{10} 1\ 000$

35.2 Logarithm Bases

Logarithms, like exponentials, also have a base and $\log_2(2)$ is not the same as $\log_{10}(2)$.

We generally use the "common" base, 10, or the *natural* base, e.

The number e is an irrational number between 2.71 and 2.72. It comes up surprisingly often in Mathematics, but for now suffice it to say that it is one of the two common bases.

Extension: Natural Logarithm
The natural logarithm (symbol $\ln$) is widely used in the sciences. The natural logarithm is to the base e which is approximately $2.71828183...$. e is like π and is another example of an irrational number.

While the notation $\log_{10}(x)$ and $\log_e(x)$ may be used, $\log_{10}(x)$ is often written $\log(x)$ in Science and $\log_e(x)$ is normally written as $\ln(x)$ in both Science and Mathematics. So, if you see the $\log$ symbol without a base, it means $\log_{10}$.

It is often necessary or convenient to convert a log from one base to another. An engineer might need an approximate solution to a log in a base for which he does not have a table or calculator function, or it may be algebraically convenient to have two logs in the same base.

Logarithms can be changed from one base to another, by using the change of base formula:

$$\log_a x = \frac{\log_b x}{\log_b a} \tag{35.3}$$

where b is any base you find convenient. Normally a and b are known, therefore $\log_b a$ is normally a known, if irrational, number.

For example, change $\log_2 12$ in base 10 is:

$$\log_2 12 = \frac{\log_{10} 12}{\log_{10} 2}$$

Activity :: Change of Base : Change the following to the indicated base:

1. $\log_2(4)$ **to base 8**
2. $\log_{10}(14)$ **to base 2**
3. $\log_{16}(4)$ **to base 10**
4. $\log_x(8)$ **to base** y
5. $\log_y(x)$ **to base** x

35.3 Laws of Logarithms

Just as for the exponents, logarithms have some laws which make working with them easier. These laws are based on the exponential laws and are summarised first and then explained in detail.

$$\log_a(1) = 0 \quad (35.4)$$
$$\log_a(a) = 1 \quad (35.5)$$
$$\log_a(x \cdot y) = \log_a(x) + \log_a(y) \quad (35.6)$$
$$\log_a\left(\frac{x}{y}\right) = \log_a(x) - \log_a(y) \quad (35.7)$$
$$\log_a(x^b) = b\log_a(x) \quad (35.8)$$
$$\log_a\left(\sqrt[b]{x}\right) = \frac{\log_a(x)}{b} \quad (35.9)$$

35.4 Logarithm Law 1: $\log_a 1 = 0$

$$\begin{aligned} \text{Since} \quad a^0 &= 1 \\ \text{Then,} \quad \log_a(1) &= \log_a(a^0) \\ &= 0 \quad \text{by definition of logarithm in Equation 35.1} \end{aligned}$$

For example,

$$\log_2 1 = 0$$

and

$$\log_2 51 = 0$$

Activity :: Logarithm Law 1: $\log_a 1 = 0$ **: Simplify the following:**

1. $\log_2(1) + 5$
2. $\log_{10}(1) \times 100$
3. $3 \times \log_{16}(1)$
4. $\log_x(1) + 2xy$
5. $\frac{\log_y(1)}{x}$

35.5 Logarithm Law 2: $\log_a(a) = 1$

$$\begin{aligned} \text{Since} \quad a^1 &= a \\ \text{Then,} \quad \log_a(a) &= \log_a(a^1) \\ &= 1 \quad \text{by definition of logarithm in Equation 35.1} \end{aligned}$$

For example,

$$\log_2 2 = 1$$

and

$$\log_{25} 25 = 1$$

Activity :: Logarithm Law 2: $\log_a(a) = 1$: Simplify the following:

1. $\log_2(2) + 5$
2. $\log_{10}(10) \times 100$
3. $3 \times \log_{16}(16)$
4. $\log_x(x) + 2xy$
5. $\frac{\log_y(y)}{x}$

Important: Useful to know and remember

When the base is 10, we do not need to state it. From the work done up to now, it is also useful to summarise the following facts:

1. $\log 1 = 0$
2. $\log 10 = 1$
3. $\log 100 = 2$
4. $\log 1000 = 3$

35.6 Logarithm Law 3: $\log_a(x \cdot y) = \log_a(x) + \log_a(y)$

The derivation of this law is a bit trickier than the first two. Firstly, we need to relate x and y to the base a. So, assume that $x = a^m$ and $y = a^n$. Then from Equation 35.1, we have that:

$$\log_a(x) = m \qquad (35.10)$$
$$\text{and} \quad \log_a(y) = n \qquad (35.11)$$

This means that we can write:

$$\begin{aligned} \log_a(x \cdot y) &= \log_a(a^m \cdot a^n) \\ &= \log_a(a^{m+n}) \quad \text{Exponential Law Equation 4.4} \\ &= \log_a(a^{\log_a(x)+\log_a(y)}) \quad \text{From Equation 35.10 and Equation 35.11} \\ &= \log_a(x) + \log_a(y) \quad \text{From Equation 35.1} \end{aligned}$$

For example, show that $\log(10 \cdot 100) = \log 10 + \log 100$. Start with calculating the left hand side:

$$\begin{aligned} \log(10 \cdot 100) &= \log(1000) \\ &= \log(10^3) \\ &= 3 \end{aligned}$$

The right hand side:

$$\begin{aligned} \log 10 + \log 100 &= 1 + 2 \\ &= 3 \end{aligned}$$

Both sides are equal. Therefore, $\log(10 \cdot 100) = \log 10 + \log 100$.

Activity :: Logarithm Law 3: $\log_a(x \cdot y) = \log_a(x) + \log_a(y)$ **: Write as seperate logs:**

1. $\log_2(8 \times 4)$
2. $\log_8(10 \times 10)$
3. $\log_{16}(xy)$
4. $\log_z(2xy)$
5. $\log_x(y^2)$

35.7 Logarithm Law 4: $\log_a\left(\frac{x}{y}\right) = \log_a(x) - \log_a(y)$

The derivation of this law is identical to the derivation of Logarithm Law 3 and is left as an exercise.

For example, show that $\log(\frac{10}{100}) = \log 10 - \log 100$. Start with calculating the left hand side:

$$\begin{aligned} \log(\frac{10}{100}) &= \log(\frac{1}{10}) \\ &= \log(10^{-1}) \\ &= -1 \end{aligned}$$

The right hand side:

$$\begin{aligned} \log 10 - \log 100 &= 1 - 2 \\ &= -1 \end{aligned}$$

Both sides are equal. Therefore, $\log(\frac{10}{100}) = \log 10 - \log 100$.

Activity :: Logarithm Law 4: $\log_a\left(\frac{x}{y}\right) = \log_a(x) - \log_a(y)$ **: Write as seperate logs:**

1. $\log_2(\frac{8}{5})$
2. $\log_8(\frac{100}{3})$
3. $\log_{16}(\frac{x}{y})$

4. $\log_z(\frac{2}{y})$

5. $\log_x(\frac{y}{2})$

35.8 Logarithm Law 5: $\log_a(x^b) = b\log_a(x)$

Once again, we need to relate x to the base a. So, we let $x = a^m$. Then,

$$\begin{aligned} \log_a(x^b) &= \log_a((a^m)^b) \\ &= \log_a(a^{m\cdot b}) \quad \text{(Exponential Law in Equation 4.8)} \\ \text{But,}\quad m &= \log_a(x) \qquad \text{(Assumption that } x = a^m\text{)} \\ \therefore\quad \log_a(x^b) &= \log_a(a^{b\cdot\log_a(x)}) \\ &= b\cdot\log_a(x) \quad \text{(Definition of logarithm in Equation 35.1)} \end{aligned}$$

For example, we can show that $\log_2(5^3) = 3\log_2(5)$.

$$\begin{aligned} \log_2(5^3) &= \log_(5\cdot 5\cdot 5) \\ &= \log_2 5 + \log_2 5 + \log_2 5 \quad (\because \log_a(x\cdot y) = \log_a(a^m\cdot a^n)) \\ &= 3\log_2 5 \end{aligned}$$

Therefore, $\log_2(5^3) = 3\log_2(5)$.

Activity :: Logarithm Law 5: $\log_a(x^b) = b\log_a(x)$: Simplify the following:

1. $\log_2(8^4)$

2. $\log_8(10^{10})$

3. $\log_{16}(x^y)$

4. $\log_z(y^x)$

5. $\log_x(y^{2x})$

35.9 Logarithm Law 6: $\log_a(\sqrt[b]{x}) = \frac{\log_a(x)}{b}$

The derivation of this law is identical to the derivation of Logarithm Law 5 and is left as an exercise.

For example, we can show that $\log_2(\sqrt[3]{5}) = \frac{\log_2 5}{3}$.

$$\begin{aligned} \log_2(\sqrt[3]{5}) &= \log_(5^{\frac{1}{3}}) \\ &= \frac{1}{3}\log_2 5 \quad (\because \log_a(x^b) = b\log_a(x)) \\ &= \frac{\log_2 5}{3} \end{aligned}$$

Therefore, $\log_2(\sqrt[3]{5}) = \frac{\log_2 5}{3}$.

Activity :: Logarithm Law 6: $\log_a(\sqrt[b]{x}) = \frac{\log_a(x)}{b}$ **: Simplify the following:**

1. $\log_2(\sqrt[4]{8})$
2. $\log_8(\sqrt[10]{10})$
3. $\log_{16}(\sqrt[y]{x})$
4. $\log_z(\sqrt[x]{y})$
5. $\log_x(\sqrt[2x]{y})$

Worked Example 155: Simplification of Logs

Question: Simplify, without use of a calculator:

$$3\log 2 + \log 125$$

Answer

Step 1 : Try to write any quantities as exponents

125 can be written as 5^3.

Step 2 : Simplify

$$\begin{aligned} 3\log 2 + \log 125 &= 3\log 2 + \log 5^3 \\ &= 3\log 2 + 3\log 5 \quad \because \log_a(x^b) = b\log_a(x) \end{aligned}$$

Step 3 : Final Answer

We cannot simplify any further. The final answer is:

$$3\log 2 + 3\log 5$$

The final answer does not have to be *that* simple.

Worked Example 156: Simplification of Logs

Question: Simplify, without use of a calculator:

$$8^{\frac{2}{3}} + \log_2 32$$

Answer

Step 1 : Try to write any quantities as exponents

8 can be written as 2^3. 32 can be written as 2^5.

Step 2 : Re-write the question using the exponential forms of the numbers

$$8^{\frac{2}{3}} + \log_2 32 = (2^3)^{\frac{2}{3}} + \log_2 2^5$$

Step 3 : Determine which laws can be used.

We can use:

$$\log_a(x^b) = b\log_a(x)$$

Step 4 : Apply log laws to simplify

$$(2^3)^{\frac{2}{3}} + \log_2 2^5 = (2)^{3\frac{2}{3}} + 5\log_2 2$$

Step 5 : Determine which laws can be used.

We can now use $\log_a a = 1$

Step 6 : Apply log laws to simplify

$$(2)^{3\frac{2}{3}} + 5\log_2 2 = (2)^2 + 5(1) = 4 + 5 = 9$$

Step 7 : Final Answer

The final answer is:

$$8^{\frac{2}{3}} + \log_2 32 = 9$$

Worked Example 157: Simplify to one log

Question: Write $2\log 3 + \log 2 - \log 5$ as the logarithm of a single number.

Answer

Step 1 : Reverse law 5

$2\log 3 + \log 2 - \log 5 = \log 3^2 + \log 2 - \log 5$

Step 2 : Apply laws 3 and 4

$= \log 3^2 \times 2 \div 5$

Step 3 : Write the final answer

$= \log 3{,}6$

35.10 Solving simple log equations

In grade 10 you solved some exponential equations by trial and error, because you did not know the great power of logarithms yet. Now it is much easier to solve these equations by using logarithms.

For example to solve x in $25^x = 50$ correct to two decimal places you simply apply the following reasoning. If the LHS = RHS then the logarithm of the LHS must be equal to the logarithm of the RHS. By applying Law 5, you will be able to use your calculator to solve for x.

Worked Example 158: Solving Log equations

Question: Solve for x: $25^x = 50$ correct to two decimal places.

Answer

Step 1 : Taking the log of both sides

$\log 25^x = \log 50$

Step 2 : Use Law 5

$x \log 25 = \log 50$

Step 3 : Solve for x

$x = \log 50 \div \log 25$

$x = 1{,}21533....$

Step 4 : Round off to required decimal place

$x = 1{,}22$

In general, the exponential equation should be simplified as much as possible. Then the aim is to make the unknown quantity (i.e. x) the subject of the equation.

For example, the equation

$$2^{(x+2)} = 1$$

is solved by moving all terms with the unknown to one side of the equation and taking all constants to the other side of the equation

$$\begin{aligned} 2^x \cdot 2^2 &= 1 \\ 2^x &= \frac{1}{2^2} \end{aligned}$$

Then, take the logarithm of each side.

$$\begin{aligned} \log(2^x) &= \log(\frac{1}{2^2}) \\ x\log(2) &= -\log(2^2) \\ x\log(2) &= -2\log(2) \quad \text{Divide both sides by } \log(2) \\ \therefore \quad x &= -2 \end{aligned}$$

Substituting into the original equation, yields

$$2^{-2+2} = 2^0 = 1 \quad \checkmark$$

Similarly, $9^{(1-2x)} = 3^4$ is solved as follows:

$$\begin{aligned} 9^{(1-2x)} &= 3^4 \\ 3^{2(1-2x)} &= 3^4 \\ 3^2 3^{-4x} &= 3^4 \\ 3^{-4x} &= 3^4 \cdot 3^{-2} \\ 3^{-4x} &= 3^2 \quad \text{take the logarithm of both sides} \\ \log(3^{-4x}) &= \log(3^2) \\ -4x\log(3) &= 2\log(3) \quad \text{divide both sides by } \log(3) \\ -4x &= 2 \\ \therefore x &= -\frac{1}{2} \end{aligned}$$

Substituting into the original equation, yields

$$9^{(1-2(\frac{-1}{2}))} = 9^{(1+1)} = 3^{2(2)} = 3^4 \quad \checkmark$$

Worked Example 159: Exponential Equation

Question: Solve for x in $7 \cdot 5^{(3x+3)} = 35$

Answer

Step 1 : Identify the base with x as an exponent

There are two possible bases: 5 and 7. x is an exponent of 5.

Step 2 : Eliminate the base with no x

In order to eliminate 7, divide both sides of the equation by 7 to give:

$$5^{(3x+3)} = 5$$

Step 3 : Take the logarithm of both sides

$$\log(5^{(3x+3)}) = \log(5)$$

Step 4 : Apply the log laws to make x the subject of the equation.

$$\begin{aligned} (3x+3)\log(5) &= log(5) \quad \text{divide both sides of the equation by } \log(5) \\ 3x+3 &= 1 \\ 3x &= -2 \\ x &= -\frac{2}{3} \end{aligned}$$

Step 5 : Substitute into the original equation to check answer.

$$7 \cdot 5^{(-3\frac{2}{3}+3)} = 7 \cdot 5^{(-2+3)} = 7 \cdot 5^1 = 35 \quad \checkmark$$

35.10.1 Exercises

Solve for x:

1. $log_3 x = 2$
2. $10^{log 27} = x$
3. $3^{2x-1} = 27^{2x-1}$

35.11 Logarithmic applications in the Real World

Logarithms are part of a number of formulae used in the Physical Sciences. There are formulae that deal with earthquakes, with sound, and pH-levels to mention a few. To work out time periods is growth or decay, logs are used to solve the particular equation.

Worked Example 160: Using the growth formula

Question: A city grows 5% every 2 years. How long will it take for the city to triple its size?

Answer

Step 1 : Use the formula

$A = P(1+i)^n$ Assume $P = x$, then $A = 3x$. For this example n represents a period of 2 years, therefore the n is halved for this purpose.

Step 2 : Substitute information given into formula

$$\begin{aligned} 3 &= (1{,}05)^{\frac{n}{2}} \\ \log 3 &= \frac{n}{2} \times \log 1.05 \quad (using law 5) \\ n &= 2\log 3 \div \log 1{,}05 \\ n &= 45{,}034 \end{aligned}$$

Step 3 : Final answer

So it will take approximately 45 years for the population to triple in size.

35.11.1 Exercises

1. The population of a certain bacteria is expected to grow exponentially at a rate of 15 % every hour. If the initial population is 5 000, how long will it take for the population to reach 100 000 ?
2. Plus Bank is offering a savings account with an interest rate if 10 % per annum compounded monthly. You can afford to save R 300 per month. How long will it take you to save up R 20 000 ? (Answer to the nearest rand)

Worked Example 161: Logs in Compound Interest

Question: I have R12 000 to invest. I need the money to grow to at least R30 000. If it is invested at a compound interest rate of 13% per annum, for how long (in full years) does my investment need to grow ?

Answer

Step 1 : The formula to use

$A = P(1+i)^n$

Step 2 : Substitute and solve for n

$$\begin{aligned}
30\ 000 &< 12\ 000(1+0{,}13)^n \\
1{,}13^n &> \frac{5}{2} \\
n\log 1{,}13 &> \log 2{,}5 \\
n &> \log 2{,}5 \div \log 1{,}13 \\
n &> 7{,}4972....
\end{aligned}$$

Step 3 : Determine the final answer

In this case we round up, because 7 years will not yet deliver the required R 30 000. The investment need to stay in the bank for at least **8 years**.

35.12 End of Chapter Exercises

1. Show that
$$\log_a\left(\frac{x}{y}\right) = \log_a(x) - \log_a(y)$$
2. Show that
$$\log_a\left(\sqrt[b]{x}\right) = \frac{\log_a(x)}{b}$$
3. Without using a calculator show that:
$$\log\frac{75}{16} - 2\log\frac{5}{9} + \log\frac{32}{243} = \log 2$$
4. Given that $5^n = x$ and $n = \log_2 y$
   - A Write y in terms of n
   - B Express $\log_8 4y$ in terms of n
   - C Express 50^{n+1} in terms of x and y
5. Simplify, without the use of a calculator:

A $8^{\frac{2}{3}} + \log_2 32$

B $\log_3 9 - \log_5 \sqrt{5}$

C $\left(\frac{5}{4^{-1} - 9^{-1}}\right)^{\dfrac{1}{2}} + \log_3 9^{2,12}$

6. Simplify to a single number, without use of a calculator:

   A $\log_5 125 + \frac{\log 32 - \log 8}{\log 8}$

   B $\log 3 - \log 0{,}3$

7. Given: $\log_3 6 = a$ and $\log_6 5 = b$

   A Express $\log_3 2$ in terms of a.

   B Hence, or otherwise, find $\log_3 10$ in terms of a and b.

8. Given: $pq^k = qp^{-1}$

   Prove: $k = 1 - 2\log_q p$

9. Evaluate without using a calculator: $(\log_7 49)^5 + \log_5\left(\frac{1}{125}\right) - 13\ \log_9 1$

10. If $\log 5 = 0{,}7$, determine, **without using a calculator**:

    A $\log_2 5$

    B $10^{-1{,}4}$

11. Given: $M = \log_2(x+3) + \log_2(x-3)$

    A Determine the values of x for which M is defined.

    B Solve for x if $M = 4$.

12. Solve: $\left(x^3\right)^{\log x} = 10x^2$ (Answer(s) may be left in surd form, if necessary.)

13. Find the value of $(\log_{27} 3)^3$ without the use of a calculator.

14. Simplify By using a calculator: $\log_4 8 + 2\log_3 \sqrt{27}$

15. Write $\log 4500$ in terms of a and b if $2 = 10^a$ and $9 = 10^b$.

16. Calculate: $\frac{5^{2006} - 5^{2004} + 24}{5^{2004} + 1}$

17. Solve the following equation for x without the use of a calculator and using the fact that $\sqrt{10} \approx 3{,}16$:

$$2\log(x+1) = \frac{6}{\log(x+1)} - 1$$

18. Solve the following equation for x: $6^{6x} = 66$ (Give answer correct to 2 decimal places.)

Chapter 36

Sequences and Series - Grade 12

36.1 Introduction

In this chapter we extend the arithmetic and quadratic sequences studied in earlier grades, to geometric sequences. We also look at series, which is the summing of the terms in a sequence.

36.2 Arithmetic Sequences

The simplest type of numerical sequence is an *arithmetic sequence*.

Definition: Arithmetic Sequence
An *arithmetic* (or *linear*) *sequence* is a sequence of numbers in which each new term is calculated by **adding** a constant value to the previous term

For example,

$$1,2,3,4,5,6,\ldots$$

is an arithmetic sequence because you add 1 to the current term to get the next term:

first term:	1
second term:	2=1+1
third term:	3=2+1
$\vdots$	
n^{th} term:	$n = (n-1) + 1$

Activity :: Common Difference : Find the constant value that is added to get the following sequences and write out the next 5 terms.

1. $2,6,10,14,18,22,\ldots$
2. $-5,-3,-1,1,3,\ldots$
3. $1,4,7,10,13,16,\ldots$
4. $-1,10,21,32,43,54,\ldots$
5. $3,0,-3,-6,-9,-12,\ldots$

36.2.1 General Equation for the n^{th}-term of an Arithmetic Sequence

More formally, the number we start out with is called a_1 (the first term), and the difference between each successive term is denoted *d*, called the *common difference*.

The general arithmetic sequence looks like:

$$\begin{aligned} a_1 &= a_1 \\ a_2 &= a_1 + d \\ a_3 &= a_2 + d = (a_1 + d) + d = a_1 + 2d \\ a_4 &= a_3 + d = (a_1 + 2d) + d = a_1 + 3d \\ \ldots & \\ a_n &= a_1 + d \cdot (n - 1) \end{aligned}$$

Thus, the equation for the n^{th}-term will be:

$$a_n = a_1 + d \cdot (n - 1) \tag{36.1}$$

Given a_1 and the common difference, d, the entire set of numbers belonging to an arithmetic sequence can be generated.

Definition: Arithmetic Sequence
An *arithmetic* (or *linear*) *sequence* is a sequence of numbers in which each new term is calculated by adding a constant value to the previous term:

$$a_n = a_{n-1} + d \tag{36.2}$$

where

- a_n represents the new term, the n^{th}-term, that is calculated;
- a_{n-1} represents the previous term, the $(n-1)^{th}$-term;
- d represents some constant.

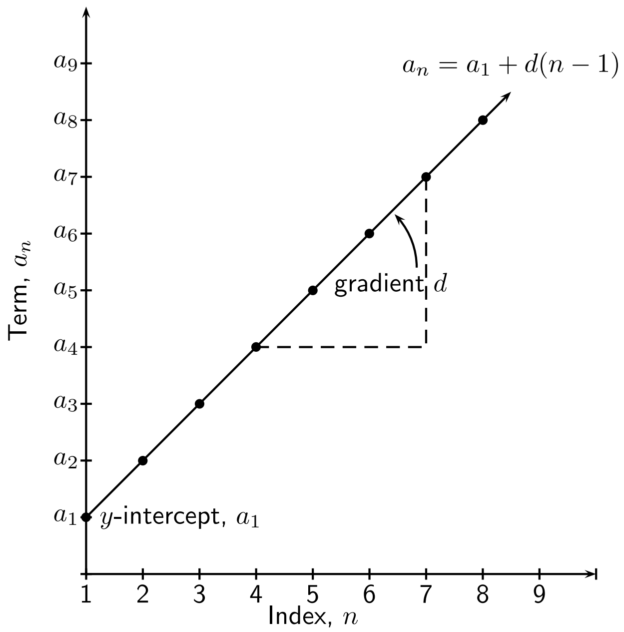
Important: Arithmetic Sequences

A simple test for an arithmetic sequence is to check that the difference between consecutive terms is constant:

$$a_2 - a_1 = a_3 - a_2 = a_n - a_{n-1} = d \tag{36.3}$$

This is quite an important equation, and is the definitive test for an arithmetic sequence. If this condition does not hold, the sequence is not an arithmetic sequence.

Extension: Plotting a graph of terms in an arithmetic sequence
Plotting a graph of the terms of sequence sometimes helps in determining the type of sequence involved. For an arithmetic sequence, plotting a_n vs. n results in:



36.3 Geometric Sequences

Definition: Geometric Sequences
A geometric sequence is a sequence in which every number in the sequence is equal to the previous number in the sequence, **multiplied** by a constant number.

This means that the *ratio* between consecutive numbers in the geometric sequence is a constant. We will explain what we mean by ratio after looking at the following example.

36.3.1 Example - A Flu Epidemic

Extension: What is influenza?
Influenza (commonly called "the flu") is caused by the influenza virus, which infects the respiratory tract (nose, throat, lungs). It can cause mild to severe illness that most of us get during winter time. The main way that the influenza virus is spread is from person to person in respiratory droplets of coughs and sneezes. (This is called "droplet spread".) This can happen when droplets from a cough or sneeze of an infected person are propelled (generally, up to a metre) through the air and deposited on the mouth or nose of people nearby. It is good practise to cover your mouth when you cough or sneeze so as not to infect others around you when you have the flu.

Assume that you have the flu virus, and you forgot to cover your mouth when two friends came to visit while you were sick in bed. They leave, and the next day they also have the flu. Let's assume that they in turn spread the virus to two of their friends by the same droplet spread the following day. Assuming this pattern continues and each sick person infects 2 other friends, we can represent these events in the following manner:

Again we can tabulate the events and formulate an equation for the general case:

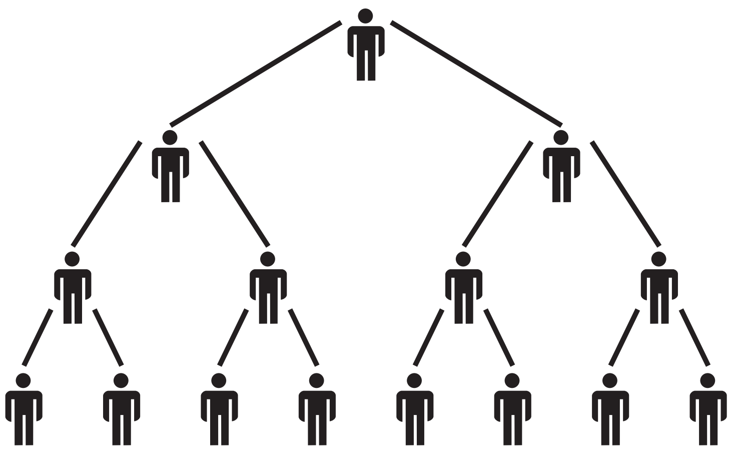


Figure 36.1: Each person infects two more people with the flu virus.

Day, n	**Number of newly-infected people**
1	$2 \quad = 2$
2	$4 \quad = 2 \times 2 = 2 \times 2^1$
3	$8 \quad = 2 \times 4 = 2 \times 2 \times 2 = 2 \times 2^2$
4	$16 \ = 2 \times 8 = 2 \times 2 \times 2 \times 2 = 2 \times 2^3$
5	$32 \ = 2 \times 16 = 2 \times 2 \times 2 \times 2 \times 2 = 2 \times 2^4$
$\vdots$	$\vdots$
n	$= 2 \times 2 \times 2 \times 2 \times \ldots \times 2 = 2 \times 2^{n-1}$

The above table represents the number of **newly-infected** people after n days since you first infected your 2 friends.

You sneeze and the virus is carried over to 2 people who start the chain ($a_1 = 2$). The next day, each one then infects 2 of their friends. Now 4 people are newly-infected. Each of them infects 2 people the third day, and 8 people are infected, and so on. These events can be written as a geometric sequence:

$$2;\ 4;\ 8;\ 16;\ 32;\ \ldots$$

Note the common factor (2) between the events. Recall from the linear arithmetic sequence how the common difference between terms were established. In the geometric sequence we can determine the *common ratio*, r, by

$$\frac{a_2}{a_1} = \frac{a_3}{a_2} = r \tag{36.4}$$

Or, more general,

$$\frac{a_n}{a_{n-1}} = r \tag{36.5}$$

Activity :: Common Factor of Geometric Sequence : Determine the common factor for the following geometric sequences:

1. $5, 10, 20, 40, 80, \ldots$
2. $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$
3. $7, 28, 112, 448, \ldots$
4. $2, 6, 18, 54, \ldots$

5. $-3, 30, -300, 3000, \ldots$

36.3.2 General Equation for the n^{th}-term of a Geometric Sequence

From the above example we know $a_1 = 2$ and $r = 2$, and we have seen from the table that the n^{th}-term is given by $a_n = 2 \times 2^{n-1}$. Thus, in general,

$$a_n = a_1 \cdot r^{n-1} \qquad (36.6)$$

where a_1 is the first term and r is called the *common ratio*.

So, if we want to know how many people are newly-infected after 10 days, we need to work out a_{10}:

$$\begin{aligned} a_n &= a_1 \cdot r^{n-1} \\ a_{10} &= 2 \times 2^{10-1} \\ &= 2 \times 2^9 \\ &= 2 \times 512 \\ &= 1024 \end{aligned}$$

That is, after 10 days, there are 1 024 newly-infected people.

Or, how many days would pass before 16 384 people become newly infected with the flu virus?

$$\begin{aligned} a_n &= a_1 \cdot r^{n-1} \\ 16\ 384 &= 2 \times 2^{n-1} \\ 16\ 384 \div 2 &= 2^{n-1} \\ 8\ 192 &= 2^{n-1} \\ 2^{13} &= 2^{n-1} \\ 13 &= n - 1 \\ n &= 14 \end{aligned}$$

That is, 14 days pass before 16 384 people are newly-infected.

Activity :: General Equation of Geometric Sequence : Determine the formula for the following geometric sequences:

1. $5, 10, 20, 40, 80, \ldots$

2. $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$

3. $7, 28, 112, 448, \ldots$

4. $2, 6, 18, 54, \ldots$

5. $-3, 30, -300, 3000, \ldots$

36.3.3 Exercises

1. What is the important characteristic of an arithmetic sequence?

2. Write down how you would go about finding the formula for the n^{th} term of an arithmetic sequence?

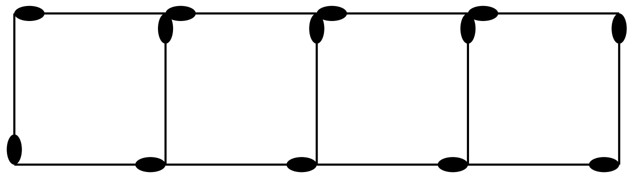
3. A single square is made from 4 matchsticks. Two squares in a row needs 7 matchsticks and 3 squares in a row needs 10 matchsticks. Determine:

   A the first term

   B the common difference

   C the formula for the general term

   D how many matchsticks are in a row of 25 squares



4. 5; x; y is an arithmetic sequence and 81; x; y is a geometric sequence. All terms in the sequences are integers. Calculate the values of x and y.

36.4 Recursive Formulae for Sequences

When discussing arithmetic and quadratic sequences, we noticed that the difference between two consecutive terms in the sequence could be written in a general way.

For an arithmetic sequence, where a new term is calculated by taking the previous term and adding a constant value, d:

$$a_n = a_{n-1} + d$$

The above equation is an example of a *recursive equation* since we can calculate the n^{th}-term only by considering the previous term in the sequence. Compare this with equation (36.1),

$$a_n = a_1 + d \cdot (n-1) \tag{36.7}$$

where one can directly calculate the n^{th}-term of an arithmetic sequence without knowing previous terms.

For quadratic sequences, we noticed the difference between consecutive terms is given by (**??**):

$$a_n - a_{n-1} = D \cdot (n-2) + d$$

Therefore, we re-write the equation as

$$a_n = a_{n-1} + D \cdot (n-2) + d \tag{36.8}$$

which is then a recursive equation for a quadratic sequence with common second difference, D.

Using (36.5), the recursive equation for a geometric sequence is:

$$a_n = r \cdot a_{n-1} \tag{36.9}$$

Recursive equations are extremely powerful: you can work out every term in the series just by knowing previous terms. As you can see from the examples above, working out a_n using the previous term a_{n-1} can be a much simpler computation than working out a_n from scratch using a general formula. This means that using a recursive formula when using a computer to work out a sequence would mean the computer would finish its calculations significantly quicker.

Activity :: Recursive Formula : Write the first 5 terms of the following sequences, given their recursive formulae:

1. $a_n = 2a_{n-1} + 3, a_1 = 1$
2. $a_n = a_{n-1}, a_1 = 11$
3. $a_n = 2a_{n-1}^2, a_1 = 2$

Extension: **The Fibonacci Sequence**
Consider the following sequence:

$$0;\ 1;\ 1;\ 2;\ 3;\ 5;\ 8;\ 13;\ 21;\ 34;\ \ldots \tag{36.10}$$

The above sequence is called the Fibonacci sequence. Each new term is calculated by adding the previous two terms. Hence, we can write down the recursive equation:

$$a_n = a_{n-1} + a_{n-2} \tag{36.11}$$

36.5 Series

In this section we simply work on the concept of *adding* up the numbers belonging to arithmetic and geometric sequences. We call the sum of *any* sequence of numbers a *series*.

36.5.1 Some Basics

If we add up the terms of a sequence, we obtain what is called a *series*. If we only sum a finite amount of terms, we get a *finite series*. We use the symbol S_n to mean the sum of the first n terms of a sequence $\{a_1;\ a_2;\ a_3; \ldots; a_n\}$:

$$S_n = a_1 + a_2 + a_3 + \ldots + a_n \tag{36.12}$$

For example, if we have the following sequence of numbers

$$1;\ 4;\ 9;\ 25;\ 36;\ 49;\ \ldots$$

and we wish to find the sum of the first 4 terms, then we write

$$S_4 = 1 + 4 + 9 + 25 = 39$$

The above is an example of a finite series since we are only summing 4 terms.

If we sum infinitely many terms of a sequence, we get an *infinite series*:

$$S_\infty = a_1 + a_2 + a_3 + \ldots \tag{36.13}$$

In the case of an infinite series, the number of terms is unknown and simply increases to ∞.

36.5.2 Sigma Notation

In this section we introduce a notation that will make our lives a little easier.

A sum may be written out using the summation symbol $\sum$. This symbol is *sigma*, which is the capital letter "S" in the Greek alphabet. It indicates that you must sum the expression to the *right* of it:

$$\sum_{i=m}^{n} a_i = a_m + a_{m+1} + \ldots + a_{n-1} + a_n \tag{36.14}$$

where

- i is the index of the sum;
- m is the lower bound (or start index), shown below the summation symbol;
- n is the upper bound (or end index), shown above the summation symbol;
- a_i are the terms of a sequence.

The index i is increased from m to n in steps of 1.

If we are summing from $n = 1$ (which implies summing from the first term in a sequence), then we can use either S_n- or $\sum$-notation since they mean the same thing:

$$S_n = \sum_{i=1}^{n} a_i = a_1 + a_2 + \ldots + a_n \tag{36.15}$$

For example, in the following sum,

$$\sum_{i=1}^{5} i$$

we have to add together all the terms in the sequence $a_i = i$ from $i = 1$ up until $i = 5$:

$$\sum_{i=1}^{5} i = 1 + 2 + 3 + 4 + 5 = 15$$

Examples

1.

$$\begin{aligned} \sum_{i=1}^{6} 2^i &= 2^1 + 2^2 + 2^3 + 2^4 + 2^5 + 2^6 \\ &= 2 + 4 + 8 + 16 + 32 + 64 \\ &= 126 \end{aligned}$$

2.

$$\sum_{i=3}^{10} (3x^i) = 3x^3 + 3x^4 + \ldots + 3x^9 + 3x^{10}$$

for any value x.

Some Basic Rules for Sigma Notation

1. Given two sequences, a_i and b_i,

$$\sum_{i=1}^{n} (a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i \tag{36.16}$$

2. For any constant c, which is any variable not dependent on the index i,

$$\begin{aligned}\sum_{i=1}^{n} c \cdot a_i &= c \cdot a_1 + c \cdot a_2 + c \cdot a_3 + \ldots + c \cdot a_n \\ &= c\,(a_1 + a_2 + a_3 + \ldots + a_n) \\ &= c \sum_{i=1}^{n} a_i \end{aligned} \quad (36.17)$$

Exercises

1. What is $\sum\limits_{k=1}^{4} 2$?
2. Determine $\sum\limits_{i=-1}^{3} i$.
3. Expand $\sum\limits_{k=0}^{5} i$.
4. Calculate the value of a if:

$$\sum_{k=1}^{3} a \cdot 2^{k-1} = 28$$

36.6 Finite Arithmetic Series

Remember that an arithmetic sequence is a set of numbers, such that the difference between any term and the previous term is a constant number, d, called the **constant difference**:

$$a_n = a_1 + d\,(n-1) \quad (36.18)$$

where

- n is the index of the sequence;
- a_n is the n^{th}-term of the sequence;
- a_1 is the first term;
- d is the common difference.

When we sum a finite number of terms in an arithmetic sequence, we get a *finite arithmetic series*.

The simplest arithmetic sequence is when $a_1 = 1$ and $d = 0$ in the general form (36.18); in other words all the terms in the sequence are 1:

$$\begin{aligned} a_i &= a_1 + d\,(i-1) \\ &= 1 + 0 \cdot (i-1) \\ &= 1 \\ \{a_i\} &= \{1;\ 1;\ 1;\ 1;\ 1;\ \ldots\} \end{aligned}$$

If we wish to sum this sequence from $i = 1$ to any positive integer n, we would write

$$\sum_{i=1}^{n} a_i = \sum_{i=1}^{n} 1 = 1 + 1 + 1 + \ldots + 1 \qquad (n \text{ times})$$

Since all the terms are equal to 1, it means that if we sum to n we will be adding n-number of 1's together, which is simply equal to n:

$$\boxed{\sum_{i=1}^{n} 1 = n} \tag{36.19}$$

Another simple arithmetic sequence is when $a_1 = 1$ and $d = 1$, which is the sequence of positive integers:

$$\begin{aligned} a_i &= a_1 + d\,(i-1) \\ &= 1 + 1 \cdot (i-1) \\ &= i \\ \{a_i\} &= \{1;\ 2;\ 3;\ 4;\ 5;\ \ldots\} \end{aligned}$$

If we wish to sum this sequence from $i = 1$ to any positive integer n, we would write

$$\sum_{i=1}^{n} i = 1 + 2 + 3 + \ldots + n \tag{36.20}$$

This is an equation with a very important solution as it gives the answer to the sum of positive integers.

Mathematician, Karl Friedrich Gauss, discovered this proof when he was only 8 years old. His teacher had decided to give his class a problem which would distract them for the entire day by asking them to add all the numbers from 1 to 100. Young Karl realised how to do this almost instantaneously and shocked the teacher with the correct answer, 5050.

We first write S_n as a sum of terms in ascending order:

$$S_n = 1 + 2 + \ldots + (n-1) + n \tag{36.21}$$

We then write the same sum but with the terms in descending order:

$$S_n = n + (n-1) + \ldots + 2 + 1 \tag{36.22}$$

We then add corresponding pairs of terms from equations (36.21) and (36.22), and we find that the sum for each pair is the same, $(n+1)$:

$$2\,S_n = (n+1) + (n+1) + \ldots + (n+1) + (n+1) \tag{36.23}$$

We then have n-number of $(n+1)$-terms, and by simplifying we arrive at the final result:

$$\begin{aligned} 2\,S_n &= n\,(n+1) \\ S_n &= \frac{n}{2}\,(n+1) \end{aligned}$$

$$\boxed{S_n = \sum_{i=1}^{n} i = \frac{n}{2}\,(n+1)} \tag{36.24}$$

36.6.1 General Formula for a Finite Arithmetic Series

If we wish to sum any arithmetic sequence, there is no need to work it out term-for-term. We will now determine the general formula to evaluate a finite arithmetic series. We start with the general formula for an arithmetic sequence and sum it from $i = 1$ to any positive integer n:

$$\begin{aligned}
\sum_{i=1}^{n} a_i &= \sum_{i=1}^{n} [a_1 + d(i-1)] \\
&= \sum_{i=1}^{n} (a_1 + di - d) \\
&= \sum_{i=1}^{n} [(a_1 - d) + di] \\
&= \sum_{i=1}^{n} (a_1 - d) + \sum_{i=1}^{n} (di) \\
&= \sum_{i=1}^{n} (a_1 - d) + d \sum_{i=1}^{n} i \\
&= (a_1 - d)\, n + \frac{dn}{2}(n+1) \\
&= \frac{n}{2}(2a_1 - 2d + dn + d) \\
&= \frac{n}{2}(2a_1 + dn - d) \\
&= \frac{n}{2}[\,2a_1 + d(n-1)\,]
\end{aligned}$$

So, the general formula for determining an arithmetic series is given by

$$\boxed{S_n = \sum_{i=1}^{n} [\,a_1 + d(i-1)\,] = \frac{n}{2}[\,2a_1 + d(n-1)\,]} \qquad (36.25)$$

For example, if we wish to know the series S_{20} for the arithmetic sequence $a_i = 3 + 7(i-1)$, we could either calculate each term individually and sum them:

$$\begin{aligned}
S_{20} &= \sum_{i=1}^{20} [3 + 7(i-1)] \\
&= 3 + 10 + 17 + 24 + 31 + 38 + 45 + 52 + \\
&\quad 59 + 66 + 73 + 80 + 87 + 94 + 101 + \\
&\quad 108 + 115 + 122 + 129 + 136 \\
&= 1390
\end{aligned}$$

or, more sensibly, we could use equation (36.25) noting that $a_1 = 3$, $d = 7$ and $n = 20$ so that

$$\begin{aligned}
S_{20} &= \sum_{i=1}^{20} [3 + 7(i-1)] \\
&= \tfrac{20}{2}[2 \cdot 3 + 7(20-1)] \\
&= 1390
\end{aligned}$$

In this example, it is clear that using equation (36.25) is beneficial.

36.6.2 Exercises

1. The sum to n terms of an arithmetic series is $S_n = \frac{n}{2}(7n + 15)$.

   A How many terms of the series must be added to give a sum of 425?

   B Determine the 6^{th} term of the series.

2. The sum of an arithmetic series is 100 times its first term, while the last term is 9 times the first term. Calculate the number of terms in the series if the first term is not equal to zero.

3. The common difference of an arithmetic series is 3. Calculate the values of n for which the n^{th} term of the series is 93, and the sum of the first n terms is 975.

4. The sum of n terms of an arithmetic series is $5n^2 - 11n$ for all values of n. Determine the common difference.

5. The sum of an arithmetic series is 100 times the value of its first term, while the last term is 9 times the first term. Calculate the number of terms in the series if the first term is not equal to zero.

6. The third term of an arithmetic sequence is -7 and the 7^th term is 9. Determine the sum of the first 51 terms of the sequence.

7. Calculate the sum of the arithmetic series $4 + 7 + 10 + \cdots + 901$.

8. The common difference of an arithmetic series is 3. Calculate the values of n for which the n^{th} term of the series is 93 and the sum of the first n terms is 975.

36.7 Finite Squared Series

When we sum a finite number of terms in a quadratic sequence, we get a *finite quadratic series*. The general form of a quadratic series is quite complicated, so we will only look at the simple case when $D = 2$ and $d = (a_2 - a_1) = 3$ in the general form (???). This is the sequence of squares of the integers:

$$\begin{aligned} a_i &= i^2 \\ \{a_i\} &= \{1^2;\ 2^2;\ 3^2;\ 4^2;\ 5^2;\ 6^2;\ \ldots\} \\ &= \{1;\ 4;\ 9;\ 16;\ 25;\ 36;\ \ldots\} \end{aligned}$$

If we wish to sum this sequence and create a series, then we write

$$S_n = \sum_{i=1}^{n} i^2 = 1 + 4 + 9 + \ldots + n^2$$

which can be written, in general, as

$$\boxed{S_n = \sum_{i=1}^{n} i^2 = \frac{n}{6}(2n+1)(n+1)} \qquad (36.26)$$

The proof for equation (36.26) can be found under the Advanced block that follows:

Extension: Derivation of the Finite Squared Series

We will now prove the formula for the finite squared series:

$$S_n = \sum_{i=1}^{n} i^2 = 1 + 4 + 9 + \ldots + n^2$$

We start off with the expansion of $(k+1)^3$.

$$\begin{aligned} (k+1)^3 &= k^3 + 3k^2 + 3k + 1 \\ (k+1)^3 - k^3 &= 3k^2 + 3k + 1 \end{aligned}$$

$$\begin{aligned}
k=1 &\quad : \quad 2^3-1^3=3(1)^2+3(1)+1\\
k=2 &\quad : \quad 3^3-2^3=3(2)^2+3(2)+1\\
k=3 &\quad : \quad 4^3-3^3=3(3)^2+3(3)+1\\
&\quad \vdots\\
k=n &\quad : \quad (n+1)^3-n^3=3n^2+3n+1
\end{aligned}$$

If we add all the terms on the right and left, we arrive at

$$\begin{aligned}
(n+1)^3-1 &= \sum_{i=1}^{n}(3i^2+3i+1)\\
n^3+3n^2+3n+1-1 &= 3\sum_{i=1}^{n} i^2+3\sum_{i=1}^{n} i+\sum_{i=1}^{n} 1\\
n^3+3n^2+3n &= 3\sum_{i=1}^{n} i^2+\frac{3n}{2}(n+1)+n\\
\sum_{i=1}^{n} i^2 &= \frac{1}{3}[n^3+3n^2+3n-\frac{3n}{2}(n+1)-n]\\
&= \frac{1}{3}(n^3+3n^2+3n-\frac{3}{2}n^2-\frac{3}{2}n-n)\\
&= \frac{1}{3}(n^3+\frac{3}{2}n^2+\frac{1}{2}n)\\
&= \frac{n}{6}(2n^2+3n+1)
\end{aligned}$$

Therefore,

$$\boxed{\sum_{i=1}^{n} i^2=\frac{n}{6}(2n+1)(n+1)}$$

36.8 Finite Geometric Series

When we sum a known number of terms in a geometric sequence, we get a *finite geometric series*. We know from (**??**) that we can write out each term of a geometric sequence in the general form:

$$a_n=a_1\cdot r^{n-1} \tag{36.27}$$

where

- n is the index of the sequence;
- a_n is the n^{th}-term of the sequence;
- a_1 is the first term;
- r is the common ratio (the ratio of any term to the previous term).

By simply adding together the first n terms, we are actually writing out the series

$$S_n=a_1+a_1\,r+a_1\,r^2+\ldots+a_1\,r^{n-2}+a_1\,r^{n-1} \tag{36.28}$$

We may multiply the above equation by r on both sides, giving us

$$rS_n=a_1\,r+a_1\,r^2+a_1\,r^3+\ldots+a_1\,r^{n-1}+a_1\,r^n \tag{36.29}$$

You may notice that all the terms on the right side of (36.28) and (36.29) are the same, except the first and last terms. If we subtract (36.28) from (36.29), we are left with just

$$rS_n - S_n = a_1\, r^n - a_1$$
$$S_n(r-1) = a_1\,(r^n - 1)$$

Dividing by $(r-1)$ on both sides, we arrive at the general form of a geometric series:

$$\boxed{S_n = \sum_{i=1}^{n} a_1 \cdot r^{i-1} = \frac{a_1\,(r^n-1)}{r-1}} \qquad (36.30)$$

36.8.1 Exercises

1. Prove that
$$a + ar + ar^2 + ... + ar^{n-1} = \frac{a\,(1-r^n)}{(1-r)}$$

2. Find the sum of the first 11 terms of the geometric series $6 + 3 + \frac{3}{2} + \frac{3}{4} + \ldots$

3. Show that the sum of the first n terms of the geometric series
$$54 + 18 + 6 + ... + 5\left(\tfrac{1}{3}\right)^{n-1}$$
is given by $81 - 3^{4-n}$.

4. The eighth term of a geometric sequence is 640. The third term is 20. Find the sum of the first 7 terms.

5. Solve for n: $\sum\limits_{t=1}^{n} 8\left(\frac{1}{2}\right)^t = 15\frac{3}{4}$.

6. The ratio between the sum of the first three terms of a geometric series and the sum of the 4^{th}-, $5^{th}-$ and 6^{th}-terms of the same series is $8 : 27$. Determine the common ratio and the first 2 terms if the third term is 8.

7. Given the geometric series:
$$2 \cdot (5)^5 + 2 \cdot (5)^4 + 2 \cdot (5)^3 + \ldots$$
   A Show that the series converges
   B Calculate the sum to infinity of the series
   C Calculate the sum of the first 8 terms of the series, correct to two decimal places.
   D Determine
$$\sum_{n=9}^{\infty} 2 \cdot 5^{6-n}$$
   correct to two decimal places using previously calculated results.

8. Given the geometric sequence $1;\ -3;\ 9;\ \ldots$ determine:
   A The 8$^{\text{th}}$ term of the sequence
   B The sum of the first 8 terms of the sequence.

9. Determine:
$$\sum_{n=1}^{4} 3 \cdot 2^{n-1}$$

36.9 Infinite Series

Thus far we have been working only with finite sums, meaning that whenever we determined the sum of a series, we only considered the sum of the first n terms. In this section, we consider what happens when we add infinitely many terms together. You might think that this is a silly question - surely the answer will be ∞ when one sums infinitely many numbers, no matter how small they are? The surprising answer is that in some cases one will reach ∞ (like when you try to add all the positive integers together), but in some cases one will get a finite answer. If you don't believe this, try doing the following sum, a geometric series, on your calculator or computer:

$$\tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \tfrac{1}{16} + \tfrac{1}{32} + \dots$$

You might think that if you keep adding more and more terms you will eventually get larger and larger numbers, but in fact you won't even get past 1 - try it and see for yourself!

We denote the sum of an infinite number of terms of a sequence by

$$S_\infty = \sum_{i=1}^{\infty} a_i$$

When we sum the terms of a series, and the answer we get after each summation gets closer and closer to some number, we say that the series *converges*. If a series does not converge, then we say that it *diverges*.

36.9.1 Infinite Geometric Series

There is a simple test for knowing instantly which geometric series converges and which diverges. When r, the common ratio, is strictly between -1 and 1, i.e. $-1 < r < 1$, the infinite series will converge, otherwise it will diverge. There is also a formula for working out what the series converges to.

Let's start off with formula (36.30) for the finite geometric series:

$$S_n = \sum_{i=1}^{n} a_1 \cdot r^{i-1} = \frac{a_1 (r^n - 1)}{r - 1}$$

Now we will investigate the value of r^n for -1

Take $r = \frac{1}{2}$:

$$\begin{aligned} n = 1 &: \quad r^n = r^1 = (\tfrac{1}{2})^1 = \tfrac{1}{2} \\ n = 2 &: \quad r^n = r^2 = (\tfrac{1}{2})^2 = \tfrac{1}{2} \cdot \tfrac{1}{2} = \tfrac{1}{4} < \tfrac{1}{2} \\ n = 3 &: \quad r^n = r^3 = (\tfrac{1}{2})^3 = \tfrac{1}{2} \cdot \tfrac{1}{2} \cdot \tfrac{1}{2} = \tfrac{1}{8} < \tfrac{1}{4} \end{aligned}$$

Since r is a fractional value in the range -1

Therefore,

$$\begin{aligned} S_n &= \frac{a_1 (r^n - 1)}{r - 1} \\ S_\infty &= \frac{a_1 (0 - 1)}{r - 1} \qquad \text{for} - 1 < r < 1 \\ &= \frac{-a_1}{r - 1} \\ &= \frac{a_1}{1 - r} \end{aligned}$$

The sum of an infinite geometric series is given by the formula

$$\boxed{S_\infty = \sum_{i=1}^{\infty} a_1.r^{i-1} = \frac{a_1}{1-r} \qquad \text{for} \qquad -1} \tag{36.31}$$

where a_1 is the first term of the series and r is the common ratio.

36.9.2 Exercises

1. What does $(\frac{2}{5})^n$ approach as n tends towards ∞?
2. Find the sum to infinity of the geometric series $3 + 1 + \frac{1}{3} + \frac{1}{9} + \ldots$
3. Determine for which values of x, the geometric series

   $$2 + \tfrac{2}{3}(x+1) + \tfrac{2}{9}(x+1)^2 + \ldots$$

   will converge.
4. The sum to infinity of a geometric series with positive terms is $4\frac{1}{6}$ and the sum of the first two terms is $2\frac{2}{3}$. Find a, the first term, and r, the common ratio between consecutive terms.

36.10 End of Chapter Exercises

1. Is $1 + 2 + 3 + 4 + \ldots$ an example of a *finite series* or an *infinite series*?
2. Calculate

   $$\sum_{k=2}^{6} 3\left(\tfrac{1}{3}\right)^{k+2}$$

3. If $x+1$; $x-1$; $2x-5$ are the first 3 terms of a convergent geometric series, calculate the:

   A Value of x.

   B Sum to infinity of the series.
4. Write the sum of the first 20 terms of the series $6 + 3 + \frac{3}{2} + \frac{3}{4} + \ldots$ in $\sum$-notation.
5. Given the geometric series: $2 \cdot 5^5 + 2 \cdot 5^4 + 2 \cdot 5^3 + \ldots$

   A Show that the series converges.

   B Calculate the sum of the first 8 terms of the series, correct to TWO decimal places.

   C Calculate the sum to infinity of the series.

   D Use your answer to 5c above to determine

   $$\sum_{n=9}^{\infty} 2 \cdot 5^{(6-n)}$$

   correct to TWO decimal places.
6. For the geometric series,

   $$54 + 18 + 6 + \ldots + 5\left(\tfrac{1}{3}\right)^{n-1}$$

   calculate the smallest value of n for which the sum of the first n terms is greater than 80.99.
7. Determine the value of $\sum\limits_{k=1}^{\infty} 12(\frac{1}{5})^{k-1}$.
8. A new soccer competition requires each of 8 teams to play every other team once.

   A Calculate the total number of matches to be played in the competition.

B If each of n teams played each other once, determine a formula for the total number of matches in terms of n.

9. The midpoints of the sides of square with length equal to 4 units are joined to form a new square. The process is repeated indefinitely. Calculate the sum of the areas of all the squares so formed.

10. Thembi worked part-time to buy a Mathematics book which cost R29,50. On 1 February she saved R1,60, and saves everyday 30 cents more than she saved the previous day. (So, on the second day, she saved R1,90, and so on.) After how many days did she have enough money to buy the book?

11. Consider the geometric series:
$$5 + 2\tfrac{1}{2} + 1\tfrac{1}{4} + \ldots$$
    A If A is the sum to infinity and B is the sum of the first n terms, write down the value of:
        i. A
        ii. B in terms of n.
    B For which values of n is $(A - B) < \frac{1}{24}$?

12. A certain plant reaches a height of 118 mm after one year under ideal conditions in a greenhouse. During the next year, the height increases by 12 mm. In each successive year, the height increases by $\frac{5}{8}$ of the previous year's growth. Show that the plant will never reach a height of more than 150 mm.

13. Calculate the value of n if $\sum_{a=1}^{n} (20 - 4a) = -20$.

14. Michael saved R400 during the first month of his working life. In each subsequent month, he saved 10% more than what he had saved in the previous month.
    A How much did he save in the 7^{th} working month?
    B How much did he save all together in his first 12 working months?
    C In which month of his working life did he save more than R1,500 for the first time?

15. A man was injured in an accident at work. He receives a disability grant of R4,800 in the first year. This grant increases with a fixed amount each year.
    A What is the annual increase if, over 20 years, he would have received a total of R143,500?
    B His initial annual expenditure is R2,600 and increases at a rate of R400 per year. After how many years does his expenses exceed his income?

16. The Cape Town High School wants to build a school hall and is busy with fundraising. Mr. Manuel, an ex-learner of the school and a successful politician, offers to donate money to the school. Having enjoyed mathematics at school, he decides to donate an amount of money on the following basis. He sets a mathematical quiz with 20 questions. For the correct answer to the first question (any learner may answer), the school will receive 1 cent, for a correct answer to the second question, the school will receive 2 cents, and so on. The donations 1, 2, 4, ... form a geometric sequence. Calculate (Give your answer to the nearest Rand)
    A The amount of money that the school will receive for the correct answer to the 20$^{\text{th}}$ question.
    B The total amount of money that the school will receive if all 20 questions are answered correctly.

17. The first term of a geometric sequence is 9, and the ratio of the sum of the first eight terms to the sum of the first four terms is $97 : 81$. Find the first three terms of the sequence, if it is given that all the terms are positive.

18. $(k - 4)$; $(k + 1)$; m; $5k$ is a set of numbers, the first three of which form an arithmetic sequence, and the last three a geometric sequence. Find k and m if both are positive.

19. Given: The sequence $6+p;\ 10+p;\ 15+p$ is geometric.

    A Determine p.

    B Show that the common ratio is $\frac{5}{4}$.

    C Determine the 10^{th} term of this sequence correct to one decimal place.

20. The second and fourth terms of a convergent geometric series are 36 and 16, respectively. Find the sum to infinity of this series, if all its terms are positive.

21. Evaluate: $\sum_{k=2}^{5} \frac{k(k+1)}{2}$

22. $S_n = 4n^2 + 1$ represents the sum of the first n terms of a particular series. Find the second term.

23. Find p if: $\sum_{k=1}^{\infty} 27p^k = \sum_{t=1}^{12} (24-3t)$

24. Find the integer that is the closest approximation to:

$$\frac{10^{2001} + 10^{2003}}{10^{2002} + 10^{2002}}$$

25. Find the pattern and hence calculate:

$$1-2+3-4+5-6\ldots+677-678+\ldots-1000$$

26. Determine $\sum_{p=1}^{\infty}(x+2)^p$, if it exists, when

    A $x = -\frac{5}{2}$

    B $x = -5$

27. Calculate: $\sum_{i=1}^{\infty} 5 \cdot 4^{-i}$

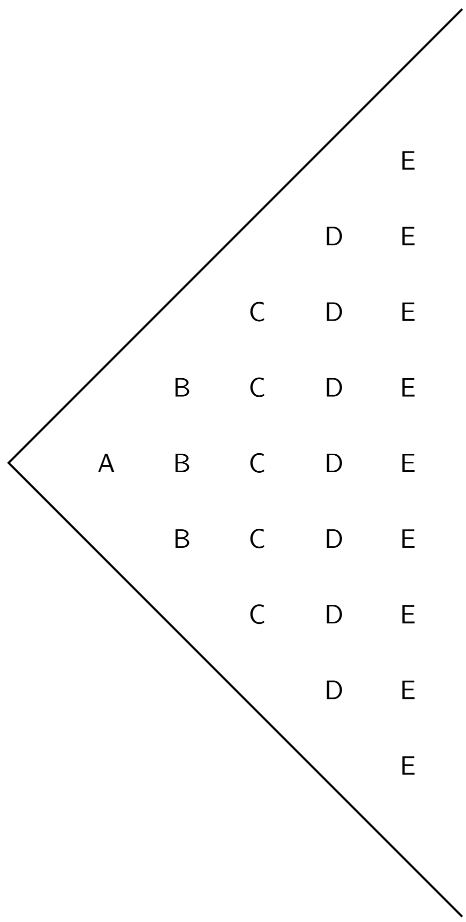
28. The sum of the first p terms of a sequence is $p\,(p+1)$. Find the 10^{th} term.

29. he powers of 2 are removed from the set of positive integers

$$1;\ 2;\ 3;\ 4;\ 5;\ 6;\ldots;1998;\ 1999;\ 2000$$

    Find the sum of remaining integers.

30. Observe the pattern below:



A If the pattern continues, find the number of letters in the column containing M's.

B If the total number of letters in the pattern is 361, which letter will the last column consist of.

31. The following question was asked in a test:

    Find the value of $2^{2005} + 2^{2005}$.

    Here are some of the students' answers:

    A Megansaid the answer is 4^{2005}.

    B Stefan wrote down 2^{4010}.

    C Nina thinks it is 2^{2006}.

    D Annatte gave the answer $2^{2005\times2005}$.

    Who is correct? ("None of them" is also a possibility.)

32. Find the pattern and hence calculate:

$$1 - 2 + 3 - 4 + 5 - 6 \ldots + 677 - 678 + \ldots - 1000$$

33. Determine $\sum_{p=1}^{\infty}(x+2)^p$, if it exists, when

    A $x = -\frac{5}{2}$

    B $x = -5$

34. Calculate: $\sum_{i=1}^{\infty} 5 \cdot 4^{-i}$

35. The sum of the first p terms of a sequence is $p\,(p+1)$. Find the 10^{th} term.

36. The powers of 2 are removed from the set of positive integers

$$1;\ 2;\ 3;\ 4;\ 5;\ 6;\ldots;1998;\ 1999;\ 2000$$

Find the sum of remaining integers.

37. A shrub of height 110 cm is planted. At the end of the first year, the shrub is 120 cm tall. Thereafter, the growth of the shrub each year is half of its growth in the previous year. Show that the height of the shrub will never exceed 130 cm.

Chapter 37

Finance - Grade 12

37.1 Introduction

In earlier grades simple interest and compound interest were studied, together with the concept of depreciation. Nominal and effective interest rates were also described. Since this chapter expands on earlier work, it would be best if you revised the work done in Chapters 8 and 21.

If you master the techniques in this chapter, when you start working and earning you will be able to apply the techniques in this chapter to critically assess how to invest your money. And when you are looking at applying for a bond from a bank to buy a home, you will confidently be able to get out the calculator and work out with amazement how much you could actually save by making additional repayments. Indeed, this chapter will provide you with the fundamental concepts you will need to confidently manage your finances and with some successful investing, sit back on your yacht and enjoy the millionaire lifestyle.

37.2 Finding the Length of the Investment or Loan

In Grade 11, we used the formula $A = P(1+i)^n$ to determine the term of the investment or loan, by trial and error. In other words, if we know what the starting sum of money is and what it grows to, and if we know what interest rate applies - then we can work out how long the money needs to be invested for all those other numbers to tie up.

Now, that you have learnt about logarithms, you are ready to work out the proper algebraic solution. If you need to remind yourself how logarithms work, go to Chapter 35 (on page 445).

The basic finance equation is:

$$A = P \cdot (1+i)^n$$

If you don't know what A, P, i and n represent, then you should definitely revise the work from Chapters 8 and 21.

Solving for n:

$$\begin{aligned} A &= P(1+i)^n \\ (1+i)^n &= (A/P) \\ \log((1+i)^n) &= \log(A/P) \\ n\log(1+i) &= \log(A/P) \\ n &= \log(A/P)/\log(1+i) \end{aligned}$$

Remember, you do not have to memorise this formula. It is very easy to derive any time you need it. It is simply a matter of writing down what you have, deciding what you need, and solving for that variable.

Worked Example 162: Term of Investment - Logarithms

Question: If we invested R3 500 into a savings account which pays 7,5% compound interest for an unknown period of time, at the end of which our account is worth R4 044,69. How long did we invest the money? How does this compare with the trial and error answer from Chapters 21.

Answer

Step 1 : Determine what is given and what is required

- P=R3 500
- i=7,5%
- A=R4 044,69

We are required to find n.

Step 2 : Determine how to approach the problem

We know that:

$$\begin{aligned} A &= P(1+i)^n \\ (1+i)^n &= (A/P) \\ \log((1+i)^n) &= \log(A/P) \\ n\log(1+i) &= \log(A/P) \\ n &= \log(A/P)/\log(1+i) \end{aligned}$$

Step 3 : Solve the problem

$$\begin{aligned} n &= \log(A/P)/\log(1+i) \\ &= \frac{\log(\frac{\text{R4 044,69}}{\text{R3 500}})}{\log(1+7{,}5\%)} \\ &= 2.0 \end{aligned}$$

Step 4 : Write final answer

The R3 500 was invested for 2 years.

37.3 A Series of Payments

By this stage, you know how to do calculations such as "If I want R1 000 in 3 years' time, how much do I need to invest now at 10% ?"

But what if we extend this as follows: If I want R1 000 next year and R1 000 the year after that and R1 000 after three years ... how much do I need to put into a bank account earning 10% p.a. right now to be able to afford that?"

The obvious way of working that out is to work out how much you need now to afford the payments individually and sum them. We'll work out how much is needed now to afford the payment of R1 000 in a year ($= R1\ 000 \times (1{,}10)^{-1} = R909{,}0909$), the amount needed now for the following year's R1 000 ($= R1\ 000 \times (1{,}10)^{-2} = R826{,}4463$) and the amount needed now for the R1 000 after 3 years ($= R1\ 000 \times (1{,}10)^{-3} = R751{,}3148$). Add these together gives you the amount needed to afford all three payments and you get R2486,85.

So, if you put R2486,85 into a 10% bank account now, you will be able to draw out R1 000 in a year, R1 000 a year after that, and R1 000 a year after that - and your bank account will come down to R0. You would have had exactly the right amount of money to do that (obviously!).

You can check this as follows:

Amount at Time 0 (i.e. Now)		= R2486,85
Amount at Time 1 (i.e. a year later)	= 2486,85(1+10%)	= R2735,54
Amount after the R1 000	= 2735,54 - 1 000	= R1735,54
Amount at Time 2 (i.e. a year later)	= 1735,54(1+10%)	= R1909,09
Amount after the R1 000	= R1909,09 - 1 000	= R909,09
Amount at Time 3 (i.e. a year later)	= 909,09(1+10%)	= R1 000
Amount after the R1 000	= 1 000 - 1 000	= R0

Perfect! Of course, for only three years, that was not too bad. But what if I asked you how much you needed to put into a bank account now, to be able to afford R100 a month for the next 15 years. If you used the above approach you would still get the right answer, but it would take you weeks!

There is - I'm sure you guessed - an easier way! This section will focus on describing how to work with:

- **annuities** - a fixed sum payable each year or each month either to provide a pre-determined sum at the end of a number of years or months (referred to as a future value annuity) or a fixed amount paid each year or each month to repay (amortise) a loan (referred to as a present value annuity).
- **bond repayments** - a fixed sum payable at regular intervals to pay off a loan. This is an example of a present value annuity.
- **sinking funds** - an accounting term for cash set aside for a particular purpose and invested so that the correct amount of money will be available when it is needed. This is an example of a future value annuity

37.3.1 Sequences and Series

Before we progress, you need to go back and read Chapter 36 (from page 457) to revise sequences and series.

In summary, if you have a series of n terms in total which looks like this:

$$a + ar + ar^2 + ... + ar^{n-1} = a[1 + r + r^2 + ...r^{n-1}]$$

this can be simplified as:

$$\frac{a(r^n - 1)}{r - 1} \quad \text{useful when } r > 1$$
$$\frac{a(1 - r^n)}{1 - r} \quad \text{useful when } 0 \leq r < 1$$

37.3.2 Present Values of a series of Payments

So having reviewed the mathematics of Sequences and Series, you might be wondering how this is meant to have any practical purpose! Given that we are in the finance section, you would be right to guess that there must be some financial use to all this Here is an example which happens in many people's lives - so you know you are learning something practical

Let us say you would like to buy a property for R300 000, so you go to the bank to apply for a mortgage bond. The bank wants it to be repaid by annually payments for the next 20 years, starting at end of this year. They will charge you 15% per annum. At the end of the 20 years the bank would have received back the total amount you borrowed together with all the interest they have earned from lending you the money. You would obviously want to work out what the annual repayment is going to be!

Let X be the annual repayment, i is the interest rate, and M is the amount of the mortgage bond you will be taking out.

Time lines are particularly useful tools for visualizing the series of payments for calculations, and we can represent these payments on a time line as:

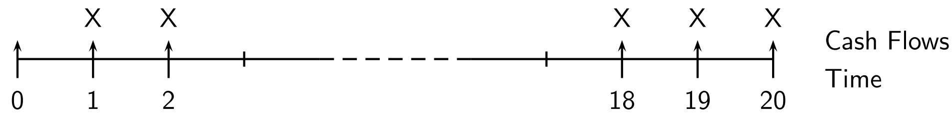


Figure 37.1: Time Line for an annuity (in arrears) of X for n periods.

The present value of all the payments (which includes interest) must equate to the (present) value of the mortgage loan amount.

Mathematically, you can write this as:

$$M = X(1+i)^{-1} + X(1+i)^{-2} + X(1+i)^{-3} + ... + X(1+i)^{-20}$$

The painful way of solving this problem would be to do the calculation for each of the terms above - which is 20 different calculations. Not only would you probably get bored along the way, but you are also likely to make a mistake.

Naturally, there is a simpler way of doing this! You can rewrite the above equation as follows:

$$M = X(v^1 + v^2 + v^3 + ... + v^{20})$$
$$\text{where } v = (1+i)^{-1} = 1/(1+i)$$

Of course, you do not have to use the method of substitution to solve this. We just find this a useful method because you can get rid of the negative exponents - which can be quite confusing! As an exercise - to show you are a real financial whizz - try to solve this without substitution. It is actually quite easy.

Now, the item in square brackets is the sum of a geometric sequence, as discussion in section 36. This can be re-written as follows, using what we know from Chapter 36 of this text book:

$$\begin{aligned} v^1 + v^2 + v^3 + ... + v^n &= v(1 + v + v^2 + ... + v^{n-1}) \\ &= v(\frac{1-v^n}{1-v}) \\ &= \frac{1-v^n}{1/v - 1} \\ &= \frac{1-(1+i)^{-n}}{i} \end{aligned}$$

Note that we took out a common factor of v before using the formula for the geometric sequence.

So we can write:

$$M = X[\frac{(1-(1+i)^{-n})}{i}]$$

This can be re-written:

$$X = \frac{M}{[\frac{(1-(1+i)^{-n})}{i}]}$$

So, this formula is useful if you know the amount of the mortgage bond you need and want to work out the repayment, or if you know how big a repayment you can afford and want to see what property you can buy.

For example, if I want to buy a house for R300 000 over 20 years, and the bank is going to charge me 15% per annum, then the annual repayment is:

$$\begin{aligned} X &= \frac{M}{[\frac{(1-(1+i)^{-n})}{i}]} \\ &= \frac{\text{R300 000}}{[\frac{(1-(1{,}15)^{-20})}{0{,}15}]} \\ &= \text{R47 928,44} \end{aligned}$$

This means, each year for the next 20 years, I need to pay the bank R47 928,44 per year before I have paid off the mortgage bond.

On the other hand, if I know I will only have R30 000 a year to repay my bond, then how big a house can I buy? That is easy ....

$$\begin{aligned} M &= X[\frac{(1-(1+i)^{-n})}{i}] \\ &= \text{R30 000}[\frac{(1-(1{,}15)^{-20})}{0{,}15}] \\ &= \text{R187 779,90} \end{aligned}$$

So, for R30 000 a year for 20 years, I can afford to buy a house of R187 800 (rounded to the nearest hundred).

The bad news is that R187 800 does not come close to the R300 000 you wanted to buy! The good news is that you do not have to memorise this formula. In fact , when you answer questions like this in an exam, you will be expected to start from the beginning - writing out the opening equation in full, showing that it is the sum of a geometric sequence, deriving the answer, and then coming up with the correct numerical answer.

Worked Example 163: Monthly mortgage repayments

Question: Sam is looking to buy his first flat, and has R15 000 in cash savings which he will use as a deposit. He has viewed a flat which is on the market for R250 000, and he would like to work out how much the monthly repayments would be. He will be taking out a 30 year mortgage with monthly repayments. The annual interest rate is 11%.

Answer

Step 1 : Determine what is given and what is needed

The following is given:

- Deposit amount = R15 000
- Price of flat = R250 000
- interest rate, $i = 11\%$

We are required to find the monthly repayment for a 30-year mortgage.

Step 2 : Determine how to approach the problem

We know that:

$$X = \frac{M}{[\frac{(1-(1+i)^{-n})}{i}]}$$

. In order to use this equation, we need to calculate M, the amount of the mortgage bond, which is the purchase price of property less the deposit which Sam pays up-front.

$$\begin{aligned} M &= R250\ 000 - R15\ 000 \\ &= R235\ 000 \end{aligned}$$

Now because we are considering monthly repayments, but we have been given an annual interest rate, we need to convert this to a monthly interest rate, $i12$. (If you are not clear on this, go back and revise section 21.8.)

$$\begin{aligned}(1+i12)^{12} &= (1+i)\\(1+i12)^{12} &= 1{,}11\\i12 &= 0{,}873459\%\end{aligned}$$

We know that the mortgage bond is for 30 years, which equates to 360 months.

Step 3 : Solve the problem

Now it is easy, we can just plug the numbers in the formula, but do not forget that you can always deduce the formula from first principles as well!

$$\begin{aligned}X &= \frac{M}{[\frac{(1-(1+i)^{-n})}{i}]}\\&= \frac{\text{R235 000}}{[\frac{(1-(1.00876459)^{-360})}{0{,}008734594}]}\\&= \text{R2 146,39}\end{aligned}$$

Step 4 : Write the final answer

That means that to buy a house for R300 000, after Sam pays a R15 000 deposit, he will make repayments to the bank each month for the next 30 years equal to R2 146,39.

Worked Example 164: Monthly mortgage repayments

Question: You are considering purchasing a flat for R200 000 and the bank's mortgage rate is currently 9% per annum payable monthly. You have savings of R10 000 which you intend to use for a deposit. How much would your monthly mortgage payment be if you were considering a mortgage over 20 years.

Answer

Step 1 : Determine what is given and what is required

The following is given:

- Deposit amount = R10 000
- Price of flat = R200 000
- interest rate, $i = 9\%$

We are required to find the monthly repayment for a 20-year mortgage.

Step 2 : Determine how to approach the problem

We are consider monthly mortgage repayments, so it makes sense to use months as our time period.

The interest rate was quoted as 9% per annum payable monthly, which means that the monthly effective rate = 9%/12 = 0,75% per month. Once we have converted 20 years into 240 months, we are ready to do the calculations!

First we need to calculate M, the amount of the mortgage bond, which is the purchase price of property less the deposit which Sam pays up-front.

$$\begin{aligned}M &= R200\ 000 - R10\ 000\\&= R190\ 000\end{aligned}$$

The present value of our mortgage payments, X, must equate to the mortgage amount that we borrow today, so

$$\begin{aligned} & X \times (1 + 0{,}75\%)^{-1} \quad + \\ & X \times (1 + 0{,}75\%)^{-2} \quad + \\ & X \times (1 + 0{,}75\%)^{-3} \quad + \\ & X \times (1 + 0{,}75\%)^{-4} \quad + \quad \ldots \\ X \times (1 + 0{,}75\%)^{-239} + & X \times (1 + 0{,}75\%)^{-240} \end{aligned}$$

But it is clearly much easier to use our formula that work out 240 factors and add them all up!

Step 3 : Solve the problem

$$\begin{aligned} X \times \frac{1 - (1 + 0{,}75\%)^{-240}}{0{,}75\%} &= \text{R190 000} \\ X \times 111{,}14495 &= \text{R190 000} \\ X &= \text{R1 709,48} \end{aligned}$$

Step 4 : Write the final answer

So to repay a R190 000 mortgage over 20 years, at 9% interest payable monthly, will cost you R1 709,48 per month for 240 months.

Show me the money

Now that you've done the calculations for the worked example and know what the monthly repayments are, you can work out some surprising figures. For example, R1 709,48 per month for 240 month makes for a total of R410 275,20 (=R1 709,48 × 240). That is more than double the amount that you borrowed! This seems like a lot. However, now that you've studied the effects of time (and interest) on money, you should know that this amount is somewhat meaningless. The value of money is dependant on its timing.

Nonetheless, you might not be particularly happy to sit back for 20 years making your R1 709,48 mortgage payment every month knowing that half the money you are paying are going toward interest. But there is a way to avoid those heavy interest charges. It can be done for less than R300 extra every month...

So our payment is now R2 000. The interest rate is still 9% per annum payable monthly (0,75% per month), and our principal amount borrowed is R190 000. Making this higher repayment amount every month, how long will it take to pay off the mortgage?

The present value of the stream of payments must be equal to R190 000 (the present value of the borrowed amount). So we need to solve for n in:

$$\begin{aligned} \text{R2 000} \times [1 - (1 + 0{,}75\%)^{-\text{n}}]/0{,}75\% &= \text{R190 000} \\ 1 - (1 + 0{,}75\%)^{-n} &= (\text{R190 000}/2\ 000) \times 0{,}75\% \\ \log(1 + 0{,}75\%)^{-n} &= \log[(1 - (\text{R190 000}/\text{R2 000}) \times 0{,}75\%] \\ -n \times \log(1 + 0{,}75\%) &= \log[(1 - (\text{R190 000}/\text{R2 000}) \times 0{,}75\%] \\ -n \times 0{,}007472 &= -1{,}2465 \\ n &= 166{,}8 \text{ months} \\ &= 13{,}9 \text{ years} \end{aligned}$$

So the mortgage will be completely repaid in less than 14 years, and you would have made a total payment of 166,8× R2 000 = R333 600.

Can you see what is happened? Making regular payments of R2 000 instead of the required R1,709,48, you will have saved R76 675,20 (= R410 275,20 - R333 600) in interest, and yet you have only paid an additional amount of R290,52 for 166,8 months, or R48 458,74. You surely know by now that the difference between the additional R48 458,74 that you have paid and the R76 675,20 interest that you have saved is attributable to, yes, you have got it, compound interest!

37.3.3 Future Value of a series of Payments

In the same way that when we have a single payment, we can calculate a present value or a future value - we can also do that when we have a series of payments.

In the above section, we had a few payments, and we wanted to know what they are worth now - so we calculated present values. But the other possible situation is that we want to look at the future value of a series of payments.

Maybe you want to save up for a car, which will cost R45 000 - and you would like to buy it in 2 years time. You have a savings account which pays interest of 12% per annum. You need to work out how much to put into your bank account now, and then again each month for 2 years, until you are ready to buy the car.

Can you see the difference between this example and the ones at the start of the chapter where we were only making a single payment into the bank account - whereas now we are making a series of payments into the same account? This is a sinking fund.

So, using our usual notation, let us write out the answer. Make sure you agree how we come up with this. Because we are making monthly payments, everything needs to be in months. So let A be the closing balance you need to buy a car, P is how much you need to pay into the bank account each month, and $i12$ is the monthly interest rate. (Careful - because 12% is the annual interest rate, so we will need to work out later what the month interest rate is!)

$$A = P(1+i12)^{24} + P(1+i12)^{23} + ... + P(1+i12)^{1}$$

Here are some important points to remember when deriving this formula:

1. We are calculating future values, so in this example we use $(1+i12)^n$ and not $(1+i12)^{-n}$. Check back to the start of the chapter is this is not obvious to you by now.
2. If you draw a timeline you will see that the time between the first payment and when you buy the car is 24 months, which is why we use 24 in the first exponent.
3. Again, looking at the timeline, you can see that the 24th payment is being made one month before you buy the car - which is why the last exponent is a 1.
4. Always check that you have got the right number of payments in the equation. Check right now that you agree that there are 24 terms in the formula above.

So, now that we have the right starting point, let us simplify this equation:

$$\begin{aligned} A &= P[(1+i12)^{24} + (1+i12)^{23} + \ldots + (1+i12)^{1}] \\ &= P[X^{24} + X^{23} + \ldots + X^{1}] \text{ using } X = (1+i12) \end{aligned}$$

Note that this time X has a positive exponent not a negative exponent, because we are doing future values. This is not a rule you have to memorise - you can see from the equation what the obvious choice of X should be.

Let us reorder the terms:

$$A = P[X^1 + X^2 + \ldots + X^{24}] = P \cdot X[1 + X + X^2 + \ldots + X^23]$$

This is just another sum of a geometric sequence, which as you know can be simplified as:

$$\begin{aligned} A &= P \cdot X[X^n - 1]/((1+i12) - 1) \\ &= P \cdot X[X^n - 1]/i12 \end{aligned}$$

So if we want to use our numbers, we know that $A =$ R45 000, n=24 (because we are looking at monthly payments, so there are 24 months involved) and $i = 12\%$ per annum.

BUT (and it is a big but) we need a monthly interest rate. Do not forget that the trick is to keep the time periods and the interest rates in the same units - so if we have monthly payments, make sure you use a monthly interest rate! Using the formula from Section 21.8, we know that $(1 + i) = (1 + i12)^{12}$. So we can show that $i12 = 0{,}0094888 = 0{,}94888\%$.

Therefore,

$$\begin{aligned} 45\ 000 &= P(1{,}0094888)[(1{,}0094888)^{24} - 1]/0{,}0094888 \\ P &= 1662{,}67 \end{aligned}$$

This means you need to invest R1 662,67 each month into that bank account to be able to pay for your car in 2 years time.

There is another way of looking at this too - in terms of present values. We know that we need an amount of R45 000 in 24 months time, and at a monthly interest rate of 0,94888%, the present value of this amount is R35 873,72449. Now the question is what monthly amount at 0,94888% interest over 24 month has a present value of R35 873,72449? We have seen this before - it is just like the mortgage questions! So let us go ahead and see if we get to the same answer

$$\begin{aligned} P &= M/[(1 - (1 + i)^{-n})/i] \\ &= \text{R35 873,72449}[(1 - (1{,}0094888)^{-24})/0{,}0094888] \\ &= \text{R1 662,67} \end{aligned}$$

37.3.4 Exercises - Present and Future Values

1. You have taken out a mortgage bond for R875 000 to buy a flat. The bond is for 30 years and the interest rate is 12% per annum payable monthly.
   A What is the monthly repayment on the bond?
   B How much interest will be paid in total over the 30 years?
2. How much money must be invested now to obtain regular annuity payments of R 5 500 per month for five years ? The money is invested at 11,1% p.a., compounded monthly. (Answer to the nearest hundred rand)

37.4 Investments and Loans

By now, you should be well equipped to perform calculations with compound interest. This section aims to allow you to use these valuable skills to critically analyse investment and load options that you will come across in your later life. This way, you will be able to make informed decisions on options presented to you.

At this stage, you should understand the mathematical theory behind compound interest. However, the numerical implications of compound interest is often subtle and far from obvious.

Recall the example in section **??**FIXTHIS. For an extra payment of R290,52 a month, we could have paid off our loan in less than 14 years instead of 20 years. This provides a good illustration of the long term effect of compound interest that is often surprising. In the following section, we'll aim to explain the reason for drastic deduction in times it takes to repay the loan.

37.4.1 Loan Schedules

So far, we have been working out loan repayment amounts by taking all the payments and discounting them back to the present time. We are not considering the repayments individually.

Think about the time you make a repayment to the bank. There are numerous questions that could be raised: how much do you still owe them? Since you are paying off the loan, surely you must owe them less money, but how much less? We know that we'll be paying interest on the money we still owe the bank. When exactly do we pay interest? How much interest are we paying?

The answer to these questions lie in something called the load schedule.

We will continue to use the example from section **??**FIXTHIS. There is a loan amount of R190 000. We are paying it off over 20 years at an interest of 9% per annum payable monthly. We worked out that the repayments should be R1 709,48.

Consider the first payment of R1 709,48 one month into the loan. First, we can work out how much interest we owe the bank at this moment. We borrowed R190 000 a month ago, so we should owe:

$$\begin{aligned} I &= M \times i12 \\ &= R190\ 000 \times 0{,}75\% \\ &= R1\ 425 \end{aligned}$$

We are paying them R1 425 in interest. We calls this the interest component of the repayment. We are only paying off R1 709,48 - R1 425 = R284.48 of what we owe! This is called the capital component. That means we still owe R190 000 - R284,48 = R189 715,52. This is called the capital outstanding. Let's see what happens at end of the second month. The amount of interest we need to pay is the interest on the capital outstanding.

$$\begin{aligned} I &= M \times i12 \\ &= R189\ 715{,}52 \times 0{,}75\% \\ &= R1\ 422{,}87 \end{aligned}$$

Since we don't owe the bank as much as we did last time, we also owe a little less interest. The capital component of the repayment is now R1 709,48 - R1 422,87 = R286,61. The capital outstanding will be R189 715,52 - R286,61 = R189 428,91. This way, we can break each of our repayments down into an interest part and the part that goes towards paying off the loan.

This is a simple and repetitive process. Table 37.1 is a table showing the breakdown of the first 12 payments. This is called a loan schedule.

Now, let's see the same thing again, but with R2 000 being repaid each year. We expect the numbers to change. However, how much will they change by? As before, we owe R1 425 in interest in interest. After one month. However, we are paying R2 000 this time. That leaves R575 that goes towards paying off the capital outstanding, reducing it to R189 425. By the end of the second month, the interest owed is R1 420,69 (That's R189 425×$i12$). Our R2 000 pays for that interest, and reduces the capital amount owed by R2 000 - R1 420,69 = R579,31. This reduces the amount outstanding to R188 845,69.

Doing the same calculations as before yields a new loan schedule shown in Table 37.2.

The important numbers to notice is the "Capital Component" column. Note that when we are paying off R2 000 a month as compared to R1 709,48 a month, this column more than doubles? In the beginning of paying off a loan, very little of our money is used to pay off the captital outstanding. Therefore, even a small incread in repayment amounts can significantly increase the speed at which we are paying off the capital.

Whatsmore, look at the amount we are still owing after one year (i.e. at time 12). When we were paying R1 709,48 a month, we still owe R186 441,84. However, if we increase the repayments to R2 000 a month, the amount outstanding decreases by over R3 000 to R182 808,14. This means we would have paid off over R7 000 in our first year instead of less than R4 000. This

Time	Repayment	Interest Component	Capital Component	Capital Outstanding
0				R 190 000,00
1	R 1 709,48	R 1 425,00	R 284,48	R 189 715,52
2	R 1 709,48	R 1 422,87	R 286,61	R 189 428,91
3	R 1 709,48	R 1 420,72	R 288,76	R 189 140,14
4	R 1 709,48	R 1 418,55	R 290,93	R 188 849,21
5	R 1 709,48	R 1 416,37	R 293,11	R 188 556,10
6	R 1 709,48	R 1 414,17	R 295,31	R 188 260,79
7	R 1 709,48	R 1 411,96	R 297,52	R 187 963,27
8	R 1 709,48	R 1 409,72	R 299,76	R 187 663,51
9	R 1 709,48	R 1 407,48	R 302,00	R 187 361,51
10	R 1 709,48	R 1 405,21	R 304,27	R 187 057,24
11	R 1 709,48	R 1 402,93	R 306,55	R 186 750,69
12	R 1 709,48	R 1 400,63	R 308,85	R 186 441,84

Table 37.1: A loan schedule with repayments of R1 709,48 per month.

Time	Repayment	Interest Component	Capital Component	Capital Outstanding
0				R 190 000,00
1	R 2 000,00	R 1 425,00	R 575,00	R 189 425,00
2	R 2 000,00	R 1 420,69	R 579,31	R 188 845,69
3	R 2 000,00	R 1 416,34	R 583,66	R 188 262,03
4	R 2 000,00	R 1 411,97	R 588,03	R 187 674,00
5	R 2 000,00	R 1 407,55	R 592,45	R 187 081,55
6	R 2 000,00	R 1 403,11	R 596,89	R 186 484,66
7	R 2 000,00	R 1 398,63	R 601,37	R 185 883,30
8	R 2 000,00	R 1 394,12	R 605,88	R 185 277,42
9	R 2 000,00	R 1 389,58	R 610,42	R 184 667,00
10	R 2 000,00	R 1 385,00	R 615,00	R 184 052,00
11	R 2 000,00	R 1 380,39	R 619,61	R 183 432,39
12	R 2 000,00	R 1 375,74	R 624,26	R 182 808,14

Table 37.2: A loan schedule with repayments of R2 000 per month.

increased speed at which we are paying off the capital portion of the loan is what allows us to pay off the whole load in around 14 years instead of the original 20. Note however, the effect of paying R2 000 instead of R1 709,48 is more significant in be beginning of the loan than near the end of the loan.

It is noted that in this instance, by paying slightly more than what the bank would ask you to pay, you can pay off a loan a lot quicker. The natural question to ask here is: why are banks asking us to pay the lower amount for much longer then? Are they trying to cheat us out of our money?

There is no simple answer to this. Banks provide a service to us in return for a fee, so they are out to make a profit. However, they need to be careful not to cheat their customers for fear that they'll simply use another bank. The central issue here is one of scale. For us, the changes involved appear big. We are paying off our loan 6 years earlier by paying just a bit more a month. To a bank, however, it doesn't matter much either way. In all likelihoxod, it doesn't affect their profit margins one bit!

Remember that a bank calculates repayment amount using the same methods as we've been learning. Therefore, they are correct amounts for given interest rates and terms. As a result, which amount is repaid does generally make a bank more or less money. It's a simple matter of less money now or more money later. Banks generally use a 20 year repayment period by default.

Learning about financial mathematics enables you to duplicate these calculations for yourself. This way, you can decide what's best for you. You can decide how much you want to repay each month and you'll know of its effects. A bank wouldn't care much either way, so you should pick something that suits you.

Worked Example 165: Monthly Payments

Question: Stefan and Marna want to buy a house that costs R 1 200 000. Their parents offer to put down a 20% payment towards the cost of the house. They need to get a moratage for the balance. What are their monthly repayments if the term of the home loan is 30 years and the interest is 7,5%, compounded monthly ?

Answer

Step 1 : Determine how much money they need to borrow

$R1\ 200\ 00 - R240\ 000 = R960\ 000$

Step 2 : Determine how to approach the problem

Use the formula:

$$P = \frac{x[1-(1+i)^{-n}]}{i}$$

Where

$P = 960\ 000$

$n = 30 \times 12 = 360 months$

$i = 0{,}075 \div 12 = 0{,}00625$

Step 3 : Solve the problem

$$\begin{aligned} R960\ 000 &= \frac{x[1-(1+0{,}00625)^{-360}]}{0{,}00625} \\ &= x(143{,}017\ 627\ 3) \\ x &= R6\ 712{,}46 \end{aligned}$$

Step 4 : Write the final answer

The monthly repayments $= R6\ 712{,}46$

37.4.2 Exercises - Investments and Loans

1. A property costs R1 800 000. Calculate the monthly repayments if the interest rate is 14% p.a. compounded monthly and the loan must be paid of in 20 years time.
2. A loan of R 4 200 is to be returned in two equal annual instalments. If the rate of interest os 10% per annum, compounded annually, calculate the amount of each instalment.

37.4.3 Calculating Capital Outstanding

As defined in Section 37.4.1, Capital outstanding is the amount we still owe the people we borrowed money from at a given moment in time. We also saw how we can calculate this using loan schedules. However, there is a significant disadvantage to this method: it is very time consuming. For example, in order to calculate how much capital is still outstanding at time 12 using the loan schedule, we'll have to first calculate how much capital is outstanding at time 1 through to 11 as well. This is already quite a bit more work than we'd like to do. Can you imagine calculating the amount outstanding after 10 years (time 120)?

Fortunately, there is an easier method. However, it is not immediately why this works, so let's take some time to examine the concept.

Prospective method for Capital Outstanding

Let's say that after a certain number of years, just after we made a repayment, we still owe amount Y. What do we know about Y? We know that using the loan schedule, we can calculate what it equals to, but that is a lot of repetitive work. We also know that Y is the amount that we are still going to pay off. In other words, all the repayments we are still going to make in the future will exactly pay off Y. This is true because in the end, after all the repayments, we won't be owing anything.

Therefore, the present value of all outstanding future payments equal the present amount outstanding. This is the prospective method for calculating capital outstanding.

Let's return to a previous example. Recall the case where we were trying to repay a loan of R200 000 over 20 years. At an interested rate of 9% compounded monthly, the monthly repayment is R1 709,48. In table 37.1, we can see that after 12 month, the amount outstanding is R186 441,84. Let's try to work this out using the the prospective method.

After time 12, there is still $19 \times 12 = 228$ repayments left of R1 709,48 each. The present value is:

$$\begin{aligned} n &= 228 \\ i &= 0{,}75\% \\ Y &= R1\ 709{,}48 \times \frac{1 - 1{,}0075^{-228}}{0{,}0075} \\ &= R186\ 441{,}92 \end{aligned}$$

Oops! This seems to be almost right, but not quite. We should have got R186 441,84. We are 8 cents out. However, this is in fact not a mistake. Remember that when we worked out the monthly repayments, we rounded to the nearest cents and arrived at R1 709,48. This was because one cannot make a payment for a fraction of a cent. Therefore, the rounding off error was carried through. That's why the two figures don't match exactly. In financial mathematics, this is largely unavoidable.

37.5 Formulae Sheet

As an easy reference, here are the key formulae that we derived and used during this chapter. While memorising them is nice (there are not many), it is the application that is useful. Financial experts are not paid a salary in order to recite formulae, they are paid a salary to use the right methods to solve financial problems.

37.5.1 Definitions

P	Principal (the amount of money at the starting point of the calculation)
i	interest rate, normally the effective rate per annum
n	period for which the investment is made
iT	the interest rate paid T times per annum, i.e. $iT = \frac{\text{Nominal Interest Rate}}{T}$

37.5.2 Equations

$$\left.\begin{array}{l}\text{Present Value - simple}\\ \text{Future Value - simple}\\ \text{Solve for } i\\ \text{Solve for } n\end{array}\right\} = P(1 + i \cdot n)$$

$$\left.\begin{array}{l}\text{Present Value - compound}\\ \text{Future Value - compound}\\ \text{Solve for } i\\ \text{Solve for } n\end{array}\right\} = P(1 + i)^n$$

Important: Always keep the interest and the time period in the same units of time (e.g. both in years, or both in months etc.).

37.6 End of Chapter Exercises

1. Thabo is about to invest his R8 500 bonus in a special banking product which will pay 1% per annum for 1 month, then 2% per annum for the next 2 months, then 3% per annum for the next 3 months, 4% per annum for the next 4 months, and 0% for the rest of the year. The are going to charge him R100 to set up the account. How much can he expect to get back at the end of the period?

2. A special bank account pays simple interest of 8% per annum. Calculate the opening balance required to generate a closing balance of R5 000 after 2 years.

3. A different bank account pays compound interest of 8% per annum. Calculate the opening balance required to generate a closing balance of R5 000 after 2 years.

4. Which of the two answers above is lower, and why?

5. After 7 months after an initial deposit, the value of a bank account which pays compound interest of 7,5% per annum is R3 650,81. What was the value of the initial deposit?

6. Suppose you invest R500 this year compounded at interest rate i for a year in Bank T. In the following year you invest the accumulation that you received for another year at the same interest rate and on the third year, you invested the accumulation you received at the same interest rate too. If P represents the present value (R500), find a pattern for this investment. [Hint: find a formula]

7. Thabani and Lungelo are both using UKZN Bank for their saving. Suppose Lungelo makes a deposit of X today at interest rate of i for six years. Thabani makes a deposit of $3X$ at an interest rate of 0.05. Thabani made his deposit 3 years after Lungelo made his first deposit. If after 6 years, their investments are equal, calculate the value of i and find X. if the sum of their investment is R20 000, use X you got to find out how much Thabani got in 6 years.

8. Sipho invests R500 at an interest rate of $\log(1{,}12)$ for 5 years. Themba, Sipho's sister invested R200 at interest rate i for 10 years on the same date that her brother made his first deposit. If after 5 years, Themba's accumulation equals Sipho's, find the interest rate i and find out whether Themba will be able to buy her favorite cell phone after 10 years which costs R2 000.

9. Moira deposits R20 000 in her saving account for 2 years at an interest rate of 0.05. After 2 years, she invested her accumulation for another 2 years, at the same interest rate. After 4 years, she invested her accumulation for which she got for another 2 years at an interest rate of 5 %. After 6 years she choose to buy a car which costs R26 000. Her husband, Robert invested the same amount at interest rate of 5 % for 6 years.

   A Without using any numbers, find a pattern for Moira's investment?

   B How Moira's investment differ from Robert's?

10. Calculate the real cost of a loan of R10 000 for 5 years at 5% capitalised monthly and half yearly.

11. Determine how long, in years, it will take for the value of a motor vehicle to decrease to 25% of its original value if the rate of depreciation, based on the reducing-balance method, is 21% per annum.

12. André and Thoko, decided to invest their winnings (amounting to R10 000) from their science project. They decided to divide their winnings according to the following: Because Andr was the head of the project and he spent more time on it, André got 65,2 % of the winnings and Thoko got 34,8%. So, Thoko decided to invest only 0,5 % of the share of her sum and Andrédecided to invest 1,5 % of the share of his sum. When they calculated how much each contributed in the investment, Thoko had 25 % and André had 75 % share. They planned to invest their money for 20 years , but, as a result of Thoko finding a job in Australia 7 years after their initial investment. They both decided to take whatever value was there and split it according to their initial investment(in terms of percentages). Find how much each will get after 7 years, if the interest rate is equal to the percentage that Thoko invested (NOT the money but the percentage).

Chapter 38

Factorising Cubic Polynomials - Grade 12

38.1 Introduction

In grades 10 and 11, you learnt how to solve different types of equations. Most of the solutions, relied on being able to factorise some expression and the factorisation of quadratics was studied in detail. This chapter focusses on the factorisation of cubic polynomials, that is expressions with the highest power equal to 3.

38.2 The Factor Theorem

The *factor theorem* describes the relationship between the root of a polynomial and a factor of the polynomial.

Definition: Factor Theorem
For any polynomial, $f(x)$, for all values of a which satisfy $f(a) = 0$, $(x - a)$ is a factor of $f(x)$. Or, more concisely:

$$\frac{f(x)}{x - a} = q(x)$$

is a polynomial.
In other words: If the remainder when dividing $f(x)$ by $(x - a)$ is zero, then $(ax + b)$ is a factor of $f(x)$.
So if $f(-\frac{b}{a}) = 0$, then $(ax + b)$ is a factor of $f(x)$.

Worked Example 166: Factor Theorem

Question: Use the Factor Theorem to determine whether $y - 1$ is a factor of $f(y) = 2y^4 + 3y^2 - 5y + 7$.
Answer
Step 1 : Determine how to approach the problem
In order for $y - 1$ to be a factor, $f(1)$ must be 0.
Step 2 : Calculate $f(1)$

$$\begin{aligned} f(y) &= 2y^4 + 3y^2 - 5y + 7 \\ \therefore f(1) &= 2(1)^4 + 3(1)^2 - 5(1) + 7 \\ &= 2 + 3 - 5 + 7 \\ &= 7 \end{aligned}$$

Step 3 : Conclusion
Since $f(1) \neq 0$, $y-1$ is not a factor of $f(y) = 2y^4 + 3y^2 - 5y + 7$.

Worked Example 167: Factor Theorem

Question: Using the Factor Theorem, verify that $y+4$ is a factor of $g(y) = 5y^4 + 16y^3 - 15y^2 + 8y + 16$.
Answer
Step 1 : Determine how to approach the problem
In order for $y+4$ to be a factor, $g(-4)$ must be 0.
Step 2 : Calculate $f(1)$

$$\begin{aligned} g(y) &= 5y^4 + 16y^3 - 15y^2 + 8y + 16 \\ \therefore g(-4) &= 5(-4)^4 + 16(-4)^3 - 15(-4)^2 + 8(-4) + 16 \\ &= 5(256) + 16(-64) - 15(16) + 8(-4) + 16 \\ &= 1280 - 1024 - 240 - 32 + 16 \\ &= 0 \end{aligned}$$

Step 3 : Conclusion
Since $g(-4) = 0$, $y+4$ is a factor of $g(y) = 5y^4 + 16y^3 - 15y^2 + 8y + 16$.

38.3 Factorisation of Cubic Polynomials

Cubic expressions have a highest power of 3 on the unknown variable. This means that there should be at least 3 factors. We have seen in Grade 10 that the sum and difference of cubes is factorised as follows.:

$$(x+y)(x^2 - xy + y^2) = x^3 + y^3$$

and

$$(x-y)(x^2 + xy + y^2) = x^3 - y^3$$

We also saw that the quadratic terms do not have rational roots.

There are many methods of factorising a cubic polynomial. The general method is similar to that used to factorise quadratic equations. If you have a cubic polynomial of the form:

$$f(x) = ax^3 + bx^2 + cx + d$$

then you should expect factors of the form:

$$(Ax+B)(Cx+D)(Ex+F). \qquad (38.1)$$

We will deal with simplest case first. When $a = 1$, then $A = C = E = 1$, and you only have to determine B, D and F. For example, find the factors of:

$$x^3 - 2x^2 - 5^x + 6.$$

In this case we have

$$\begin{aligned} a &= 1 \\ b &= -2 \\ c &= -5 \\ d &= 6 \end{aligned}$$

The factors will have the general form shown in (38.1), with $A = C = E = 1$. We can then use values for a, b, c and d to determine values for B, D and F. We can re-write (38.1) with $A = C = E = 1$ as:

$$(x + B)(x + D)(x + F).$$

If we multiply this out we get:

$$\begin{aligned}(x + B)(x + D)(x + F) &= (x + B)(x^2 + Dx + Fx + DF)\\ &= x^3 + Dx^2 + Fx^2 + Bx^2 + DFx + BDx + BFx + BDF\\ &= x^3 + (D + F + B)x^2 + (DF + BD + BF)x + BDF\end{aligned}$$

We can therefore write:

$$b = -2 = D + F + B \qquad (38.2)$$

$$c = -5 = DF + BD + BF \qquad (38.3)$$

$$d = 6 = BDF. \qquad (38.4)$$

This is a set of three equations in three unknowns. However, we know that B, D and F are factors of 6 because $BDF = 6$. Therefore we can use a trial and error method to find B, D and F.

This can become a very tedious method, therefore the **Factor Theorem** can be used to find the factors of cubic polynomials.

Worked Example 168: Factorisation of Cubic Polynomials

Question: Factorise $f(x) = x^3 + x^2 - 9x - 9$ into three linear factors.

Answer

Step 1 : By trial and error using the factor theorem to find a factor

Try

$$f(1) = (1)^3 + (1)^2 - 9(1) - 9 = 1 + 1 - 9 - 9 = -16$$

Therefore $(x - 1)$ is not a factor

Try

$$f(-1) = (-1)^3 + (-1)^2 9(-1) 9 = 1 + 1 + 99 = 0$$

Thus $(x + 1)$ is a factor, because $f(-1) = 0$.

Now divide $f(x)$ by $(x + 1)$ using division by inspection:
Write $x^3 + x^2 - 9x - 9 = (x + 1)(\qquad)$
The first term in the second bracket must be x^2 to give x^3 if one works backwards.
The last term in the second bracket must be -9 because $+1 \times -9 = -9$.
So we have $x^3 + x^2 - 9x - 9 = (x + 1)(x^2 ?x - 9)$.
Now, we must find the coefficient of the middelterm (x).
$(+1)(x^2)$ gives x^2. So, the coefficient of the x-term must be 0.
So $f(x) = (x + 1)(x^2 - 9)$.

Step 2 : Factorise fully

$x^2 - 9$ can be further factorised to $(x - 3)(x + 3)$,
and we are now left with $f(x) = (x + 1)(x - 3)(x + 3)$

In general, to factorise a cubic polynomial, you find one factor by trial and error. Use the factor theorem to confirm that the guess is a root. Then divide the cubic polynomial by the factor to obtain a quadratic. Once you have the quadratic, you can apply the standard methods to factorise the quadratic.

For example the factors of $x^3 - 2x^2 - 5x + 6$ can be found as follows: There are three factors which we can write as

$$(x - a)(x - b)(x - c).$$

Worked Example 169: Factorisation of Cubic Polynomials

Question: Use the Factor Theorem to factorise

$$x^3 - 2x^2 - 5^x + 6.$$

Answer

Step 1 : Find one factor using the Factor Theorem

Try

$$f(1) = (1)^3 - 2(1)^2 - 5(1) + 6 = 1 - 2 - 5 + 6 = 0$$

Therefore $(x - 1)$ is a factor.

Step 2 : Division by expection

$x^3 - 2x^2 - 5x + 6 = (x - 1)(\qquad)$

The first term in the second bracket must be x^2 to give x^3 if one works backwards.

The last term in the second bracket must be -6 because $-1 \times -6 = +6$.

So we have $x^3 - 2x^2 - 5x + 6 = (x - 1)(x^2 ? x - 6)$.

Now, we must find the coefficient of the middelterm (x).

$(-1)(x^2)$ gives $-x^2$. So, the coefficient of the x-term must be -1.

So $f(x) = (x - 1)(x^2 - x - 6)$.

Step 3 : Factorise fully

$x^2 - x - 6$ can be further factorised to $(x - 3)(x + 2)$,

and we are now left with $x^3 - 2x^2 - 5^x + 6 = (x - 1)(x - 3)(x + 2)$

38.4 Exercises - Using Factor Theorem

1. Find the remainder when $4x^3 - 4x^2 + x - 5$ is divided by $(x + 1)$.
2. Use the factor theorem to factorise $x^3 - 3x^2 + 4$ completely.
3. $f(x) = 2x^3 + x^2 - 5x + 2$

   A Find $f(1)$.

   B Factorise $f(x)$ completely

4. Use the Factor Theorem to determine all the factors of the following expression:

$$x^3 + x^2 - 17x + 15$$

5. Complete: If $f(x)$ is a polynomial and p is a number such that $f(p) = 0$, then $(x - p)$ is .....

38.5 Solving Cubic Equations

Once you know how to factorise cubic polynomials, it is also easy to solve cubic equations of the kind

$$ax^3 + bx^2 + cx + d = 0$$

Worked Example 170: Solution of Cubic Equations

Question: Solve

$$6x^3 - 5x^2 - 17x + 6 = 0.$$

Answer

Step 1 : Find one factor using the Factor Theorem

Try

$$f(1) = 6(1)^3 - 5(1)^2 - 17(1) + 6 = 6 - 5 - 17 + 6 = -10$$

Therefore $(x - 1)$ is NOT a factor.
Try

$$f(2) = 6(2)^3 - 5(2)^2 - 17(2) + 6 = 48 - 20 - 34 + 6 = 0$$

Therefore $(x - 2)$ IS a factor.

Step 2 : Division by expection

$6x^3 - 5x^2 - 17^x + 6 = (x - 2)(\qquad)$
The first term in the second bracket must be $6x^2$ to give $6x^3$ if one works backwards.
The last term in the second bracket must be -3 because $-2 \times -3 = +6$.
So we have $6x^3 - 5x^2 - 17^x + 6 = (x - 2)(6x^2?x - 3)$.
Now, we must find the coefficient of the middelterm (x).
$(-2)(6x^2)$ gives $-12x^2$. So, the coefficient of the x-term must be 7.
So, $6x^3 - 5x^2 - 17^x + 6 = (x - 2)(6x^2 + 7x - 3)$.

Step 3 : Factorise fully

$6x^2 + 7x - 3$ can be further factorised to $(2x + 3)(3x - 1)$,
and we are now left with $x^3 - 2x^2 - 5x + 6 = (x - 2)(2x + 3)(3x - 1)$

Step 4 : Solve the equation

$$\begin{aligned} 6x^3 - 5x^2 - 17x + 6 &= 0 \\ (x - 2)(2x + 3)(3x - 1) &= 0 \\ x &= 2; \frac{1}{3}; -\frac{3}{2} \end{aligned}$$

Sometimes it is not possible to factorise the trinomial ("second bracket"). This is when the quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

can be used to solve the cubic equation fully.
For example:

Worked Example 171: Solution of Cubic Equations

Question: Solve for x: $x^3 - 2x^2 - 6x + 4 = 0$.

Answer

Step 1 : Find one factor using the Factor Theorem

Try

$$f(1) = (1)^3 - 2(1)^2 - 6(1) + 4 = 3 - 2 - 6 + 4 = -1$$

Therefore $(x - 1)$ is NOT a factor.
Try

$$f(2) = (2)^3 - 2(2)^2 - 6(2) + 4 = 8 - 8 - 12 + 4 = -8$$

Therefore $(x-2)$ is NOT a factor.

$$f(-2) = (-2)^3 - 2(-2)^2 - 6(-2) + 4 = -8 - 8 + 12 + 4 = 0$$

Therefore $(x+2)$ IS a factor.

Step 2 : Division by expection

$x^3 - 2x^2 - 6x + 4 = (x+2)(\qquad)$

The first term in the second bracket must be x^2 to give x^3.

The last term in the second bracket must be 2 because $2 \times 2 = +4$.

So we have $x^3 - 2x^2 - 6x + 4 = (x+2)(x^2 ? x + 2)$.

Now, we must find the coefficient of the middelterm (x).

$(2)(x^2)$ gives $2x^2$. So, the coefficient of the x-term must be -4. $(2x^2 - 4x^2 = -2x^2)$

So $x^3 - 2x^2 - 6x + 4 = (x+2)(x^2 - 4x + 2)$.

$x^2 - 4x + 2$ cannot be factorised any futher and we are now left with

$(x+2)(x^2 - 4x + 2) = 0$

Step 3 : Solve the equation

$$\begin{aligned} (x+2)(x^2-4x+2) &= 0 \\ (x+2) = 0 \quad &or \quad (x^2-4x+2) = 0 \end{aligned}$$

Step 4 : Apply the quadratic formula for the second bracket

Always write down the formula first and then substitute the values of a, b and c.

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2-4ac}}{2a} \\ &= \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(2)}}{2(1)} \\ &= \frac{4 \pm \sqrt{8}}{2} \\ &= 2 \pm \sqrt{2} \end{aligned}$$

Step 5 : Final solutions

$x = -2 \; or \; x = 2 \pm \sqrt{2}$

38.5.1 Exercises - Solving of Cubic Equations

1. Solve for x: $x^3 + x^2 - 5x + 3 = 0$
2. Solve for y: $y^3 - 3y^2 - 16y - 12 = 0$
3. Solve for m: $m^3 - m^2 - 4m - 4 = 0$
4. Solve for x: $x^3 - x^2 = 3(3x+2)$

Important: :

Remove brackets and write as an equation equal to zero.

5. Solve for x if $2x^3 - 3x^2 - 8x = 3$

38.6 End of Chapter Exercises

1. Solve for x: $16(x+1) = x^2(x+1)$

2. A Show that $x - 2$ is a factor of $3x^3 - 11x^2 + 12x - 4$

   B Hence, by factorising completely, solve the equation

$$3x^3 - 11x^2 + 12x - 4 = 0$$

3. $2x^3 - x^2 - 2x + 2 = Q(x).(2x - 1) + R$ for all values of x. What is the value of R?

4. A Use the factor theorem to solve the following equation for m:

$$8m^3 + 7m^2 - 17m + 2 = 0$$

   B Hence, or otherwise, solve for x:

$$2^{3x+3} + 7 \cdot 2^{2x} + 2 = 17 \cdot 2^x$$

5. **A challenge**:
   Determine the values of p for which the function

$$f(x) = 3p^3 - (3p - 7)x^2 + 5x - 3$$

   leaves a remainder of 9 when it is divided by $(x - p)$.

Chapter 39

Functions and Graphs - Grade 12

39.1 Introduction

In grades 10 and 11 you have learnt about linear functions and quadratic functions as well as the hyperbolic functions and exponential functions and many more. In grade 12 you are expected to demonstrate the ability to work with various types of functions and relations including the inverses of some functions and generate graphs of the inverse relations of functions, in particular the inverses of:

$$y = ax + q$$

$$y = ax^2$$

$$y = ax; a > 0$$

.

39.2 Definition of a Function

A *function* is a relation for which there is only one value of y corresponding to any value of x. We sometimes write $y = f(x)$, which is notation meaning 'y is a function of x'. This definition makes complete sense when compared to our real world examples — each person has only one height, so height is a function of people; on each day, in a specific town, there is only one average temperature.

However, some very common mathematical constructions are not functions. For example, consider the relation $x^2 + y^2 = 4$. This relation describes a circle of radius 2 centred at the origin, as in figure 39.1. If we let $x = 0$, we see that $y^2 = 4$ and thus either $y = 2$ or $y = -2$. Since there are two y values which are possible for the same x value, the relation $x^2 + y^2 = 4$ is **not** a function.

There is a simple test to check if a relation is a function, by looking at its graph. This test is called the *vertical line test*. If it is possible to draw any vertical line (a line of constant x) which crosses the relation more than once, then the relation is **not** a function. If more than one intersection point exists, then the intersections correspond to multiple values of y for a single value of x.

We can see this with our previous example of the circle by looking at its graph again in Figure 39.1.

We see that we can draw a vertical line, for example the dotted line in the drawing, which cuts the circle more than once. Therefore this is **not** a function.

39.2.1 Exercises

1. State whether each of the following equations are functions or not:

   A $x + y = 4$

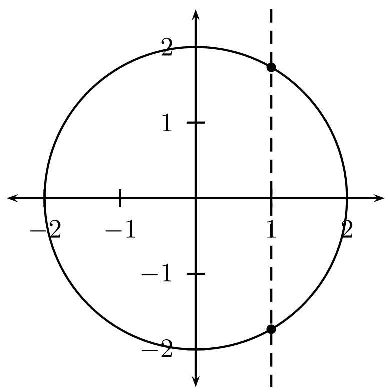


Figure 39.1: Graph of $y^2 + x^2 = 4$

B $y = \frac{x}{4}$
C $y = 2^x$
D $x^2 + y^2 = 4$

2. The table gives the average per capita income, d, in a region of the country as a function of the percent unemployed, u. Write down the equation to show that income is a function of the persent unemployed.

u	1	2	3	4
d	22500	22000	21500	21000

39.3 Notation used for Functions

In grade 10 you were introduced to the notation used to "name" a function. In a function $y = f(x)$, y is called the *dependent variable*, because the value of y depends on what you choose as x. We say x is the *independent variable*, since we can choose x to be any number. Similarly if $g(t) = 2t + 1$, then t is the independent variable and g is the function name. If $f(x) = 3x - 5$ and you are ask to determine $f(3)$, then you have to work out the value for $f(x)$ when $x = 3$. For example,

$$\begin{aligned} f(x) &= 3x - 5 \\ f(3) &= 3(3) - 5 \\ &= 4 \end{aligned}$$

39.4 Graphs of Inverse Functions

In earlier grades, you studied various types of functions and understood the effect of various parameters in the general equation. In this section, we will consider *inverse functions*.

An inverse function is a function which "does the reverse" of a given function. More formally, if f is a function with domain X, then f^{-1} is its inverse function if and only if for every $x \in X$ we have:

$$f^{-1}(f(x)) = f(f^{-1}(x)) = x \tag{39.1}$$

For example, if the function $x \to 3x + 2$ is given, then its inverse function is $x \to \frac{(x-2)}{3}$. This is usually written as:

$$f \; : \; x \to 3x + 2 \tag{39.2}$$

$$f^{-1} \; : \; x \to \frac{(x-2)}{3} \tag{39.3}$$

The superscript "-1" is not an exponent.

If a function f has an inverse then f is said to be invertible.

If f is a real-valued function, then for f to have a valid inverse, it must pass the **horizontal line test**, that is a horizontal line $y = k$ placed anywhere on the graph of f must pass through f exactly once for all real k.

It is possible to work around this condition, by defining a multi-valued function as an inverse.

If one represents the function f graphically in a xy-coordinate system, then the graph of f^{-1} is the reflection of the graph of f across the line $y = x$.

Algebraically, one computes the inverse function of f by solving the equation

$$y = f(x)$$

for x, and then exchanging y and x to get

$$y = f^{-1}(x)$$

39.4.1 Inverse Function of $y = ax + q$

The inverse function of $y = ax + q$ is determined by solving for x as:

$$y = ax + q \quad (39.4)$$
$$ax = y - q \quad (39.5)$$
$$x = \frac{y-q}{a} \quad (39.6)$$
$$= \frac{1}{a}y - \frac{q}{a} \quad (39.7)$$

Therefore the inverse of $y = ax + q$ is $y = \frac{1}{a}x - \frac{q}{a}$.

The inverse function of a straight line is also a straight line.

For example, the straight line equation given by $y = 2x - 3$ has as inverse the function, $y = \frac{1}{2}x + \frac{3}{2}$. The graphs of these functions are shown in Figure 39.2. It can be seen that the two graphs are reflections of each other across the line $y = x$.

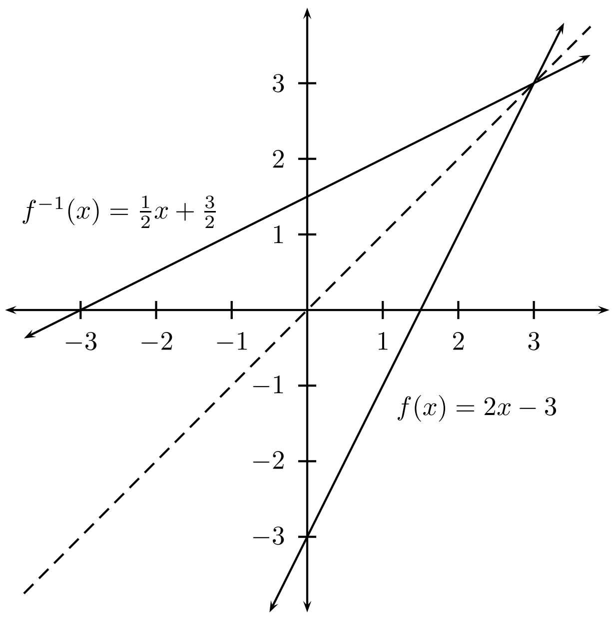


Figure 39.2: The function $f(x) = 2x - 3$ and its inverse $f^{-1}(x) = \frac{1}{2}x + \frac{3}{2}$. The line $y = x$ is shown as a dashed line.

Domain and Range

We have seen that the domain of a function of the form $y = ax+q$ is $\{x : x \in \mathbb{R}\}$ and the range is $\{y : y \in \mathbb{R}\}$. Since the inverse function of a straight line is also a straight line, the inverse function will have the same domain and range as the original function.

Intercepts

The general form of the inverse function of the form $y = ax + q$ is $y = \frac{1}{a}x - \frac{q}{a}$.

By setting $x = 0$ we have that the y-intercept is $y_{int} = -\frac{q}{a}$. Similarly, by setting $y = 0$ we have that the x-intercept is $x_{int} = q$.

It is interesting to note that if $f(x) = ax + q$, then $f^{-1}(x) = \frac{1}{a}x - \frac{q}{a}$ and the y-intercept of $f(x)$ is the x-intercept of $f^{-1}(x)$ and the x-intercept of $f(x)$ is the y-intercept of $f^{-1}(x)$.

39.4.2 Exercises

1. Given $f(x) = 2x - 3$, find $f^{-1}(x)$
2. Consider the function $f(x) = 3x - 7$.
   A Is the relation a function?
   B Identify the domain and range.
3. Sketch the graph of the function $f(x) = 3x - 1$ and its inverse on the same set of axes.
4. The inverse of a function is $f^{-1}(x) = 2x - 4$, what is the function $f(x)$?

39.4.3 Inverse Function of $y = ax^2$

The inverse function of $y = ax^2$ is determined by solving for x as:

$$y = ax^2 \quad (39.8)$$

$$x^2 = \frac{y}{a} \quad (39.9)$$

$$x = \sqrt{\frac{y}{a}} \quad (39.10)$$

We see that the inverse function of $y = ax^2$ is not a function because it fails the vertical line test. If we draw a vertical line through the graph of $f^{-1}(x) = \pm\sqrt{x}$, the line intersects the graph more than once. There has to be a restriction on the domain of a parabola for the inverse to also be a function. Consider the function $f(x) = -x^2 + 9$. The inverse of f can be found by witing $f(y) = x$. Then

$$\begin{aligned} x &= -y^2 + 9 \\ y^2 &= 9 - x \\ y &= \pm\sqrt{9 - x} \end{aligned}$$

If $x \geq 0$, then $\sqrt{9 - x}$ is a function. If the restriction on the domain of f is $x \leq 0$ then $-\sqrt{9 - x}$ would be a function.

39.4.4 Exercises

1. The graph of f^{-1} is shown. Find the equation of f, given that the graph of f is a parabola. (Do not simplify your answer)

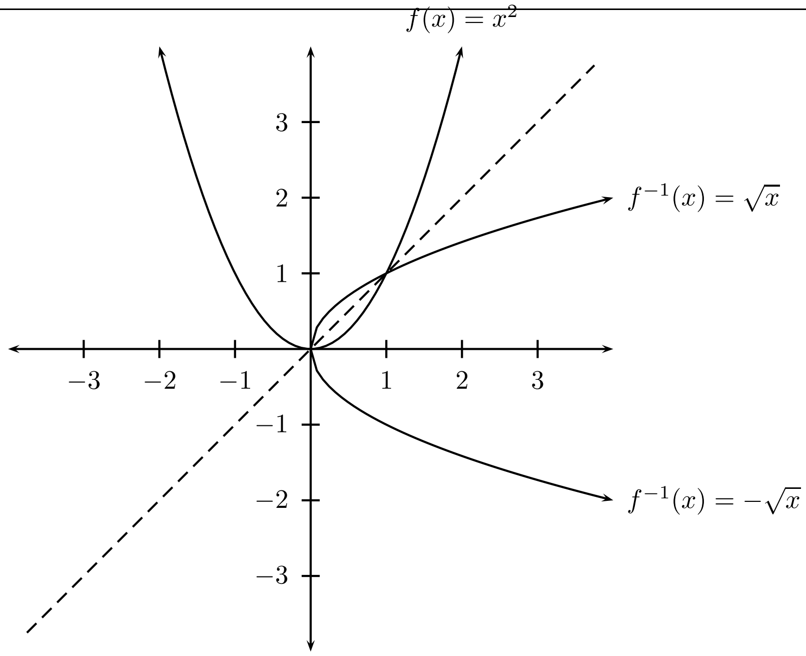
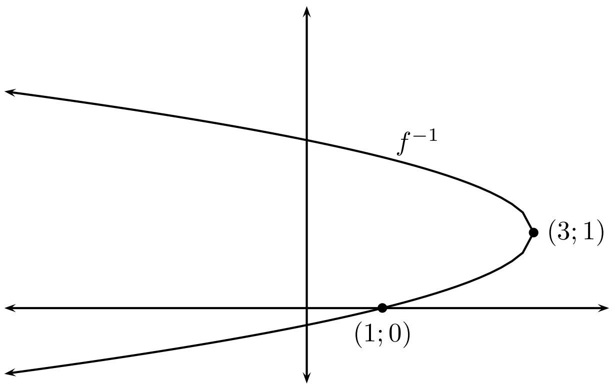


Figure 39.3: The function $f(x) = x^2$ and its inverse $f^{-1}(x) = \pm\sqrt{x}$. The line $y = x$ is shown as a dashed line.



2. $f(x) = 2x^2$.

   A Draw the graph of f and state its domain and range.

   B Find f^{-1} and state the domain and range.

   C What must the domain of f be, so that f^{-1} is a function ?

3. Sketch the graph of $x = -\sqrt{10 - y^2}$. Label a point on the graph other than the intercepts with the axes.

4. A Sketch the graph of $y = x^2$ labelling a point other than the origin on your graph.

   B Find the equation of the inverse of the above graph in the form $y = \ldots$.

   C Now sketch the $y = \sqrt{x}$.

   D The tangent to the graph of $y = \sqrt{x}$ at the point A(9;3) intersects the x-axis at B. Find the equation of this tangent and hence or otherwise prove that the y-axis bisects the straight line AB.

5. Given: $g(x) = -1 + \sqrt{x}$, find the inverse of $g(x)$ in the form $g^{-1}(x)$.

39.4.5 Inverse Function of $y = a^x$

The inverse function of $y = ax^2$ is determined by solving for x as:

$$y = a^x \quad (39.11)$$
$$\log(y) = \log(a^x) \quad (39.12)$$
$$= x\log(a) \quad (39.13)$$
$$\therefore \quad x = \frac{\log(y)}{\log(a)} \quad (39.14)$$

The inverse of $y = 10^x$ is $x = 10^y$, which we write as $y = logx$. Therefore, if $f(x) = 10^x$, then $f^{-1} = logx$.

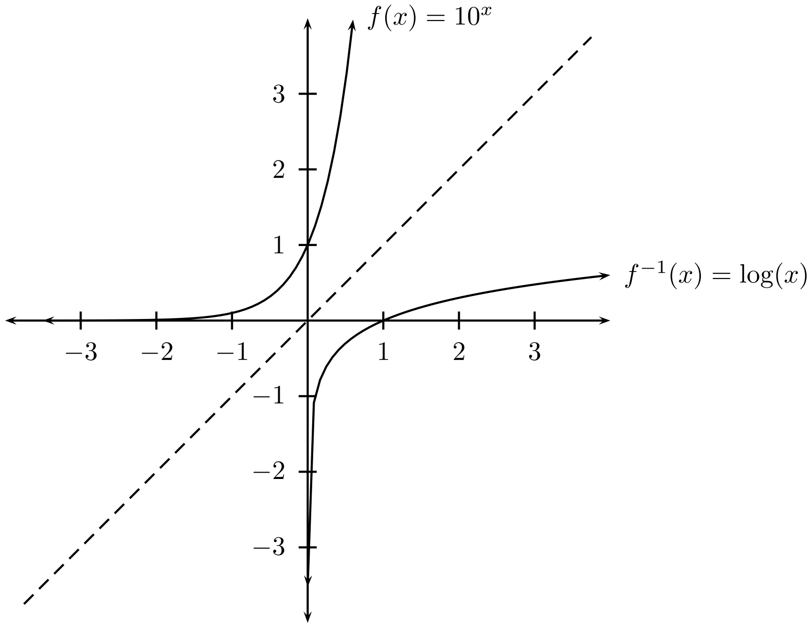


Figure 39.4: The function $f(x) = 10^x$ and its inverse $f^{-1}(x) = \log(x)$. The line $y = x$ is shown as a dashed line.

The exponential function and the logarithmic function are inverses of each other; the graph of the one is the graph of the other, reflected in the line $y = x$. The domain of the function is equal to the range of the inverse. The range of the function is equal to the domain of the inverse.

39.4.6 Exercises

1. Given that $f(x) = [\frac{1}{5}]^x$, sketch the graphs of f and f^{-1} on the same system of axes indicating a point on each graph (other than the intercepts) and showing clearly which is f and which is f^{-1}.
2. Given that $f(x) = 4^{-x}$,
   A Sketch the graphs of f and f^{-1} on the same system of axes indicating a point on each graph (other than the intercepts) and showing clearly which is f and which is f^{-1}.
   B Write f^{-1} in the form $y = \ldots$.
3. Given $g(x) = -1 + \sqrt{x}$, find the inverse of $g(x)$ in the form $g^{-1}(x) = \ldots$
4. A Sketch the graph of $y = x^2$, labeling a point other than the origin on your graph.
   B Find the equation of the inverse of the above graph in the form $y = \ldots$
   C Now, sketch $y = \sqrt{x}$.
   D The tangent to the graph of $y = \sqrt{x}$ at the point $A(9; 3)$ intersects the x-axis at B. Find the equation of this tangent, and hence, or otherwise, prove that the y-axis bisects the straight line AB.

39.5 End of Chapter Exercises

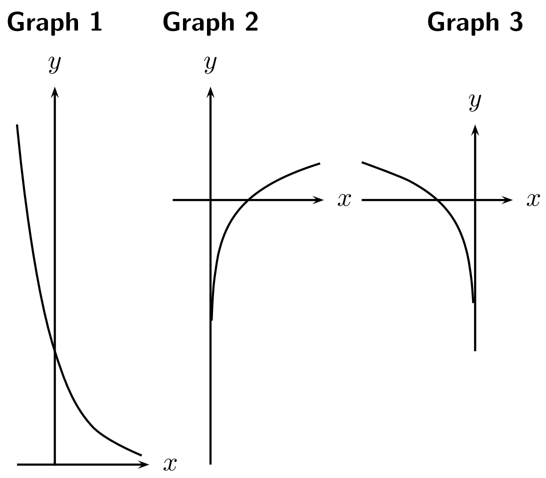
1. Sketch the graph of $x = -\sqrt{10 - y^2}$. Is this graph a function ? Verify your answer.

2. $f(x) = \frac{1}{x-5}$,

   A determine the y-intercept of $f(x)$

   B determine x if $f(x) = -1$.

3. Below, you are given 3 graphs and 5 equations.



   A $y = \log_3 x$

   B $y = -\log_3 x$

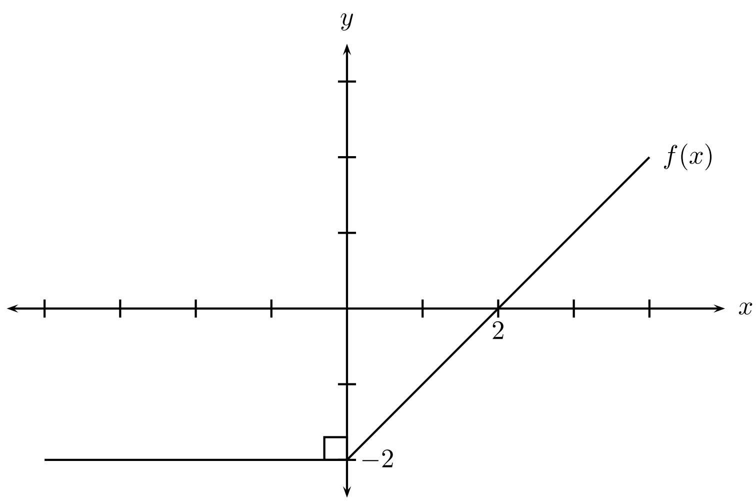
   C $y = \log_3(-x)$

   D $y = 3^{-x}$

   E $y = 3^x$

   Write the equation that best describes each graph.

4. The graph of $y = f(x)$ is shown in the diagram below.



   A Find the value of x such that $f(x) = 0$.

   B Evaluate $f(3) + f(-1)$.

5. Given $g(x) = -1 + \sqrt{x}$, find the inverse of $g(x)$ in the form $g^{-1}(x) = \ldots$

6. Given the equation $h(x) = 3^x$

   A Write down the inverse in the form $h^{-1}(x) = ...$

   B Sketch the graphs of $h(x)$ and $h^{-1}(x)$ on teh same set of axes, labelling the intercepts with the axes.

   C For which values of x is $h^{-1}(x)$ undefined ?

7. A Sketch the graph of $y = x^2$, labelling a point other than the origin on your graph.

   B Find the equation of the inverse of the above graph in the form $y = \ldots$

   C Now, sketch $y = \sqrt{x}$.

   D The tangent to the graph of $y = \sqrt{x}$ at the point $A(9; 3)$ intersects the x-axis at B. Find the equation of this tangent, and hence, or otherwise, prove that the y-axis bisects the straight line AB.

Chapter 40

Differential Calculus - Grade 12

40.1 Why do I have to learn this stuff?

Calculus is one of the central branches of mathematics and was developed from algebra and geometry. Calculus is built on the concept of limits, which will be discussed in this chapter. Calculus consists of two complementary ideas: differential calculus and integral calculus. Only differential calculus will be studied. Differential calculus is concerned with the instantaneous rate of change of quantities with respect to other quantities, or more precisely, the local behaviour of functions. This can be illustrated by the slope of a function's graph. Examples of typical differential calculus problems include: finding the acceleration and velocity of a free-falling body at a particular moment and finding the optimal number of units a company should produce to maximize its profit.

Calculus is fundamentally different from the mathematics that you have studied previously. Calculus is more dynamic and less static. It is concerned with change and motion. It deals with quantities that approach other quantities. For that reason it may be useful to have an overview of the subject before beginning its intensive study.

Calculus is a tool to understand many natural phenomena like how the wind blows, how water flows, how light travels, how sound travels and how the planets move. However, other human activities such as economics are also made easier with calculus.

In this section we give a glimpse of some of the main ideas of calculus by showing how limits arise when we attempt to solve a variety of problems.

Extension: Integral Calculus

Integral calculus is concerned with the accumulation of quantities, such as areas under a curve, linear distance traveled, or volume displaced. Differential and integral calculus act inversely to each other. Examples of typical integral calculus problems include finding areas and volumes, finding the amount of water pumped by a pump with a set power input but varying conditions of pumping losses and pressure and finding the amount of rain that fell in a certain area if the rain fell at a specific rate.

Interesting Fact

Both Isaac Newton (4 January 1643 – 31 March 1727) and Gottfried Liebnitz (1 July 1646 – 14 November 1716 (Hanover, Germany)) are credited with the 'invention' of calculus. Newton was the first to apply calculus to general physics, while Liebnitz developed most of the notation that is still in use today.

When Newton and Leibniz first published their results, there was some controversy over whether Leibniz's work was independent of Newton's. While Newton derived his results years before Leibniz, it was only some time after Leibniz published in 1684 that Newton published. Later, Newton would claim that Leibniz got the idea from Newton's notes on the subject; however examination of the papers of Leibniz and Newton show they arrived at their results independently, with Leibniz starting first with integration and Newton with differentiation. This controversy between Leibniz and Newton divided English-speaking mathematicians from those in Europe for many years, which slowed the development of mathematical analysis. Today, both Newton and Leibniz are given credit for independently developing calculus. It is Leibniz, however, who is credited with giving the new discipline the name it is known by today: "calculus". Newton's name for it was "the science of fluxions".

40.2 Limits

40.2.1 A Tale of Achilles and the Tortoise

Zeno (circa 490 BC - circa 430 BC) was a pre-Socratic Greek philosopher of southern Italy who is famous for his paradoxes.

One of Zeno's paradoxes can be summarised by:

> Achilles and a tortoise agree to a race, but the tortoise is unhappy because Achilles is very fast. So, the tortoise asks Achilles for a head-start. Achilles agrees to give the tortoise a 1 000 m head start. Does Achilles overtake the tortoise?

We know how to solve this problem. We start by writing:

$$x_A = v_A t \qquad (40.1)$$

$$x_t = 1000\,\mathrm{m} + \mathrm{v_t t} \qquad (40.2)$$

where

x_A	distance covered by Achilles
v_A	Achilles' speed
t	time taken by Achilles to overtake tortoise
x_t	distance covered by the tortoise
v_t	the tortoise's speed

If we assume that Achilles runs at 2 m·s^{-1} and the tortoise runs at 0,25 m·s^{-1} then Achilles will overtake the tortoise when both of them have covered the same distance. This means that

Achilles overtakes the tortoise at a time calculated as:

$$\begin{aligned}
x_A &= x_t && (40.3)\\
v_A t &= 1000 + v_t t && (40.4)\\
(2\,\mathrm{m\cdot s^{-1}})t &= 1000\,\mathrm{m} + (0{,}25\,\mathrm{m\cdot s^{-1}})\mathrm{t} && (40.5)\\
(2\,\mathrm{m\cdot s^{-1}} - 0{,}25\,\mathrm{m\cdot s^{-1}})t &= 1000\,\mathrm{m} && (40.6)\\
t &= \frac{1000\,\mathrm{m}}{1\frac{3}{4}\,\mathrm{m\cdot s^{-1}}} && (40.7)\\
&= \frac{1000\,\mathrm{m}}{\frac{7}{4}\,\mathrm{m\cdot s^{-1}}} && (40.8)\\
&= \frac{(4)(1000)}{7}\,\mathrm{s} && (40.9)\\
&= \frac{4000}{7}\,\mathrm{s} && (40.10)\\
&= 571\frac{3}{7}\,\mathrm{s} && (40.11)
\end{aligned}$$

However, Zeno (the Greek philosopher who thought up this problem) looked at it as follows: Achilles takes

$$t = \frac{1000}{2} = 500\,\mathrm{s}$$

to travel the 1 000 m head start that the tortoise had. However, in this 500 s, the tortoise has travelled a further

$$x = (500)(0{,}25) = 125\,\mathrm{m}.$$

Achilles then takes another

$$t = \frac{125}{2} = 62{,}5\,\mathrm{s}$$

to travel the 125 m. In this 62,5 s, the tortoise travels a further

$$x = (62{,}5)(0{,}25) = 15{,}625\,\mathrm{m}.$$

Zeno saw that Achilles would always get closer but wouldn't actually overtake the tortoise.

40.2.2 Sequences, Series and Functions

So what does Zeno, Achilles and the tortoise have to do with calculus?

Well, in Grades 10 and 11 you studied sequences. For the sequence

$$0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots$$

which is defined by the expression

$$a_n = 1 - \frac{1}{n}$$

the terms get closer to 1 as n gets larger. Similarly, for the sequence

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots$$

which is defined by the expression

$$a_n = \frac{1}{n}$$

the terms get closer to 0 as n gets larger. We have also seen that the infinite geometric series has a finite total. The infinite geometric series is

$$S_\infty = \sum_{i=1}^{\infty} a_1.r^{i-1} = \frac{a_1}{1-r} \qquad \text{for} \qquad -1$$

where a_1 is the first term of the series and r is the common ratio.

We see that there are some functions where the value of the function gets close to or **approaches** a certain value.

Similarly, for the function:

$$y = \frac{x^2 + 4x - 12}{x + 6}$$

The numerator of the function can be factorised as:

$$y = \frac{(x + 6)(x - 2)}{x + 6}.$$

Then we can cancel the $x - 6$ from numerator and denominator and we are left with:

$$y = x - 2.$$

However, we are only able to cancel the $x + 6$ term if $x \neq -6$. If $x = -6$, then the denominator becomes 0 and the function is not defined. This means that the domain of the function does not include $x = -6$. But we can examine what happens to the values for y as x gets close to -6. These values are listed in Table 40.1 which shows that as x gets closer to -6, y gets close to 8.

Table 40.1: Values for the function $y = \frac{(x+6)(x-2)}{x+6}$ as x gets close to -6.

x	$y = \frac{(x+6)(x-2)}{x+6}$
-9	-11
-8	-10
-7	-9
-6.5	-8.5
-6.4	-8.4
-6.3	-8.3
-6.2	-8.2
-6.1	-8.1
-6.09	-8.09
-6.08	-8.08
-6.01	-8.01
-5.9	-7.9
-5.8	-7.8
-5.7	-7.7
-5.6	-7.6
-5.5	-7.5
-5	-7
-4	-6
-3	-5

The graph of this function is shown in Figure 40.1. The graph is a straight line with slope 1 and intercept -2, but with a missing section at $x = -6$.

Extension: Continuity

We say that a function is continuous if there are no values of the independent variable for which the function is undefined.

40.2.3 Limits

We can now introduce a new notation. For the function $y = \frac{(x+6)(x-2)}{x+6}$, we can write:

$$\lim_{x \to -6} \frac{(x+6)(x-2)}{x+6} = -8.$$

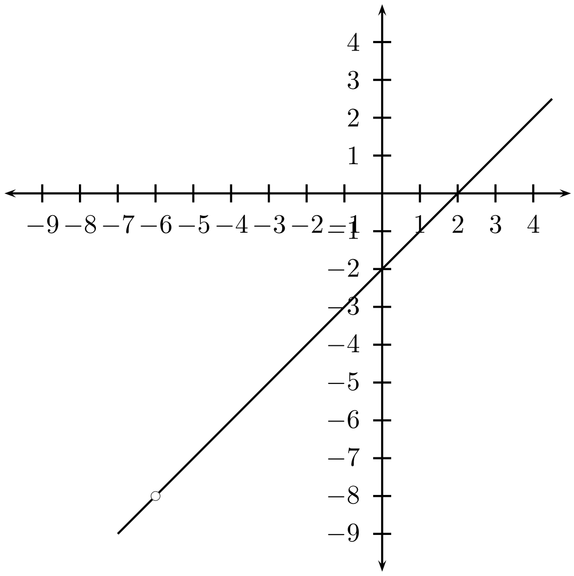


Figure 40.1: Graph of $y = \frac{(x+6)(x-2)}{x+6}$.

This is read: *the limit of* $\frac{(x+6)(x-2)}{x+6}$ *as* x *tends to -6 is 8.*

Activity :: Investigation : Limits
If $f(x) = x + 1$, determine:

f(-0.1)	
f(-0.05)	
f(-0.04)	
f(-0.03)	
f(-0.02)	
f(-0.01)	
f(0.00)	
f(0.01)	
f(0.02)	
f(0.03)	
f(0.04)	
f(0.05)	
f(0.1)	

What do you notice about the value of $f(x)$ as x gets close to 0.

Worked Example 172: Limits Notation

Question: Summarise the following situation by using limit notation: As x gets close to 1, the value of the function

$$y = x + 2$$

gets close to 3.

Answer

This is written as:

$$\lim_{x \to 1} x + 2 = 3$$

in limit notation.

We can also have the situation where a function has a different value depending on whether x approaches from the left or the right. An example of this is shown in Figure 40.2.

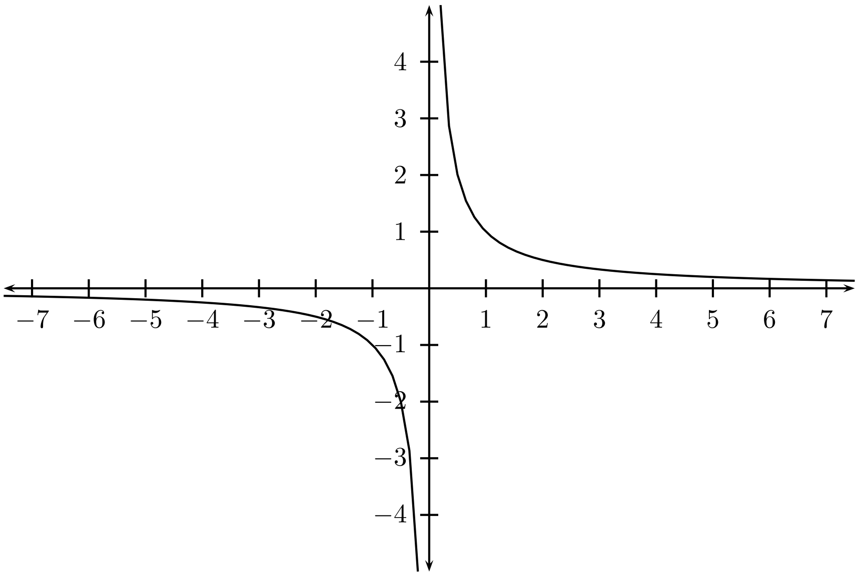


Figure 40.2: Graph of $y = \frac{1}{x}$.

As $x \to 0$ from the left, $y = \frac{1}{x}$ approaches $-\infty$. As $x \to 0$ from the right, $y = \frac{1}{x}$ approaches $+\infty$. This is written in limits notation as:

$$\lim_{x \to 0^-} \frac{1}{x} = -\infty$$

for x approaching zero from the left and

$$\lim_{x \to 0^+} \frac{1}{x} = \infty$$

for x approaching zero from the right. You can calculate the limit of many different functions using a set method.

Method:

Limits If you are required to calculate a limit like $\lim_{x \to a}$ then:

1. Simplify the expression completely.
2. If it is possible, cancel all common terms.
3. Let x approach the a.

Worked Example 173: Limits

Question: Determine

$$\lim_{x\to 1} 10$$

Answer

Step 1 : Simplify the expression

There is nothing to simplify.

Step 2 : Cancel all common terms

There are no terms to cancel.

Step 3 : Let $x \to 1$ and write final answer

$$\lim_{x\to 1} 10 = 10$$

Worked Example 174: Limits

Question: Determine

$$\lim_{x\to 2} x$$

Answer

Step 1 : Simplify the expression

There is nothing to simplify.

Step 2 : Cancel all common terms

There are no terms to cancel.

Step 3 : Let $x \to 2$ and write final answer

$$\lim_{x\to 2} x = 2$$

Worked Example 175: Limits

Question: Determine

$$\lim_{x\to 10} \frac{x^2 - 100}{x - 10}$$

Answer

Step 1 : Simplify the expression

The numerator can be factorised.

$$\frac{x^2 - 100}{x - 10} = \frac{(x + 10)(x - 10)}{x - 10}$$

Step 2 : Cancel all common terms

$x - 10$ can be cancelled from the numerator and denominator.

$$\frac{(x + 10)(x - 10)}{x - 10} = x + 10$$

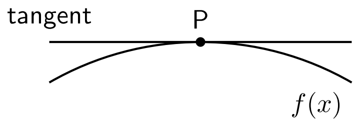
Step 3 : Let $x \to 1$ and write final answer

$$\lim_{x\to 10} \frac{x^2 - 100}{x - 10} = 20$$

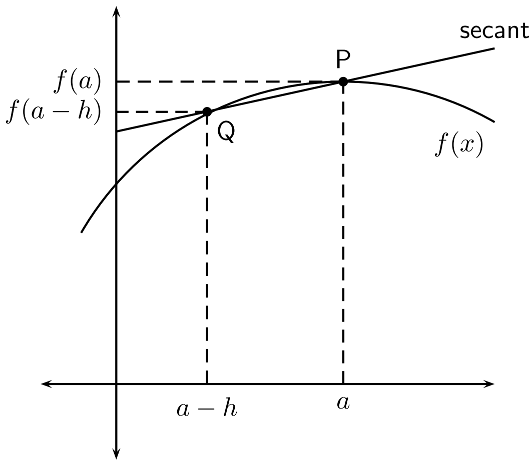
40.2.4 Average Gradient and Gradient at a Point

In Grade 10 you learnt about average gradients on a curve. The average gradient between any two points on a curve is given by the gradient of the straight line that passes through both points. In Grade 11 you were introduced to the idea of a gradient at a single point on a curve. We saw that this was the gradient of the tangent to the curve at the given point, but we did not learn how to determine the gradient of the tangent.

Now let us consider the problem of trying to find the gradient of a tangent t to a curve with equation $y = f(x)$ at a given point P.



We know how to calculate the average gradient between two points on a curve, but we need two points. The problem now is that we only have one point, namely P. To get around the problem we first consider a secant to the curve that passes through point P and another point on the curve Q. We can now find the average gradient of the curve between points P and Q.



If the x-coordinate of P is a, then the y-coordinate is $f(a)$. Similarly, if the x-coordinate of Q is $a - h$, then the y-coordinate is $f(a - h)$. If we choose a as x_2 and $a - h$ as x_1, then:

$$y_1 = f(a-h)$$

$$y_2 = f(a).$$

We can now calculate the average gradient as:

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{f(a) - f(a-h)}{a - (a-h)} \quad (40.12)$$

$$= \frac{f(a) - f(a-h)}{h} \quad (40.13)$$

Now imagine that Q moves along the curve toward P. The secant line approaches the tangent line as its limiting position. This means that the average gradient of the secant *approaches* the gradient of the tangent to the curve at P. In (40.13) we see that as point Q approaches point P, h gets closer to 0. When $h = 0$, points P and Q are equal. We can now use our knowledge of limits to write this as:

$$\text{gradient at } P = \lim_{h \to 0} \frac{f(a) - f(a-h)}{h}. \quad (40.14)$$

and we say that the gradient at point P is the limit of the average gradient as Q approaches P along the curve.

Activity :: Investigation : Limits
The gradient at a point x on a curve defined by $f(x)$ can also be written as:

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \tag{40.15}$$

Show that this is equivalent to (40.14).

Worked Example 176: Limits

Question: For the function $f(x)=2x^2-5x$, determine the gradient of the tangent to the curve at the point $x=2$.

Answer

Step 1 : Calculating the gradient at a point

We know that the gradient at a point x is given by:

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

In our case $x=2$. It is simpler to substitute $x=2$ at the end of the calculation.

Step 2 : Write $f(x+h)$ and simplify

$$\begin{aligned} f(x+h) &= 2(x+h)^2-5(x+h) \\ &= 2(x^2+2xh+h^2)-5x-5h \\ &= 2x^2+4xh+2h^2-5x-5h \end{aligned}$$

Step 3 : Calculate limit

$$\begin{aligned} \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} &= \frac{2x^2+4xh+2h^2-5x-5h-(2x^2-5x)}{h} \\ &= \lim_{h\to 0}\frac{2x^2+4xh+2h^2-5x-5h-2x^2+5x}{h} \\ &= \lim_{h\to 0}\frac{4xh+2h^2-5h}{h} \\ &= \lim_{h\to 0}\frac{h(4x+2h-5)}{h} \\ &= \lim_{h\to 0} 4x+2h-5 \\ &= 4x-5 \end{aligned}$$

Step 4 : Calculate gradient at $x=2$

$$4x-5=4(2)-5=3$$

Step 5 : Write the final answer

The gradient of the tangent to the curve $f(x)=2x^2-5x$ at $x=2$ is 3.

Worked Example 177: Limits

Question: For the function $f(x) = 5x^2 - 4x + 1$, determine the gradient of the tangent to curve at the point $x = a$.
Answer
Step 1 : Calculating the gradient at a point
We know that the gradient at a point x is given by:

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

In our case $x = a$. It is simpler to substitute $x = a$ at the end of the calculation.
Step 2 : Write $f(x + h)$ and simplify

$$\begin{aligned} f(x+h) &= 5(x+h)^2 - 4(x+h) + 1 \\ &= 5(x^2 + 2xh + h^2) - 4x - 4h + 1 \\ &= 5x^2 + 10xh + 5h^2 - 4x - 4h + 1 \end{aligned}$$

Step 3 : Calculate limit

$$\begin{aligned} \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} &= \frac{5x^2 + 10xh + 5h^2 - 4x - 4h + 1 - (5x^2 - 4x + 1)}{h} \\ &= \lim_{h\to 0}\frac{5x^2 + 10xh + 5h^2 - 4x - 4h + 1 - 5x^2 + 4x - 1}{h} \\ &= \lim_{h\to 0}\frac{10xh + 5h^2 - 4h}{h} \\ &= \lim_{h\to 0}\frac{h(10x + 5h - 4)}{h} \\ &= \lim_{h\to 0} 10x + 5h - 4 \\ &= 10x - 4 \end{aligned}$$

Step 4 : Calculate gradient at $x = a$

$$10x - 4 = 10a - 5$$

Step 5 : Write the final answer
The gradient of the tangent to the curve $f(x) = 5x^2 - 4x + 1$ at $x = 1$ is $10a - 5$.

Exercise: Limits
Determine the following

1. $$\lim_{x\to 3}\frac{x^2 - 9}{x + 3}$$

2. $$\lim_{x\to 3}\frac{x + 3}{x^2 + 3x}$$

3. $$\lim_{x\to 2}\frac{3x^2 - 4x}{3 - x}$$

4.

$$\lim_{x\to 4} \frac{x^2 - x - 12}{x - 4}$$

5.

$$\lim_{x\to 2} 3x + \frac{1}{3x}$$

40.3 Differentiation from First Principles

The tangent problem has given rise to the branch of calculus called **differential calculus** and the equation:

$$\lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

defines the **derivative of the function** $f(x)$**.** Using (40.15) to calculate the derivative is called **finding the derivative from first principles**.

Definition: Derivative
The derivative of a function $f(x)$ is written as $f'(x)$ and is defined by:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \qquad (40.16)$$

There are a few different notations used to refer to derivatives. If we use the traditional notation $y = f(x)$ to indicate that the dependent variable is y and the independent variable is x, then some common alternative notations for the derivative are as follows:

$$f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx}f(x) = Df(x) = D_x f(x)$$

The symbols D and $\frac{d}{dx}$ are called **differential operators** because they indicate the operation of **differentiation**, which is the process of calculating a derivative. It is very important that you learn to identify these different ways of denoting the derivative, and that you are consistent in your usage of them when answering questions.

Important: Though we choose to use a fractional form of representation, $\frac{dy}{dx}$ is a limit and **is not** a fraction, i.e. $\frac{dy}{dx}$ does not mean $dy \div dx$. $\frac{dy}{dx}$ means y differentiated with respect to x. Thus, $\frac{dp}{dx}$ means p differentiated with respect to x. The '$\frac{d}{dx}$' is the "operator", operating on some function of x.

Worked Example 178: Derivatives - First Principles

Question: Calculate the derivative of $g(x) = x - 1$ from first principles.
Answer
Step 1 : Calculating the gradient at a point
We know that the gradient at a point x is given by:

$$g'(x) = \lim_{h\to 0} \frac{g(x+h) - g(x)}{h}$$

Step 2 : Write $g(x+h)$ and simplify

$$g(x+h) = x + h - 1$$

Step 3 : Calculate limit

$$\begin{aligned} g'(x) &= \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} \\ &= \lim_{h\to 0} \frac{x + h - 1 - (x - 1)}{h} \\ &= \lim_{h\to 0} \frac{x + h - 1 - x + 1}{h} \\ &= \lim_{h\to 0} \frac{h}{h} \\ &= \lim_{h\to 0} 1 \\ &= 1 \end{aligned}$$

Step 4 : Write the final answer

The derivative $g'(x)$ of $g(x) = x - 1$ is 1.

Worked Example 179: Derivatives - First Principles

Question: Calculate the derivative of $h(x) = x^2 - 1$ from first principles.

Answer

Step 1 : Calculating the gradient at a point

We know that the gradient at a point x is given by:

$$g'(x) = \lim_{h\to 0} \frac{g(x+h) - g(x)}{h}$$

Step 2 : Write $g(x+h)$ and simplify

$$g(x+h) = x + h - 1$$

Step 3 : Calculate limit

$$\begin{aligned} g'(x) &= \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} \\ &= \lim_{h\to 0} \frac{x + h - 1 - (x - 1)}{h} \\ &= \lim_{h\to 0} \frac{x + h - 1 - x + 1}{h} \\ &= \lim_{h\to 0} \frac{h}{h} \\ &= \lim_{h\to 0} 1 \\ &= 1 \end{aligned}$$

Step 4 : Write the final answer

The derivative $g'(x)$ of $g(x) = x - 1$ is 1.

?

Exercise: Derivatives

1. Given $g(x) = -x^2$
   A determine $\frac{g(x+h)-g(x)}{h}$
   B hence, determine
   $$\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}$$
   C explain the meaning of your answer in (b).
2. Find the derivative of $f(x) = -2x^2 + 3x$ using first principles.
3. Determine the derivative of $f(x) = \frac{1}{x-2}$ using first principles.
4. Determine $f'(3)$ from first principles if $f(x) = -5x^2$.
5. If $h(x) = 4x^2 - 4x$, determine $h'(x)$ using first principles.

40.4 Rules of Differentiation

Calculating the derivative of a function from first principles is very long, and it is easy to make mistakes. Fortunately, there are rules which make calculating the derivative simple.

Activity :: Investigation : Rules of Differentiation
From first principles, determine the derivatives of the following:

1. $f(x) = b$
2. $f(x) = x$
3. $f(x) = x^2$
4. $f(x) = x^3$
5. $f(x) = 1/x$

You should have found the following:

$f(x)$	$f'(x)$
b	0
x	1
x^2	$2x$
x^3	$3x^2$
$1/x = x^{-1}$	$-x^{-2}$

If we examine these results we see that there is a pattern, which can be summarised by:

$$\frac{d}{dx}(x^n) = nx^{n-1} \tag{40.17}$$

There are two other rules which make differentiation simpler. For any two functions $f(x)$ and $g(x)$:

$$\frac{d}{dx}[f(x) \pm g(x)] = f'(x) \pm g'(x) \tag{40.18}$$

This means that we differentiate each term separately.

The final rule applies to a function $f(x)$ that is multiplied by a constant k.

$$\frac{d}{dx}[k.f(x)] = kf'(x) \tag{40.19}$$

Worked Example 180: Rules of Differentiation

Question: Determine the derivative of $x-1$ using the rules of differentiation.

Answer

Step 1 : Identify the rules that will be needed

We will apply two rules of differentiation:

$$\frac{d}{dx}\,(x^n) = nx^{n-1}$$

and

$$\frac{d}{dx}[f(x)-g(x)] = \frac{d}{dx}[f(x)] - \frac{d}{dx}[g(x)]$$

Step 2 : Determine the derivative

In our case $f(x) = x$ and $g(x) = 1$.

$$f'(x) = 1$$

and

$$g'(x) = 0$$

Step 3 : Write the final answer

The derivative of $x-1$ is 1 which is the same result as was obtained earlier, from first principles.

40.4.1 Summary of Differentiation Rules

$$\tfrac{d}{dx}b = 0$$

$$\tfrac{d}{dx}(x^n) = nx^{n-1}$$

$$\tfrac{d}{dx}(kf) = k\tfrac{df}{dx}$$

$$\tfrac{d}{dx}(f+g) = \tfrac{df}{dx} + \tfrac{dg}{dx}$$

?

Exercise: Rules of Differentiation

1. Find $f'(x)$ if $f(x) = \dfrac{x^2-5x+6}{x-2}$.
2. Find $f'(y)$ if $f(y) = \sqrt{y}$.
3. Find $f'(z)$ if $f(z) = (z-1)(z+1)$.
4. Determine $\frac{dy}{dx}$ if $y = \dfrac{x^3+2\sqrt{x}-3}{x}$.

5. Determine the derivative of $y = \sqrt{x^3} + \dfrac{1}{3x^3}$.

40.5 Applying Differentiation to Draw Graphs

Thus far we have learnt about how to differentiate various functions, but I am sure that you are beginning to ask, *What is the point of learning about derivatives?* Well, we know one important fact about a derivative: it is a gradient. So, any problems involving the calculations of gradients or rates of change can use derivatives. One simple application is to draw graphs of functions by firstly determine the gradients of straight lines and secondly to determine the turning points of the graph.

40.5.1 Finding Equations of Tangents to Curves

In section 40.2.4 we saw that finding the gradient of a tangent to a curve is the same as finding the slope of the same curve at the point of the tangent. We also saw that the gradient of a function at a point is just its derivative.

Since we have the gradient of the tangent and the point on the curve through which the tangent passes, we can find the equation of the tangent.

Worked Example 181: Finding the Equation of a Tangent to a Curve

Question: Find the equation of the tangent to the curve $y = x^2$ at the point (1,1) and draw both functions.
Answer
Step 1 : Determine what is required
We are required to determine the equation of the tangent to the curve defined by $y = x^2$ at the point (1,1). The tangent is a straight line and we can find the equation by using derivatives to find the gradient of the straight line. Then we will have the gradient and one point on the line, so we can find the equation using:

$$y - y_1 = m(x - x_1)$$

from grade 11 Coordinate Geometry.
Step 2 : Differentiate the function
Using our rules of differentiation we get:

$$y' = 2x$$

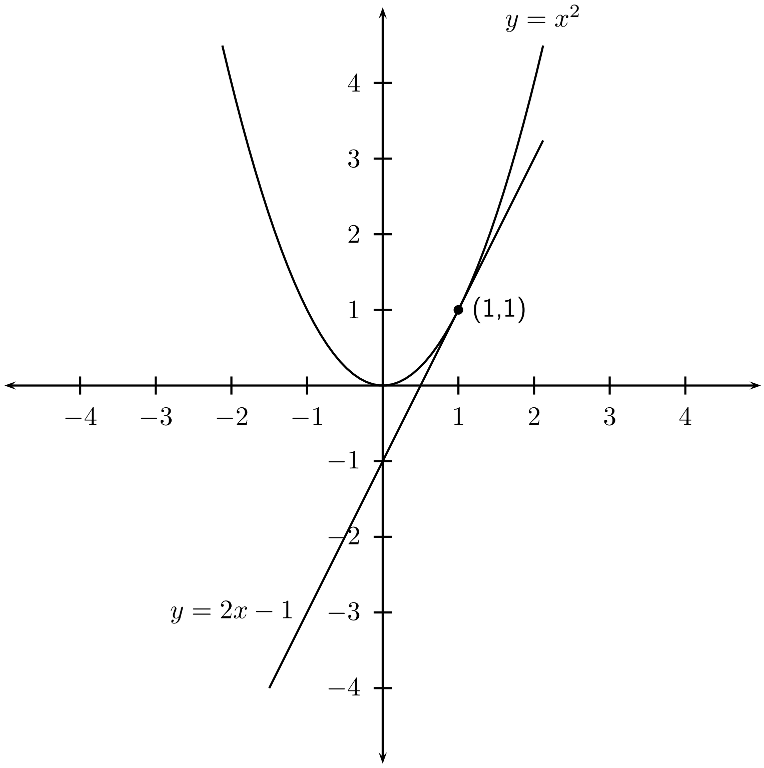
Step 3 : Find the gradient at the point (1,1)
In order to determine the gradient at the point (1,1), we substitute the x-value into the equation for the derivative. So, y' at $x = 1$ is:

$$2(1) = 2$$

Step 4 : Find equation of tangent

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 1 &= (2)(x - 1) \\ y &= 2x - 2 + 1 \\ y &= 2x - 1 \end{aligned}$$

Step 5 : Write the final answer
The equation of the tangent to the curve defined by $y = x^2$ at the point (1,1) is $y = 2x - 1$.
Step 6 : Sketch both functions



40.5.2 Curve Sketching

Differentiation can be used to sketch the graphs of functions, by helping determine the turning points. We know that if a graph is increasing on an interval and reaches a turning point, then the graph will start decreasing after the turning point. The turning point is also known as a stationary point because the gradient at a turning point is 0. We can then use this information to calculate turning points, by calculating the points at which the derivative of a function is 0.

Important: If $x = a$ is a turning point of $f(x)$, then:

$$f'(a) = 0$$

This means that the derivative is 0 at a turning point.

Take the graph of $y = x^2$ as an example. We know that the graph of this function has a turning point at (0,0), but we can use the derivative of the function:

$$y' = 2x$$

and set it equal to 0 to find the x-value for which the graph has a turning point.

$$\begin{aligned} 2x &= 0 \\ x &= 0 \end{aligned}$$

We then substitute this into the equation of the graph (i.e. $y = x^2$) to determine the y-coordinate of the turning point:

$$f(0) = (0)^2 = 0$$

This corresponds to the point that we have previously calculated.

Worked Example 182: Calculation of Turning Points

Question: Calculate the turning points of the graph of the function

$$f(x) = 2x^3 - 9x^2 + 12x - 15$$

.

Answer

Step 1 : Determine the derivative of $f(x)$

Using the rules of differentiation we get:

$$f'(x) = 6x^2 - 18x + 12$$

Step 2 : Set $f'(x) = 0$ and calculate x-coordinate of turning point

$$\begin{aligned} 6x^2 - 18x + 12 &= 0 \\ x^2 - 3x + 2 &= 0 \\ (x-2)(x-1) &= 0 \end{aligned}$$

Therefore, the turning points are at $x = 2$ and $x = 1$.

Step 3 : Substitute x-coordinate of turning point into $f(x)$ to determine y-coordinates

$$\begin{aligned} f(2) &= 2(2)^3 - 9(2)^2 + 12(2) - 15 \\ &= 16 - 36 + 24 - 15 \\ &= -11 \end{aligned}$$

$$\begin{aligned} f(1) &= 2(1)^3 - 9(1)^2 + 12(1) - 15 \\ &= 2 - 9 + 12 - 15 \\ &= -10 \end{aligned}$$

Step 4 : Write final answer

The turning points of the graph of $f(x) = 2x^3 - 9x^2 + 12x - 15$ are (2,-11) and (1,-10).

We are now ready to sketch graphs of functions.

Method:

Sketching GraphsSuppose we are given that $f(x) = ax^3 + bx^2 + cx + d$, then there are **five** steps to be followed to sketch the graph of the function:

1. If $a > 0$, then the graph is increasing from left to right, and has a maximum and then a minimum. As x increases, so does $f(x)$. If $a < 0$, then the graph decreasing is from left to right, and has first a minimum and then a maximum. $f(x)$ decreases as x increases.

2. Determine the value of the y-intercept by substituting $x = 0$ into $f(x)$

3. Determine the x-intercepts by factorising $ax^3 + bx^2 + cx + d = 0$ and solving for x. First try to eliminate constant common factors, and to group like terms together so that the expression is expressed as economically as possible. Use the factor theorem if necessary.

4. Find the turning points of the function by working out the derivative $\frac{df}{dx}$ and setting it to zero, and solving for x.

5. Determine the y-coordinates of the turning points by substituting the x values obtained in the previous step, into the expression for $f(x)$.

6. Draw a neat sketch.

Worked Example 183: Sketching Graphs

Question: Draw the graph of $g(x) = x^2 - x + 2$

Answer

Step 1 : Determine the y-intercept

y-intercept is obtained by setting $x = 0$.

$$g(0) = (0)^2 - 0 + 2 = 2$$

Step 2 : Determine the x-intercepts

The x-intercepts are found by setting $g(x) = 0$.

$$\begin{aligned} g(x) &= x^2 - x + 2 \\ 0 &= x^2 - x + 2 \end{aligned}$$

which does not have real roots. Therefore, the graph of $g(x)$ does not have any x-intercepts.

Step 3 : Find the turning points of the function

Work out the derivative $\frac{dg}{dx}$ and set it to zero to for the x coordinate of the turning point.

$$\frac{dg}{dx} = 2x - 1$$

$$\begin{aligned} \frac{dg}{dx} &= 0 \\ 2x - 1 &= 0 \\ 2x &= 1 \\ x &= \frac{1}{2} \end{aligned}$$

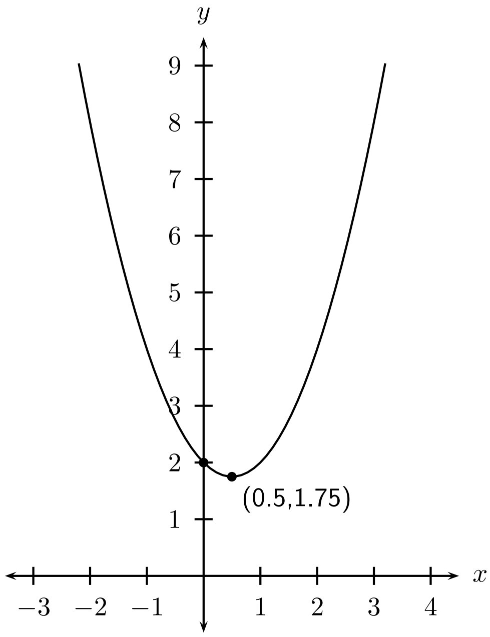
Step 4 : Determine the y-coordinates of the turning points by substituting the x values obtained in the previous step, into the expression for $f(x)$.

y coordinate of turning point is given by calculating $g(\frac{1}{2})$.

$$\begin{aligned} g(\frac{1}{2}) &= (\frac{1}{2})^2 - (\frac{1}{2}) + 2 \\ &= \frac{1}{4} - \frac{1}{2} + 2 \\ &= \frac{7}{4} \end{aligned}$$

The turning point is at $(\frac{1}{2}, \frac{7}{4})$

Step 5 : Draw a neat sketch



Worked Example 184: Sketching Graphs

Question: Sketch the graph of $g(x) = -x^3 + 6x^2 - 9x + 4$.
Answer
Step 1 : Calculate the turning points
Find the turning points by setting $g'(x) = 0$.
If we use the rules of differentiation we get

$$g'(x) = -3x^2 + 12x - 9$$

$$\begin{aligned} g'(x) &= 0 \\ -3x^2 + 12x - 9 &= 0 \\ x^2 - 4x + 3 &= 0 \\ (x-3)(x-1) &= 0 \end{aligned}$$

The x-coordinates of the turning points are: $x = 1$ and $x = 3$.
The y-coordinates of the turning points are calculated as:

$$\begin{aligned} g(x) &= -x^3 + 6x^2 - 9x + 4 \\ g(1) &= -(1)^3 + 6(1)^2 - 9(1) + 4 \\ &= -1 + 6 - 9 + 4 \\ &= 0 \end{aligned}$$

$$\begin{aligned} g(x) &= -x^3 + 6x^2 - 9x + 4 \\ g(3) &= -(3)^3 + 6(3)^2 - 9(3) + 4 \\ &= -27 + 54 - 27 + 4 \\ &= 4 \end{aligned}$$

Therefore the turning points are: (1,0) and (3,4).

Step 2 : Determine the y-intercepts

We find the y-intercepts by finding the value for $g(0)$.

$$\begin{aligned} g(x) &= -x^3 + 6x^2 - 9x + 4 \\ y_{int} = g(0) &= -(0)^3 + 6(0)^2 - 9(0) + 4 \\ &= 4 \end{aligned}$$

Step 3 : Determine the x-intercepts

We find the x-intercepts by finding the points for which the function $g(x) = 0$.

$$g(x) = -x^3 + 6x^2 - 9x + 4$$

Use the factor theorem to confirm that $(x - 1)$ is a factor. If $g(1) = 0$, then $(x - 1)$ is a factor.

$$\begin{aligned} g(x) &= -x^3 + 6x^2 - 9x + 4 \\ g(1) &= -(1)^3 + 6(1)^2 - 9(1) + 4 \\ &= -1 + 6 - 9 + 4 \\ &= 0 \end{aligned}$$

Therefore, $(x - 1)$ is a factor.

If we divide $g(x)$ by $(x - 1)$ we are left with:

$$-x^2 + 5x - 4$$

This has factors

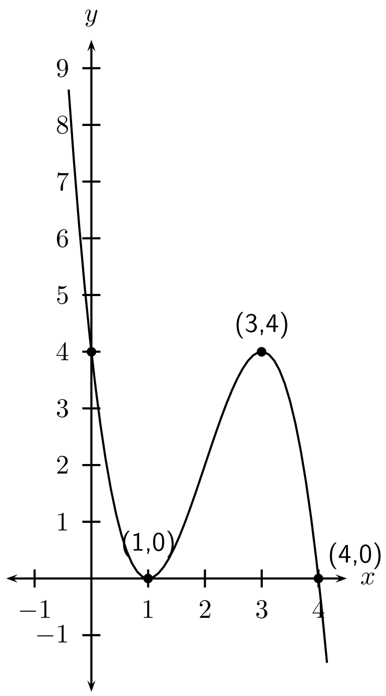
$$-(x - 4)(x - 1)$$

Therefore:

$$g(x) = -(x - 1)(x - 1)(x - 4)$$

The x-intercepts are: $x_{int} = 1, 4$

Step 4 : Draw a neat sketch



?

Exercise: Sketching Graphs

1. Given $f(x) = x^3 + x^2 - 5x + 3$:

   A Show that $(x - 1)$ is a factor of $f(x)$ and hence fatorise $f(x)$ fully.

   B Find the coordinates of the intercepts with the axes and the turning points and sketch the graph

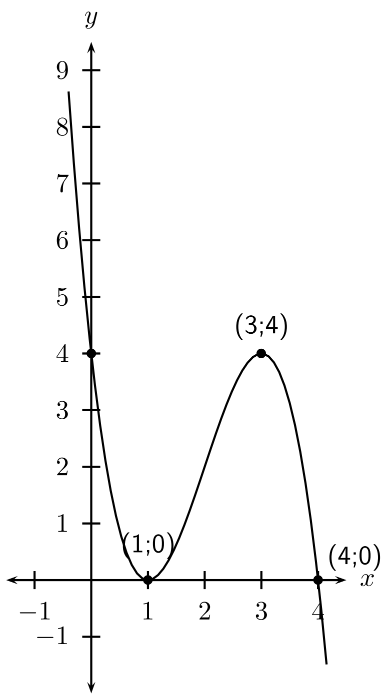
2. Sketch the graph of $f(x) = x^3 - 4x^2 - 11x + 30$ showing all the relative turning points and intercepts with the axes.

3. A Sketch the graph of $f(x) = x^3 - 9x^2 + 24x - 20$, showing all intercepts with the axes and turning points.

   B Find the equation of the tangent to $f(x)$ at $x = 4$.

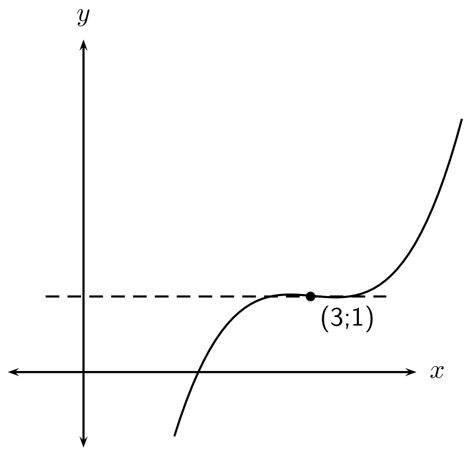
40.5.3 Local minimum, Local maximum and Point of Inflextion

If the derivative ($\frac{dy}{dx}$) is zero at a point, the gradient of the tangent at that point is zero. It means that a turning point occurs as seen in the previous example.



From the drawing the point (1;0) represents a **local minimum** and the point (3;4) the **local maximum**.

A graph has a horizontal **point of inflexion** where the derivative is zero but the sign of the sign of the gradient does not change. That means the graph always increases or always decreases.



From this drawing, the point (3;1) is a horizontal point of inflexion, because the sign of the derivative stays positive.

40.6 Using Differential Calculus to Solve Problems

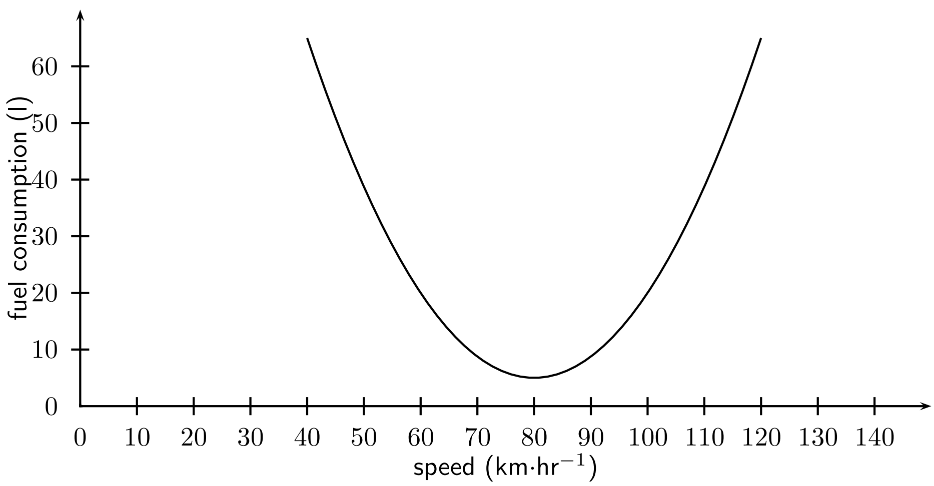
We have seen that differential calculus can be used to determine the stationary points of functions, in order to sketch their graphs. However, determining stationary points also lends itself to the solution of problems that require some variable to be *optimised*.

For example, if fuel used by a car is defined by:

$$f(v) = \frac{3}{80}v^2 - 6v + 245 \tag{40.20}$$

where v is the travelling speed, what is the most economical speed (that means the speed that uses the least fuel)?

If we draw the graph of this function we find that the graph has a minimum. The speed at the minimum would then give the most economical speed.



We have seen that the coordinates of the turning point can be calculated by differentiating the function and finding the x-coordinate (speed in the case of the example) for which the derivative is 0.

Differentiating (40.20), we get:

$$f'(v) = \frac{3}{40}v - 6$$

If we set $f'(v) = 0$ we can calculate the speed that corresponds to the turning point.

$$\begin{aligned} f'(v) &= \frac{3}{40}v - 6 \\ 0 &= \frac{3}{40}v - 6 \\ v &= \frac{6 \times 40}{3} \\ &= 80 \end{aligned}$$

This means that the most economical speed is 80 km·hr^{-1}.

Worked Example 185: Optimisation Problems

Question: The sum of two positive numbers is 10. One of the numbers is multiplied by the square of the other. If each number is greater than 0, find the numbers that make this product a maximum.

Answer

Step 1 : Examine the problem and formulate the equations that are required

Let the two numbers be a and b. Then we have:

$$a + b = 10 \tag{40.21}$$

We are required to minimise the product of a and b. Call the product P. Then:

$$P = a \cdot b \tag{40.22}$$

We can solve for b from (40.21) to get:

$$b = 10 - a \tag{40.23}$$

Substitute this into (40.22) to write P in terms of a only.

$$P = a(10 - a) = 10a - a^2 \tag{40.24}$$

Step 2 : Differentiate

The derivative of (40.24) is:

$$P'(a) = 10 - 2a$$

Step 3 : Find the stationary point

Set $P'(a) = 0$ to find the value of a which makes P a maximum.

$$\begin{aligned} P'(a) &= 10 - 2a \\ 0 &= 10 - 2a \\ 2a &= 10 \\ a &= \frac{10}{2} \\ a &= 5 \end{aligned}$$

Substitute into (40.27) to solve for the width.

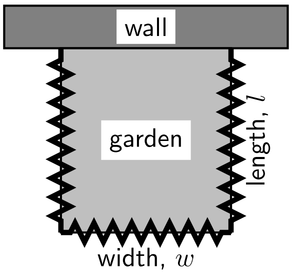
$$\begin{aligned} b &= 10 - a \\ &= 10 - 5 \\ &= 5 \end{aligned}$$

Step 4 : Write the final answer

The product is maximised if a and b are both equal to 5.

Worked Example 186: Optimisation Problems

Question: Michael wants to start a vegetable garden, which he decides to fence off in the shape of a rectangle from the rest of the garden. Michael only has 160 m of fencing, so he decides to use a wall as one border of the vegetable garden. Calculate the width and length of the garden that corresponds to largest possible area that Michael can fence off.



Answer

Step 1 : Examine the problem and formulate the equations that are required

The important pieces of information given are related to the area and modified perimeter of the garden. We know that the area of the garden is:

$$A = w \cdot l \tag{40.25}$$

We are also told that the fence covers only 3 sides and the three sides should add up to 160 m. This can be written as:

$$160 = w + l + l \tag{40.26}$$

However, we can use (40.26) to write w in terms of l:

$$w = 160 - 2l \tag{40.27}$$

Substitute (40.27) into (40.25) to get:

$$A = (160 - 2l)l = 160l - 2l^2 \tag{40.28}$$

Step 2 : Differentiate

Since we are interested in maximising the area, we differentiate (40.28) to get:

$$A'(l) = 160 - 4l$$

Step 3 : Find the stationary point

To find the stationary point, we set $A'(l) = 0$ and solve for the value of l that maximises the area.

$$\begin{aligned} A'(l) &= 160 - 4l \\ 0 &= 160 - 4l \\ \therefore 4l &= 160 \\ l &= \frac{160}{4} \\ l &= 40\,\text{m} \end{aligned}$$

Substitute into (40.27) to solve for the width.

$$\begin{aligned} w &= 160 - 2l \\ &= 160 - 2(40) \\ &= 160 - 80 \\ &= 80\,\text{m} \end{aligned}$$

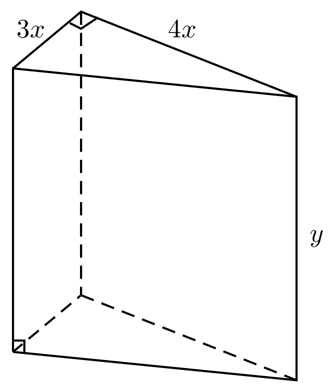
Step 4 : Write the final answer

A width of 80 m and a length of 40 m will yield the maximal area fenced off.

?

Exercise: Solving Optimisation Problems using Differential Calculus

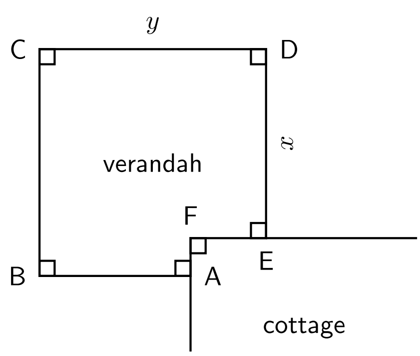
1. The sum of two positive numbers is 20. One of the numbers is multiplied by the square of the other. Find the numbers that make this products a maximum.
2. A wooden block is made as shown in the diagram. The ends are right-angled triangles having sides $3x$, $4x$ and $5x$. The length of the block is y. The total surface area of the block is $3\,600\,\mathrm{cm}^2$.



   A Show that $y = \dfrac{300 - x^2}{x}$.

   B Find the value of x for which the block will have a maximum volume. (Volume = area of base × height.)

3. The diagram shows the plan for a verandah which is to be built on the corner of a cottage. A railing $ABCDE$ is to be constructed around the four edges of the verandah.



   If $AB = DE = x$ and $BC = CD = y$, and the length of the railing must be 30 metres, find the values of x and y for which the verandah will have a maximum area.

40.6.1 Rate of Change problems

Two concepts were discussed in this chapter: **Average** rate of change $= \frac{f(b)-f(a)}{b-a}$ and **Instantaneous** rate of change $= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$. When we mention *rate of change*, the latter is implied. Instantaneous rate of change is the **derivative**. When *Average rate of change* is required, it will be specifically refer to as **average** rate of change.

Velocity is one of the most common forms of rate of change. Again, **average** velocity = **average** rate of change and **instantaneous** velocity = **instantaneous** rate of change = **derivative**. Velocity refers to the increase of distance(s) for a corresponding increade in time (t). The notation commonly used for this is:

$$v(t) = \frac{ds}{dt} = s'(t)$$

Acceleration is the change in velocity for a corersponding increase in time. Therefore, acceleration is the derivative of velocity

$$a(t) = v'(t)$$

This implies that acceleration is the second derivative of the distance(s).

Worked Example 187: Rate of Change

Question: The height (in metres) of a golf ball that is hit into the air after t seconds, is given by $h(t) = 20t = 5t^2$. Determine

1. the average velocity of the ball during the first two seconds
2. the velocity of the ball after 1,5 seconds
3. when the velocity is zero
4. the velocity at which the ball hits the ground
5. the acceleration of the ball

Answer
Step 1 : Average velocity

$$\begin{aligned} Ave\ velocity &= \frac{h(2)-h(0)}{2-0} \\ &= \frac{[20(2)-5(2)^2]-[20(0)-5(0)^2]}{2} \\ &= \frac{40-20}{2} \\ &= 10\ ms^{-1} \end{aligned}$$

Step 2 : Instantaneous Velocity

$$\begin{aligned} v(t) &= \frac{dh}{dt} \\ &= 20-10t \end{aligned}$$

Velocity after 1,5 seconds:

$$\begin{aligned} v(1{,}5) &= 20-10(1{,}5) \\ &= 5\ ms^{-1} \end{aligned}$$

Step 3 : Zero velocity

$$\begin{aligned} v(t) &= 0 \\ 20 - 10t &= 0 \\ 10t &= 20 \\ t &= 2 \end{aligned}$$

Therefore the velocity is zero after 2 seconds
Step 4 : Ground velocity
The ball hits the ground when $h(t) = 0$

$$\begin{aligned} 20t - 5t^2 &= 0 \\ 5t(4 - t) &= 0 \\ t = 0 \quad &or \quad t = 4 \end{aligned}$$

The ball hits the ground after 4 seconds. The velocity after 4 seconds will be:

$$\begin{aligned} v(4) &= h'(4) \\ &= 20 - 10(4) \\ &= 20\ ms^{-1} \end{aligned}$$

The ball hits the gound at a speed of $20ms^{-1}$
Step 5 : Acceleration

$$\begin{aligned} a &= v'(t) \\ &= -10\ ms^{-1} \end{aligned}$$

40.7 End of Chapter Exercises

1. Determine $f'(x)$ from **first principles** if:

$$f(x) = x^2 - 6x$$
$$f(x) = 2x - x^2$$

2. Given: $f(x) = -x^2 + 3x$, find $f'(x)$ using *first principles*.

3. Determine $\frac{dx}{dy}$ if:

   A
   $$y = (2x)^2 - \frac{1}{3x}$$

   B
   $$y = \frac{2\sqrt{x} - 5}{\sqrt{x}}$$

4. Given: $f(x) = x^3 - 3x^2 + 4$

   A Calculate $f(-1)$, and hence solve the equation$f(x) = 0$

   B Determine $f'(x)$

   C Sketch the graph of f neatly and clearly, showing the co-ordinates of the turning points as well as the intercepts on both axes.

D Determine the co-ordinates of the points on the graph of f where the gradient is 9.

5. Given: $f(x) = 2x^3 - 5x^2 - 4x + 3$. The x-intercepts of f are: (-1;0) ($\frac{1}{2}$;0) and (3;0).

   A Determine the co-ordinates of the turning points of f.

   B Draw a neat sketch graph of f. Clearly indicate the co-ordinates of the intercepts with the axes, as well as the co-ordinates of the turning points.

   C For which values of k will the equation $f(x) = k$, have exactly two real roots?

   D Determine the equation of the tangent to the graph of $f(x) = 2x^3 - 5x^2 - 4x + 3$ at the point where $x = 1$.

6. A Sketch the graph of $f(x) = x^3 - 9x^2 + 24x - 20$, showing all intercepts with the axes and turning points.

   B Find the equation of the tangent to $f(x)$ at $x = 4$.

7. Calculate:

$$\lim_{x\to 1} \frac{1 - x^3}{1 - x}$$

8. Given:

$$f(x) = 2x^2 - x$$

   A Use the definition of the derivative to calculate $f'(x)$.

   B Hence, calculate the co-ordinates of the point at which the gradient of the tangent to the graph of f is 7.

9. If $xy - 5 = \sqrt{x^3}$, determine $\frac{dx}{dy}$

10. Given: $g(x) = (x^{-2} + x^2)^2$. Calculate $g'(2)$.

11. Given: $f(x) = 2x - 3$

   A Find: $f^{-1}(x)$

   B Solve: $f^{-1}(x) = 3f'(x)$

12. Find $f'(x)$ for each of the following:

   A $f(x) = \dfrac{\sqrt[5]{x^3}}{3} + 10$

   B $f(x) = \dfrac{(2x^2 - 5)(3x + 2)}{x^2}$

13. Determine the minimum value of the sum of a *positive* number and its reciprocal.

14. If the displacement s (in metres) of a particle at time t (in seconds) is governed by the equation $s = \frac{1}{2}t^3 - 2t$, find its acceleration after 2 seconds. (Acceleration is the rate of change of velocity, and velocity is the rate of change of displacement.)

15. A After doing some research, a transport company has determined that the rate at which petrol is consumed by one of its large carriers, travelling at an average speed of x km per hour, is given by:

$$P(x) = \frac{55}{2x} + \frac{x}{200} \quad \text{litres per kilometre}$$

   i. Assume that the petrol costs R4,00 per litre and the driver earns R18,00 per hour (travelling time). Now deduce that the total cost, C, in Rands, for a 2 000 km trip is given by:

$$C(x) = \frac{256000}{x} + 40x$$

   ii. Hence determine the average speed to be maintained to effect a minimum cost for a 2 000 km trip.

B During an experiment the temperature T (in degrees Celsius), varies with time t (in hours), according to the formula:

$$T(t) = 30 + 4t - \frac{1}{2}t^2 \quad t \in [1; 10]$$

i. Determine an expression for the rate of change of temperature with time.

ii. During which time interval was the temperature dropping?

16. The depth, d, of water in a kettle t minutes after it starts to boil, is given by $d = 86 - \frac{1}{8}t - \frac{1}{4}t^3$, where d is measured in millimetres.

A How many millimetres of water are there in the kettle just before it starts to boil?

B As the water boils, the level in the kettle drops. Find the *rate* at which the water level is decreasing when $t = 2$ minutes.

C How many minutes after the kettle starts boiling will the water level be dropping at a rate of $12\frac{1}{8}$ mm/minute?

Chapter 41

Linear Programming - Grade 12

41.1 Introduction

In Grade 11 you were introduced to linear programming and solved problems by looking at points on the edges of the feasible region. In Grade 12 you will look at how to solve linear programming problems in a more general manner.

41.2 Terminology

Here is a recap of some of the important concepts in linear programming.

41.2.1 Feasible Region and Points

Constraints mean that we cannot just take any x and y when looking for the x and y that optimise our objective function. If we think of the variables x and y as a point (x,y) in the xy-plane then we call the set of all points in the xy-plane that satisfy our constraints the **feasible region**. Any point in the feasible region is called a **feasible point**.

For example, the constraints

$$x \geq 0$$
$$y \geq 0$$

mean that every (x,y) we can consider must lie in the first quadrant of the xy plane. The constraint

$$x \geq y$$

means that every (x,y) must lie on or below the line $y = x$ and the constraint

$$x \leq 20$$

means that x must lie on or to the left of the line $x = 20$.

We can use these constraints to draw the feasible region as shown by the shaded region in Figure 41.1.

Important: The constraints are used to create bounds of the solution.

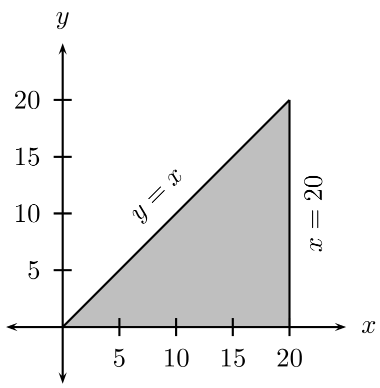


Figure 41.1: The feasible region corresponding to the constraints $x \geq 0$, $y \geq 0$, $x \geq y$ and $x \leq 20$.

Important:

$ax + by = c$	If $b \neq 0$, feasible points must lie *on* the line $y = -\frac{a}{b}x + \frac{c}{b}$ If $b = 0$, feasible points must lie *on* the line $x = c/a$
$ax + by \leq c$	If $b \neq 0$, feasible points must lie *on* or *below* the line $y = -\frac{a}{b}x + \frac{c}{b}$. If $b = 0$, feasible points must lie *on* or *to the left of* the line $x = c/a$.

When a constraint is linear, it means that it requires that any feasible point (x,y) lies on one side of or on a line. Interpreting constraints as graphs in the xy plane is very important since it allows us to construct the feasible region such as in Figure 41.1.

41.3 Linear Programming and the Feasible Region

If the objective function and all of the constraints are linear then we call the problem of optimising the objective function subject to these constraints a **linear program**. All optimisation problems we will look at will be linear programs.

The major consequence of the constraints being linear is that *the feasible region is always a polygon.* This is evident since the constraints that define the feasible region all contribute a line segment to its boundary (see Figure 41.1). It is also always true that the feasible region is a convex polygon.

The objective function being linear means that *the feasible point(s) that gives the solution of a linear program always lies on one of the vertices of the feasible region.* This is very important since, as we will soon see, it gives us a way of solving linear programs.

We will now see why the solutions of a linear program always lie on the vertices of the feasible region. Firstly, note that if we think of $f(x,y)$ as lying on the z axis, then the function $f(x,y) = ax + by$ (where a and b are real numbers) is the definition of a plane. If we solve for y in the equation defining the objective function then

$$\begin{aligned} f(x,y) &= ax + by \\ \therefore \quad y &= \frac{-a}{b}x + \frac{f(x,y)}{b} \end{aligned} \tag{41.1}$$

What this means is that if we find all the points where $f(x,y) = c$ for any real number c (i.e. $f(x,y)$ is constant with a value of c), then we have the equation of a line. This line we call a **level line** of the objective function.

Consider again the feasible region described in Figure 41.1. Lets say that we have the objective function $f(x,y) = x - 2y$ with this feasible region. If we consider Equation **??** corresponding to

$$f(x,y) = -20$$

then we get the level line

$$y = \frac{1}{2}x + 10$$

which has been drawn in Figure 41.2. Level lines corresponding to

$$\begin{aligned} f(x,y) = -10 \quad &\text{or} \quad y = \frac{x}{2} + 5 \\ f(x,y) = 0 \quad &\text{or} \quad y = \frac{x}{2} \\ f(x,y) = 10 \quad &\text{or} \quad y = \frac{x}{2} - 5 \\ f(x,y) = 20 \quad &\text{or} \quad y = \frac{x}{2} - 10 \end{aligned}$$

have also been drawn in. It is very important to realise that these are not the only level lines; in fact, there are infinitely many of them and they are *all parallel to each other*. Remember that if we look at any one level line $f(x,y)$ has the *same* value for every point (x,y) that lies on that line. Also, $f(x,y)$ will always have different values on different level lines.

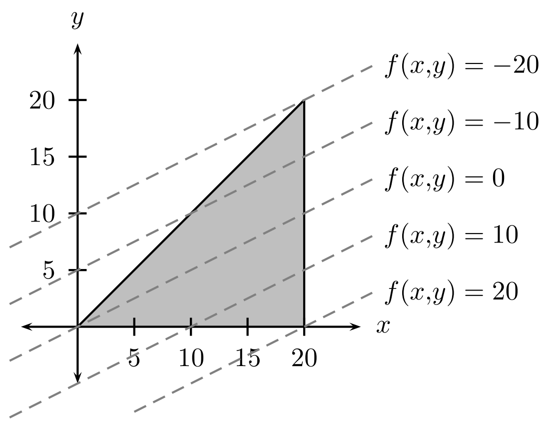


Figure 41.2: The feasible region corresponding to the constraints $x \geq 0$, $y \geq 0$, $x \geq y$ and $x \leq 20$ with objective function $f(x,y) = x - 2y$. The dashed lines represent various level lines of $f(x,y)$.

If a ruler is placed on the level line corresponding to $f(x,y) = -20$ in Figure 41.2 and moved down the page parallel to this line then it is clear that the ruler will be moving over level lines which correspond to *larger* values of $f(x,y)$. So if we wanted to maximise $f(x,y)$ then we simply move the ruler down the page until we reach the "lowest" point in the feasible region. This point will then be the feasible point that maximises $f(x,y)$. Similarly, if we wanted to minimise $f(x,y)$ then the "highest" feasible point will give the minimum value of $f(x,y)$.

Since our feasible region is a polygon, these points *will always lie on vertices in the feasible region*. The fact that the value of our objective function along the line of the ruler increases as we move it down and decreases as we move it up depends on this particular example. Some other examples might have that the function increases as we move the ruler up and decreases as we move it down.

It is a general property, though, of linear objective functions that they will consistently increase or decrease as we move the ruler up or down. Knowing which direction to move the ruler in order to maximise/minimise $f(x,y) = ax + by$ is as simple as looking at the sign of b (i.e. "is b negative, positive or zero?"). If b is *positive*, then $f(x,y)$ *increases* as we move the ruler up and $f(x,y)$ *decreases* as we move the ruler down. The opposite happens for the case when b is negative: $f(x,y)$ *decreases* as we move the ruler up and $f(x,y)$ *increases* as we move the ruler down. If $b = 0$ then we need to look at the sign of a.

If a is positive then $f(x,y)$ increases as we move the ruler to the right and decreases if we move the ruler to the left. Once again, the opposite happens for a negative. If we look again at the objective function mentioned earlier,

$$f(x,y) = x - 2y$$

with $a = 1$ and $b = -2$, then we should find that $f(x,y)$ increases as we move the ruler down the page since $b = -2 < 0$. This is exactly what we found happening in Figure 41.2.

The main points about linear programming we have encountered so far are

- The feasible region is always a polygon.
- Solutions occur at vertices of the feasible region.
- Moving a ruler parallel to the level lines of the objective function up/down to the top/bottom of the feasible region shows us which of the vertices is the solution.
- The direction in which to move the ruler is determined by the sign of b and also possibly by the sign of a.

These points are sufficient to determine a method for solving any linear program.

Method: Linear Programming

If we wish to maximise the objective function $f(x,y)$ then:

1. Find the gradient of the level lines of $f(x,y)$ (this is always going to be $-\frac{a}{b}$ as we saw in Equation **??**)
2. Place your ruler on the xy plane, making a line with gradient $-\frac{a}{b}$ (i.e. b units on the x-axis and $-a$ units on the y-axis)
3. The solution of the linear program is given by appropriately moving the ruler. Firstly we need to check whether b is negative, positive or zero.

   A If $b > 0$, move the ruler up the page, keeping the ruler parallel to the level lines all the time, until it touches the "highest" point in the feasible region. This point is then the solution.

   B If $b < 0$, move the ruler in the opposite direction to get the solution at the "lowest" point in the feasible region.

   C If $b = 0$, check the sign of a

      i. If $a < 0$ move the ruler to the "leftmost" feasible point. This point is then the solution.

      ii. If $a > 0$ move the ruler to the "rightmost" feasible point. This point is then the solution.

Worked Example 188: Prizes!

Question: As part of their opening specials, a furniture store has promised to give away at least 40 prizes with a total value of at least R2 000. The prizes are kettles and toasters.

1. If the company decides that there will be at least 10 of each prize, write down two more inequalities from these constraints.
2. If the cost of manufacturing a kettle is R60 and a toaster is R50, write down an objective function C which can be used to determine the cost to the company of both kettles and toasters.

3. Sketch the graph of the feasibility region that can be used to determine all the possible combinations of kettles and toasters that honour the promises of the company.
4. How many of each prize will represent the cheapest option for the company?
5. How much will this combination of kettles and toasters cost?

Answer
Step 1 : Identify the decision variables
Let the number of kettles be x_k and the number of toasters be y_t and write down two constraints apart from $x_k \geq 0$ and $y_t \geq 0$ that must be adhered to.
Step 2 : Write constraint equations
Since there will be at least 10 of each prize we can write:

$$x_k \geq 10$$

and

$$y_t \geq 10$$

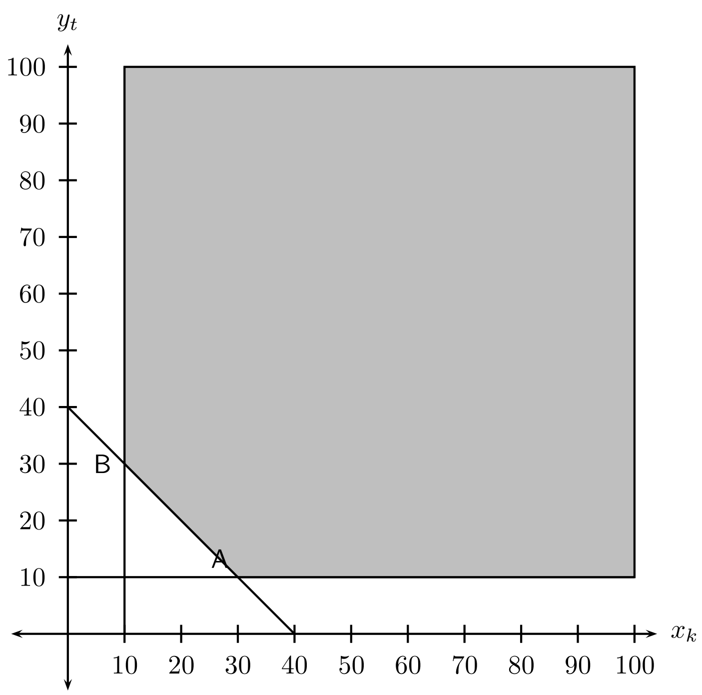
Also the store has promised to give away at least 40 prizes in total. Therefore:

$$x_k + y_t \geq 40$$

Step 3 : Write the objective function
The cost of manufacturing a kettle is R60 and a toaster is R50. Therefore the cost the total cost C is:

$$C = 60x_k + 50y_t$$

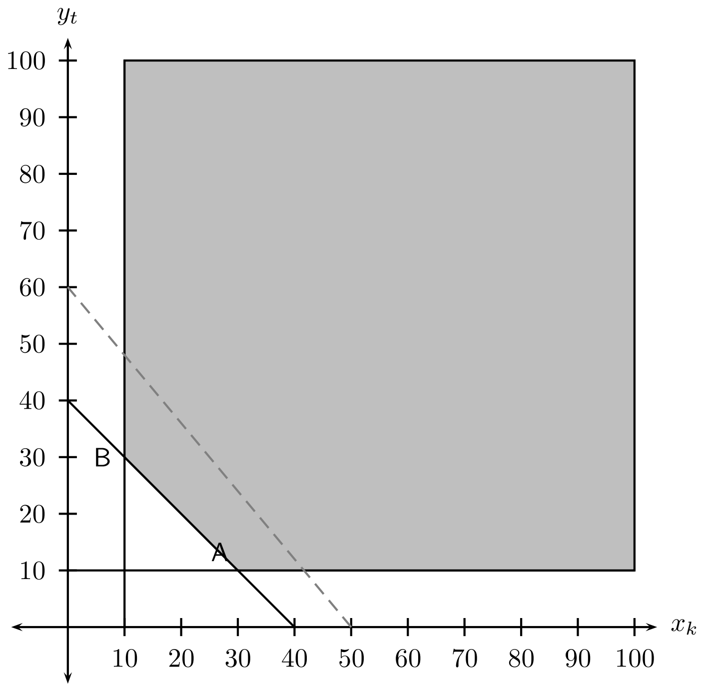
Step 4 : Sketch the graph of the feasible region



Step 5 : Determine vertices of feasible region
From the graph, the coordinates of vertex A is (3,1) and the coordinates of vertex B are (1,3).
Step 6 : Draw in the search line
The seach line is the gradient of the objective function. That is, if the equation $C = 60x + 50y$ is now written in the standard form $y = ...$, then the gradient is:

$$m = -\frac{6}{5},$$

which is shown with the broken line on the graph.



Step 7 : Calculate cost at each vertex

At vertex A, the cost is:

$$\begin{aligned} C &= 60x_k + 50y_t \\ &= 60(30) + 50(10) \\ &= 1800 + 500 \\ &= 2300 \end{aligned}$$

At vertex B, the cost is:

$$\begin{aligned} C &= 60x_k + 50y_t \\ &= 60(10) + 50(30) \\ &= 600 + 1500 \\ &= 2100 \end{aligned}$$

Step 8 : Write the final answer

The cheapest combination of prizes is 10 kettles and 30 toasters, costing the company R2 100.

Worked Example 189: Search Line Method

Question: As a production planner at a factory manufacturing lawn cutters your job will be to advise the management on how many of each model should be produced per week in order to maximise the profit on the local production. The factory is producing two types of lawn cutters: Quadrant and Pentagon. Two of the production processes that the lawn cutters must go through are: bodywork and engine work.

- The factory cannot operate for less than 360 hours on engine work for the lawn cutters.
- The factory has a maximum capacity of 480 hours for bodywork for the lawn cutters.

- Half an hour of engine work and half an hour of bodywork is required to produce one Quadrant.
- One third of an hour of engine work andone fifth of an hour of bodywork is required to produce one Pentagon.
- The ratio of Pentagon lawn cutters to Quadrant lawn cutters produced per week must be at least 3:2.
- A minimum of 200 Quadrant lawn cutters must be produced per week.

Let the number of Quadrant lawn cutters manufactured in a week be x.
Let the number of Pentagon lawn cutters manufactured in a week be y.
Two of the constraints are:

$$x \geq 200$$

$$3x + 2y \geq 2\ 160$$

1. Write down the remaining constraints in terms of x and y to represent the abovementioned information.
2. Use graph paper to represent the constraints graphically.
3. Clearly indicate the feasible region by shading it.
4. If the profit on one Quadrant lawn cutter is R1 200 and the profit on one Pentagon lawn cutter is R400, write down an equation that will represent the profit on the lawn cutters.
5. Using a search line and your graph, determine the number of Quadrant and Pentagon lawn cutters that will yield a maximum profit.
6. Determine the maximum profit per week.

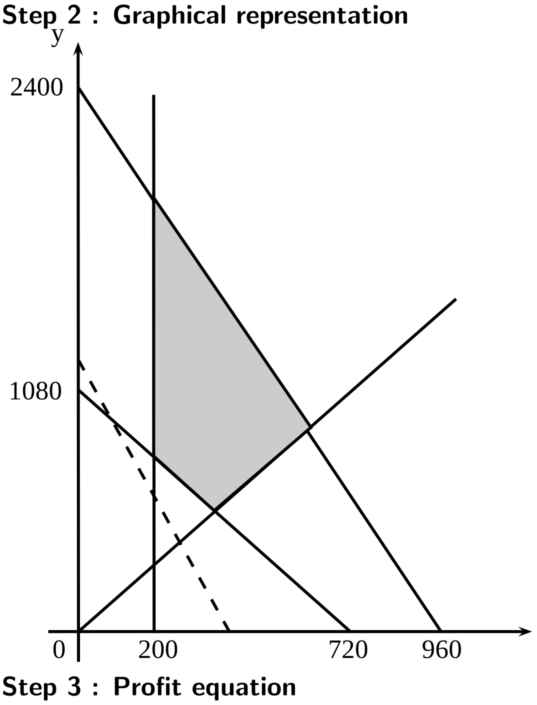
Answer

Step 1 : Remaining constraints:

$$\frac{1}{2}x + \frac{1}{5} \leq 480$$

$$\frac{y}{x} \geq \frac{3}{2}$$

Step 2 : Graphical representation



Step 3 : Profit equation

$$P = 1\ 200x + 400y$$

Step 4 : Maximum profit

$$P = 1\ 200(600) + 400(900)$$

$$P = R1\ 080\ 000$$

41.4 End of Chapter Exercises

1. Polkadots is a small company that makes two types of cards, type X and type Y. With the available labour and material, the company can make not more than 150 cards of type X and not more than 120 cards of type Y per week. Altogether they cannot make more than 200 cards per week.

   There is an order for at least 40 type X cards and 10 type Y cards per week. Polkadots makes a profit of R5 for each type X card sold and R10 for each type Y card.

   Let the number of type X cards be x and the number of type Y cards be y, manufactured per week.

   A One of the constraint inequalities which represents the restrictions above is $x \leq 150$. Write the other constraint inequalities.

   B Represent the constraints graphically and shade the feasible region.

   C Write the equation that represents the profit P (the objective function), in terms of x and y.

   D On your graph, draw a straight line which will help you to determine how many of each type must be made weekly to produce the maximum P

   E Calculate the maximum weekly profit.

2. A brickworks produces "face bricks" and "clinkers". Both types of bricks are produced and sold in batches of a thousand. Face bricks are sold at R150 per thousand, and clinkers at R100 per thousand, where an income of at least R9,000 per month is required to cover costs. The brickworks is able to produce at most 40,000 face bricks and 90,000 clinkers per month, and has transport facilities to deliver at most 100,000 bricks per month. The number of clinkers produced must be at least the same number of face bricks produced.

   Let the number of face bricks *in thousands* be x, and the number of clinkers *in thousands* be y.

   A List all the constraints.

   B Graph the feasible region.

   C If the sale of face bricks yields a profit of R25 per thousand and clinkers R45 per thousand, use your graph to determine the maximum profit.

   D If the profit margins on face bricks and clinkers are interchanged, use your graph to determine the maximum profit.

3. A small cell phone company makes two types of cell phones: *Easyhear* and *Longtalk*. Production figures are checked weekly. At most, 42 *Easyhear* and 60 *Longtalk* phones can be manufactured each week. At least 30 cell phones must be produced each week to cover costs. In order not to flood the market, the number of *Easyhear* phones cannot be more than twice the number of *Longtalk* phones. It takes $\frac{2}{3}$ hour to assemble an *Easyhear* phone and $\frac{1}{2}$ hour to put together a *Longtalk* phone. The trade unions only allow for a 50-hour week.
   Let x be the number of *Easyhear* phones and y be the number of *Longtalk* phones manufactured each week.

   A Two of the constraints are:

   $$0 \leq x \leq 42 \quad \text{and} \quad 0 \leq y \leq 60$$

   Write down the other three constraints.

B Draw a graph to represent the feasible region

C If the profit on an *Easyhear* phone is R225 and the profit on a *Longtalk* is R75, determine the maximum profit per week.

4. *Hair for Africa* is a firm that specialises in making two kinds of up-market shampoo, *Glowhair* and *Longcurls*. They must produce at least two cases of *Glowhair* and one case of *Longcurls* per day to stay in the market. Due to a limited supply of chemicals, they cannot produce more than 8 cases of *Glowhair* and 6 cases of *Longcurls* per day. It takes half-an-hour to produce one case of *Glowhair* and one hour to produce a case of *Longcurls*, and due to restrictions by the unions, the plant may operate for at most 7 hours per day. The workforce at *Hair for Africa*, which is still in training, can only produce a maximum of 10 cases of shampoo per day.

   Let x be the number of cases of *Glowhair* and y the number of cases of *Longcurls* produced per day.

   A Write down the inequalities that represent all the constraints.

   B Sketch the feasible region.

   C If the profit on a case of *Glowhair* is R400 and the profit on a case of *Longcurls* is R300, determine the maximum profit that *Hair for Africa* can make per day.

5. A transport contracter has 6 5-ton trucks and 8 3-ton trucks. He must deliver at least 120 tons of sand per day to a construction site, but he may not deliver more than 180 tons per day. The 5-ton trucks can each make three trips per day at a cost of R30 per trip, and the 3-ton trucks can each make four trips per day at a cost of R120 per trip. How must the contracter utilise his trucks so that he has minimum expense ?

Chapter 42

Geometry - Grade 12

42.1 Introduction

Activity :: Discussion : Discuss these Research Topics
Research one of the following geometrical ideas and describe it to your group:

1. taxicab geometry,
2. sperical geometry,
3. fractals,
4. the Koch snowflake.

42.2 Circle Geometry

42.2.1 Terminology

The following is a recap of terms that are regularly used when referring to circles.

arc An arc is a part of the circumference of a circle.

chord A chord is defined as a straight line joining the ends of an arc.

radius The radius, r, is the distance from the centre of the circle to any point on the circumference.

diameter The diameter, $\varnothing$, is a special chord that passes through the centre of the circle. The diameter is the straight line from a point on the circumference to another point on the circumference, that passes through the centre of the circle.

segment A segment is the part of the circle that is cut off by a chord. A chord divides a circle into two segments.

tangent A tangent is a line that makes contact with a circle at one point on the circumference. (AB is a tangent to the circle at point P.

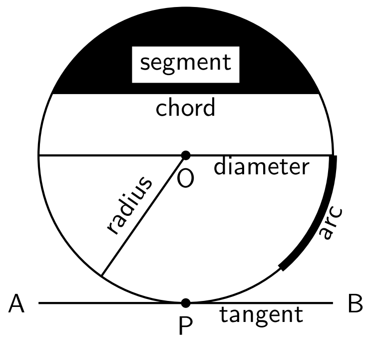


Figure 42.1: Parts of a Circle

42.2.2 Axioms

An axiom is an established or accepted principle. For this section, the following are accepted as axioms.

1. The Theorem of Pythagoras, which states that the square on the hypotenuse of a right-angled triangle is equal to the sum of the squares on the other two sides. In $\triangle ABC$, this means that $AB^2 + BC^2 = AC^2$

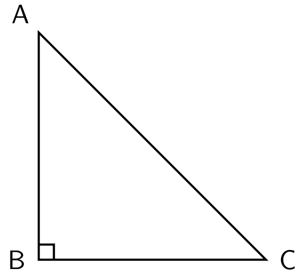


Figure 42.2: A right-angled triangle

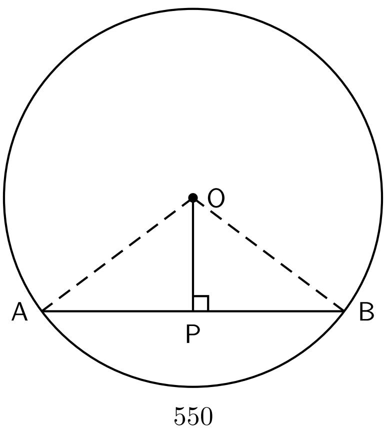
2. A tangent is perpendicular to the radius, drawn at the point of contact with the circle.

42.2.3 Theorems of the Geometry of Circles

A theorem is a general proposition that is not self-evident but is proved by reasoning (these proofs need not be learned for examination purposes).

Theorem 6. *The line drawn from the centre of a circle, perpendicular to a chord, bisects the chord.*

Proof:



Consider a circle, with centre O. Draw a chord AB and draw a perpendicular line from the centre of the circle to intersect the chord at point P.

The aim is to prove that $AP = BP$

1. $\triangle OAP$ and $\triangle OBP$ are right-angled triangles.
2. $OA = OB$ as both of these are radii and OP is common to both triangles.

Apply the Theorem of Pythagoras to each triangle, to get:

$$\begin{aligned} OA^2 &= OP^2 + AP^2 \\ OB^2 &= OP^2 + BP^2 \end{aligned}$$

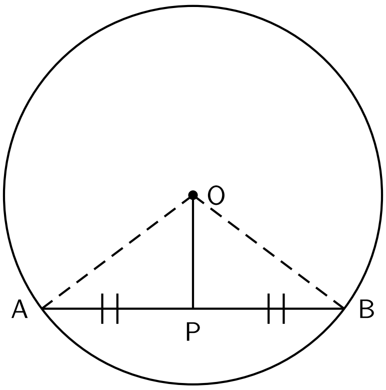
However, $OA = OB$. So,

$$\begin{aligned} OP^2 + AP^2 &= OP^2 + BP^2 \\ \therefore AP^2 &= BP^2 \\ \text{and } AP &= BP \end{aligned}$$

This means that OP bisects AB.

Theorem 7. *The line drawn from the centre of a circle, that bisects a chord, is perpendicular to the chord.*

Proof:



Consider a circle, with centre O. Draw a chord AB and draw a line from the centre of the circle to bisect the chord at point P.

The aim is to prove that $OP \perp AB$

In $\triangle OAP$ and $\triangle OBP$,

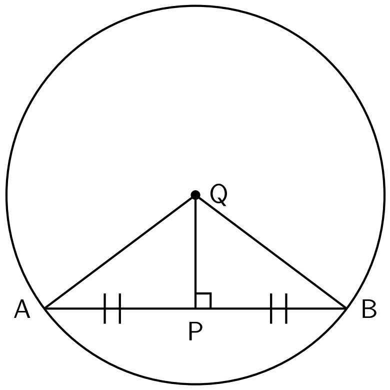
1. $AP = PB$ (given)
2. $OA = OB$ (radii)
3. OP is common to both triangles.

$\therefore \triangle OAP \equiv \triangle OBP$ (SSS).

$$\begin{aligned} O\hat{A}P &= O\hat{B}P \\ O\hat{A}P + O\hat{B}P &= 180^\circ \quad (APB \text{ is a str. line}) \\ \therefore O\hat{A}P &= O\hat{B}P = 90^\circ \\ \therefore OP &\perp AB \end{aligned}$$

Theorem 8. *The perpendicular bisector of a chord passes through the centre of the circle.*

Proof:



Consider a circle. Draw a chord AB. Draw a line PQ perpendicular to AB such that PQ bisects AB at point P. Draw lines AQ and BQ.

The aim is to prove that Q is the centre of the circle, by showing that $AQ = BQ$.

In $\triangle OAP$ and $\triangle OBP$,

1. $AP = PB$ (given)

2. $\angle QPA = \angle QPB$ ($QP \perp AB$)

3. QP is common to both triangles.

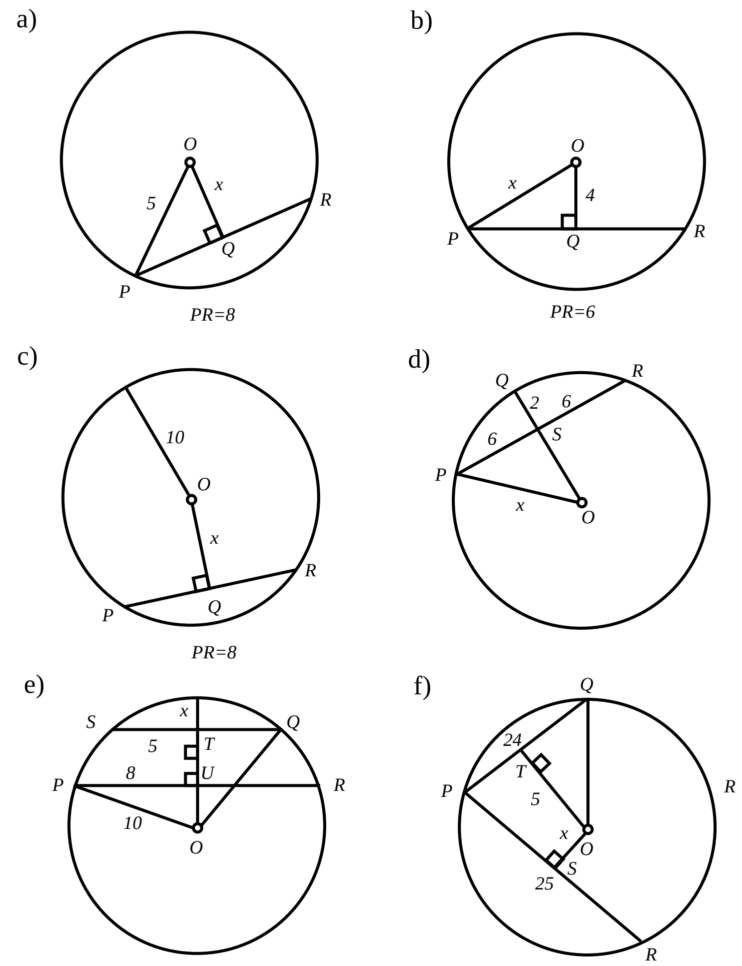
$\therefore \triangle QAP \equiv \triangle QBP$ (SAS).

From this, $QA = QB$. Since the centre of a circle is the only point inside a circle that has points on the circumference at an equal distance from it, Q must be the centre of the circle.

?

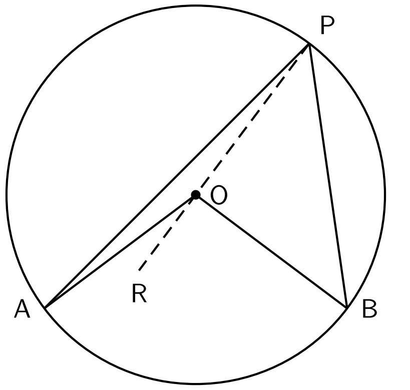
Exercise: Circles I

1. Find the value of x:



Theorem 9. *The angle subtended by an arc at the centre of a circle is double the size of the angle subtended by the same arc at the circumference of the circle.*

Proof:



Consider a circle, with centre O and with A and B on the circumference. Draw a chord AB. Draw radii OA and OB. Select any point P on the circumference of the circle. Draw lines PA and PB. Draw PO and extend to R.

The aim is to prove that $A\hat{O}B = 2 \cdot A\hat{P}B$**.**

$A\hat{O}R = P\hat{A}O + A\hat{P}O$ (exterior angle = sum of interior opp. angles)

But, $P\hat{A}O = A\hat{P}O$ ($\triangle AOP$ is an isosceles $\triangle$)

$\therefore A\hat{O}R = 2A\hat{P}O$

Similarly, $B\hat{O}R = 2B\hat{P}O$.

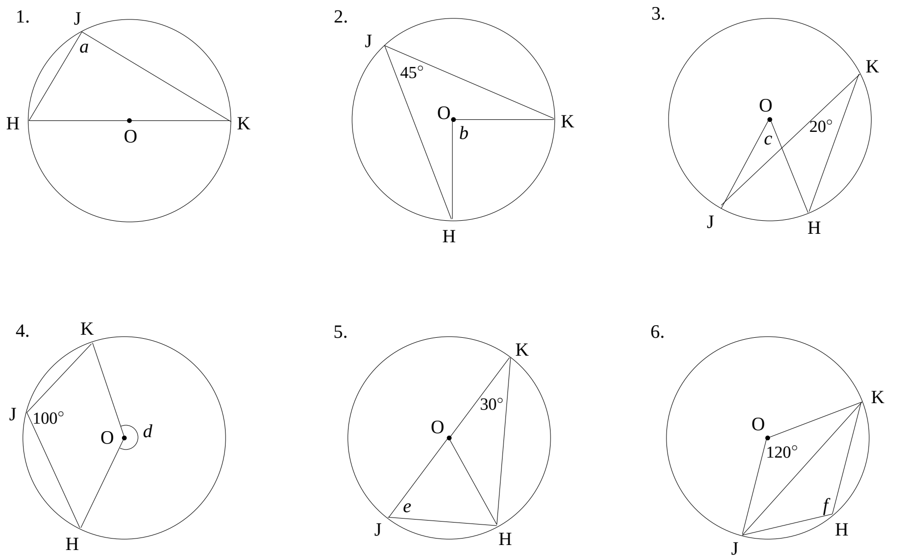
So,

$$\begin{aligned} A\hat{O}B &= A\hat{O}R + B\hat{O}R \\ &= 2A\hat{P}O + 2B\hat{P}O \\ &= 2(A\hat{P}O + B\hat{P}O) \\ &= 2(A\hat{P}B) \end{aligned}$$

?

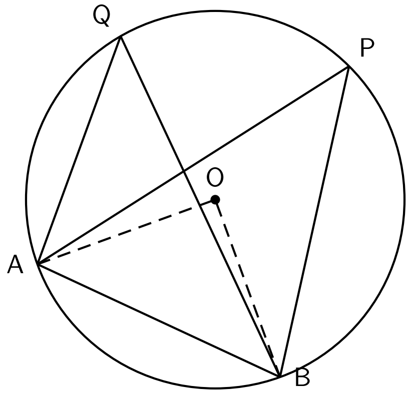
Exercise: Circles II

1. Find the angles (a to f) indicated in each diagram:



Theorem 10. *The angles subtended by a chord at the circumference of a circle on the same side of the chord are equal.*

Proof:



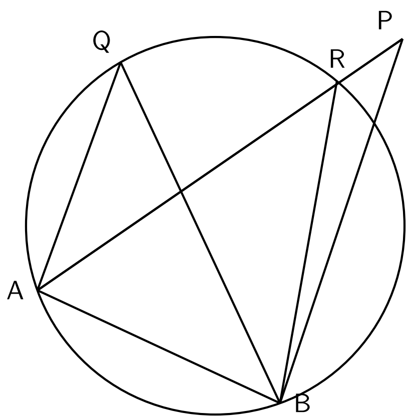
Consider a circle, with centre O. Draw a chord AB. Select any points P and Q on the circumference of the circle, such that both P and Q are on the same side of the chord. Draw lines PA, PB, QA and QB.

The aim is to prove that $A\hat{Q}B = A\hat{P}B$.

$$\begin{aligned} A\hat{O}B &= 2A\hat{Q}B \text{ } \angle \text{ at centre} = \text{twice } \angle \text{ at circumference} \\ \text{and } A\hat{O}B &= 2A\hat{P}B \text{ } \angle \text{ at centre} = \text{twice } \angle \text{ at circumference} \\ \therefore 2A\hat{Q}B &= 2A\hat{P}B \\ \therefore A\hat{Q}B &= A\hat{P}B \end{aligned}$$

Theorem 11. *(Converse of Theorem 10) If a line segment subtends equal angles at two other points on the same side of the line, then these four points lie on a circle.*

Proof:



Consider a line segment AB, that subtends equal angles at points P and Q on the same side of AB.

The aim is to prove that points A, B, P and Q lie on the circumference of a circle.

By contradiction. Assume that point P does not lie on a circle drawn through points A, B and Q. Let the circle cut AP (or AP extended) at point R.

$$\begin{aligned} A\hat{Q}B &= A\hat{R}B \text{ } \angle\text{s on same side of chord} \\ \text{but } A\hat{Q}B &= A\hat{P}B \text{ (given)} \\ \therefore A\hat{R}B &= A\hat{P}B \\ \text{but this cannot be true since } A\hat{R}B &= A\hat{P}B + R\hat{B}P \text{(ext. } \angle \text{ of } \triangle) \end{aligned}$$

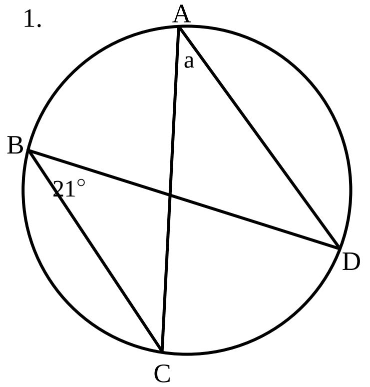
$\therefore$ the assumption that the circle does not pass through P, must be false, and A, B, P and Q lie on the circumference of a circle.

?

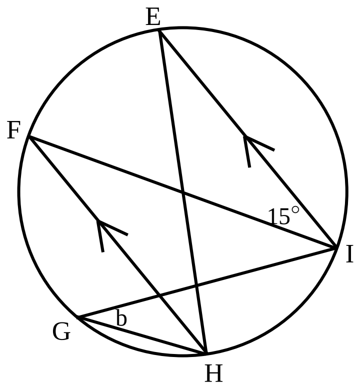
Exercise: Circles III

1. Find the values of the unknown letters.

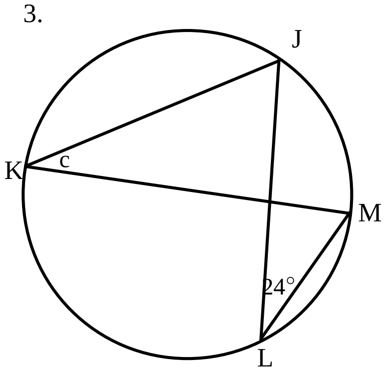
1.



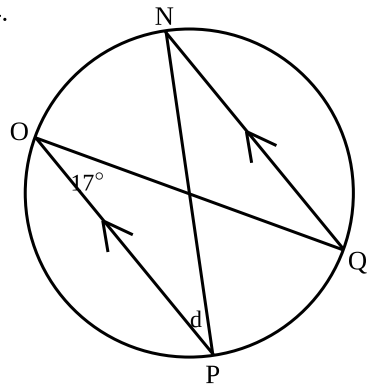
2.



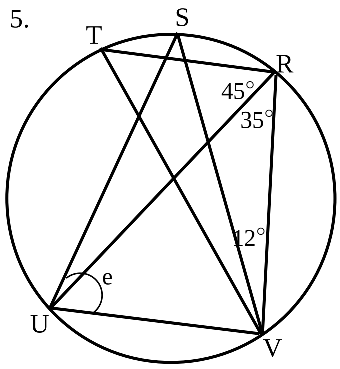
3.



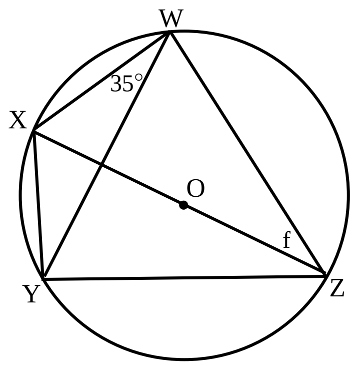
4.



5.



6.

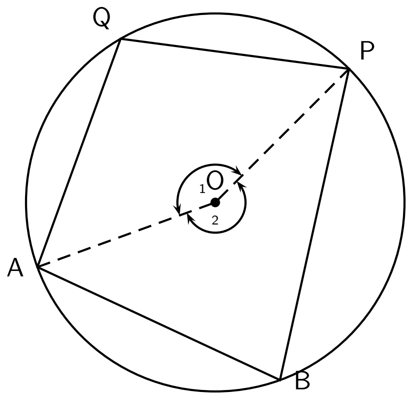


Cyclic Quadrilaterals

Cyclic quadrilaterals are quadrilaterals with all four vertices lying on the circumference of a circle. The vertices of a cyclic quadrilateral are said to be *concyclic*.

Theorem 12. *The opposite angles of a cyclic quadrilateral are supplementary.*

Proof:



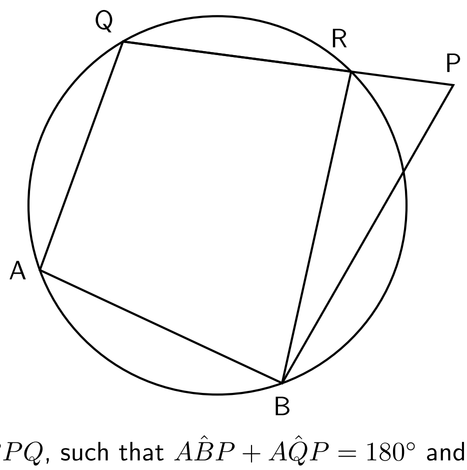
Consider a circle, with centre O. Draw a cyclic quadrilateral $ABPQ$. Draw AO and PO.

The aim is to prove that $A\hat{B}P + A\hat{Q}P = 180°$ and $Q\hat{A}B + Q\hat{P}B = 180°$.

$$\begin{aligned}
\hat{O}_1 &= 2A\hat{B}P \ \angle\text{s at centre} \\
\hat{O}_2 &= 2A\hat{Q}P \ \angle\text{s at centre} \\
\text{But, } \hat{O}_1 + \hat{O}_2 &= 360° \\
\therefore 2A\hat{B}P + 2A\hat{Q}P &= 360° \\
\therefore A\hat{B}P + A\hat{Q}P &= 180° \\
\text{Similarly, } Q\hat{A}B + Q\hat{P}B &= 180°
\end{aligned}$$

Theorem 13. *(Converse of Theorem 12) If the opposite angles of a quadrilateral are supplementary, then the quadrilateral is cyclic.*

Proof:



Consider a quadrilateral $ABPQ$, such that $A\hat{B}P + A\hat{Q}P = 180°$ and $Q\hat{A}B + Q\hat{P}B = 180°$.

The aim is to prove that points A, B, P and Q lie on the circumference of a circle.

By contradiction. Assume that point P does not lie on a circle drawn through points A, B and Q. Let the circle cut AP (or AP extended) at point R. Draw BR.

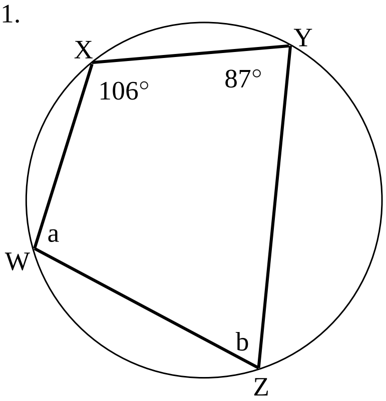
$$\begin{aligned}
Q\hat{A}B + Q\hat{R}B &= 180° \text{ opp. } \angle\text{s of cyclic quad.} \\
\text{but } Q\hat{A}B + Q\hat{P}B &= 180° \text{ (given)} \\
\therefore Q\hat{R}B &= Q\hat{P}B \\
\text{but this cannot be true since } Q\hat{R}B &= Q\hat{P}B + R\hat{B}P (\text{ext. } \angle \text{ of } \triangle)
\end{aligned}$$

$\therefore$ the assumption that the circle does not pass through P, must be false, and A, B, P and Q lie on the circumference of a circle and $ABPQ$ is a cyclic quadrilateral.

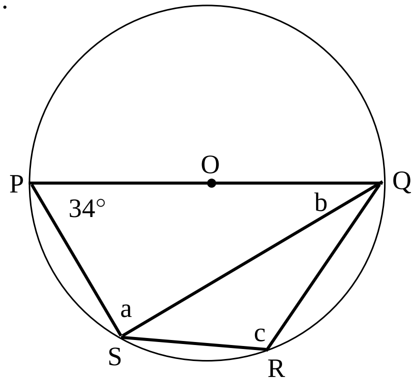
Exercise: Circles IV

1. Find the values of the unknown letters.

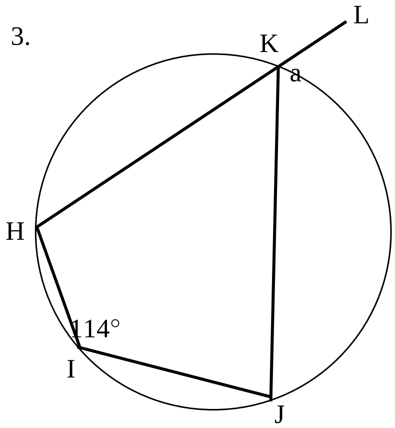
1.



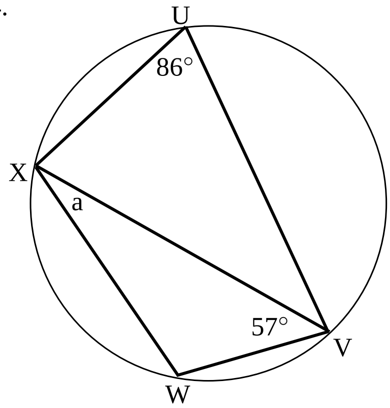
2.



3.

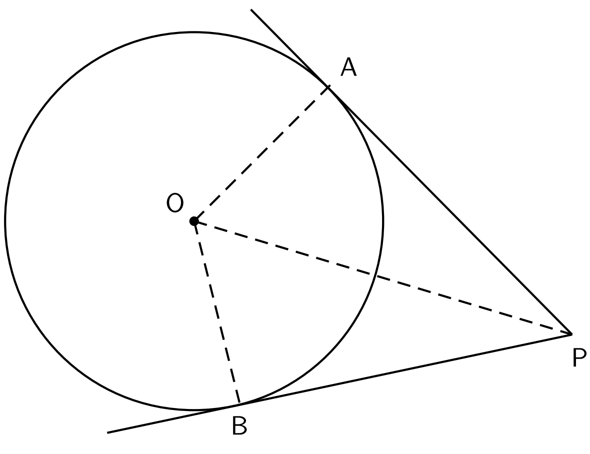


4.



Theorem 14. *Two tangents drawn to a circle from the same point outside the circle are equal in length.*

Proof:



Consider a circle, with centre O. Choose a point P outside the circle. Draw two tangents to the circle from point P, that meet the circle at A and B. Draw lines OA, OB and OP.

The aim is to prove that $AP = BP$**.**

In $\triangle OAP$ and $\triangle OBP$,

1. $OA = OB$ (radii)
2. $\angle OAP = \angle OPB = 90°$ ($OA \perp AP$ and $OB \perp BP$)
3. OP is common to both triangles.

$\triangle OAP \equiv \triangle OBP$ (right angle, hypotenuse, side)

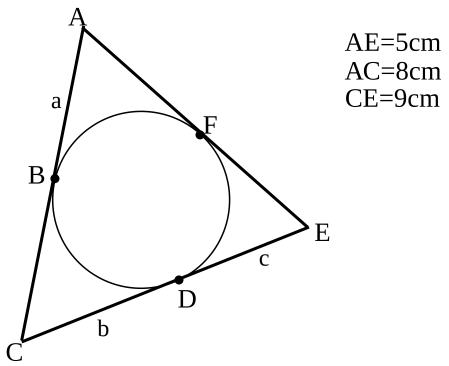
$\therefore AP = BP$

?

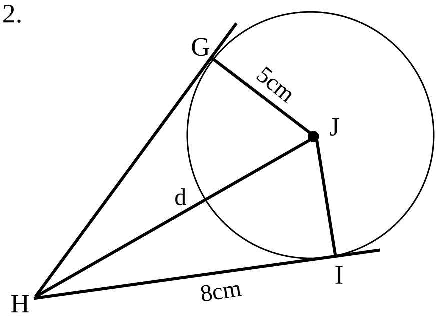
Exercise: Circles V

1. Find the value of the unknown lengths.

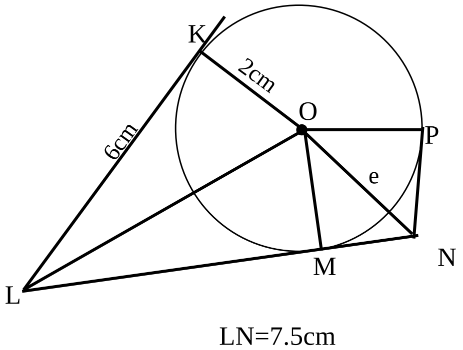
1.



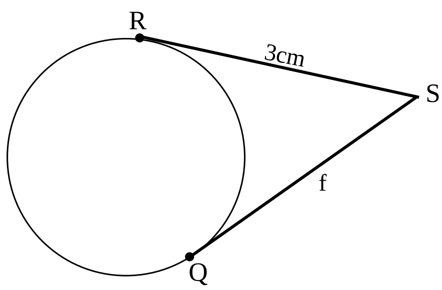
2.



3.

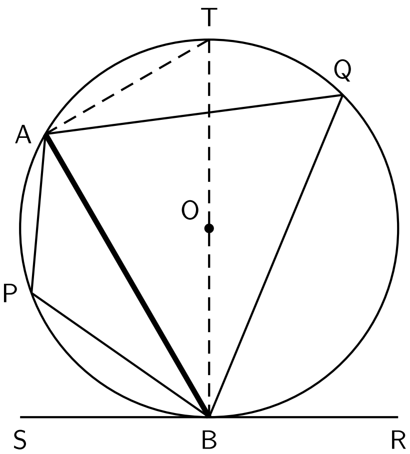


4.



Theorem 15. *The angle between a tangent and a chord, drawn at the point of contact of the chord, is equal to the angle which the chord subtends in the alternate segment.*

Proof:



Consider a circle, with centre O. Draw a chord AB and a tangent SR to the circle at point B. Chord AB subtends angles at points P and Q on the minor and major arcs, respectively.

Draw a diameter BT and join A to T.

The aim is to prove that $A\hat{P}B = A\hat{B}R$ and $A\hat{Q}B = A\hat{B}S$.

First prove that $A\hat{Q}B = A\hat{B}S$ as this result is needed to prove that $A\hat{P}B = A\hat{B}R$.

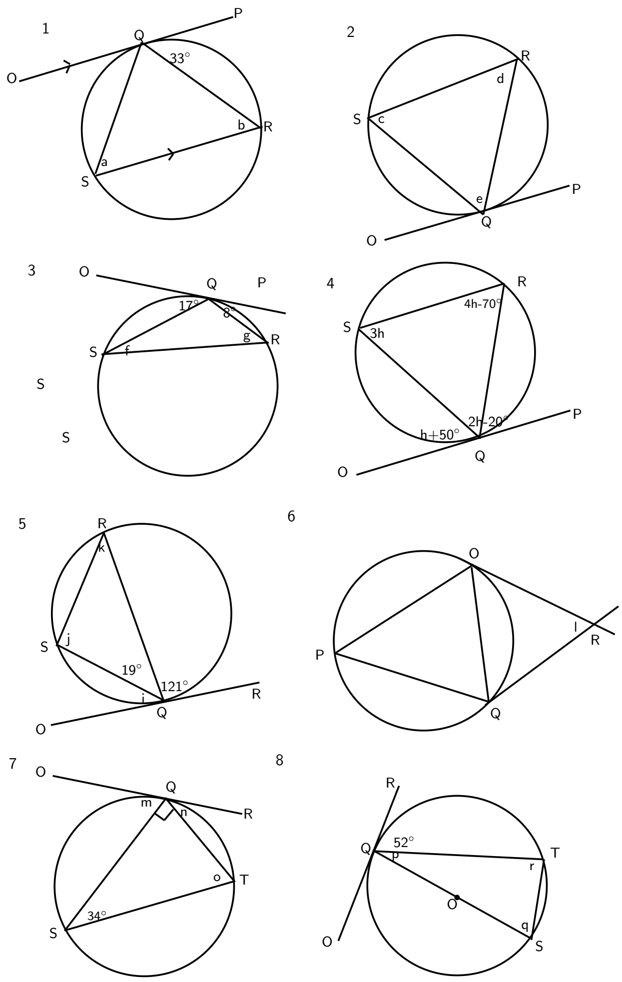
$$\begin{aligned}
A\hat{B}S + A\hat{B}T &= 90^\circ \ (TB \perp SR) \\
B\hat{A}T &= 90^\circ \ (\angle\text{s at centre}) \\
\therefore A\hat{B}T + A\hat{T}B &= 90^\circ \ (\text{sum of angles in } \triangle BAT) \\
\therefore A\hat{B}S &= A\hat{B}T \\
\text{However, } A\hat{Q}B &= A\hat{T}B (\text{ angles subtended by same chord } AB) \\
\therefore A\hat{Q}B &= A\hat{B}S
\end{aligned} \qquad (42.1)$$

$$\begin{aligned}
S\hat{B}Q + Q\hat{B}R &= 180^\circ \quad (SBT \text{ is a str. line}) \\
A\hat{P}B + A\hat{Q}B &= 180^\circ \quad (ABPQ \text{ is a cyclic quad.}) \\
\therefore S\hat{B}Q + Q\hat{B}R &= A\hat{P}B + A\hat{Q}B \\
\text{From (42.1),}\quad A\hat{Q}B &= A\hat{B}S \\
\therefore A\hat{P}B &= A\hat{B}R
\end{aligned}$$

?

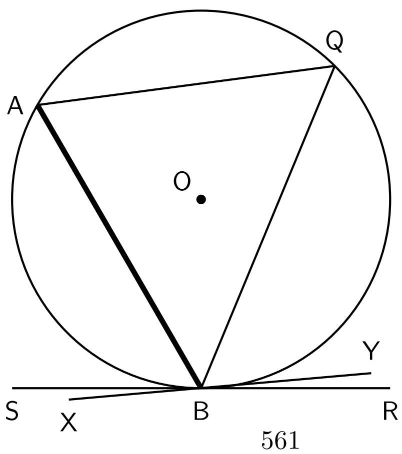
Exercise: Circles VI

1. Find the values of the unknown letters.



Theorem 16. *(Converse of 15) If the angle formed between a line, that is drawn through the end point of a chord, and the chord, is equal to the angle subtended by the chord in the alternate segment, then the line is a tangent to the circle.*

Proof:



Consider a circle, with centre O and chord AB. Let line SR pass through point B. Chord AB subtends an angle at point Q such that $A\hat{B}S = A\hat{Q}B$.

The aim is to prove that SBR is a tangent to the circle.

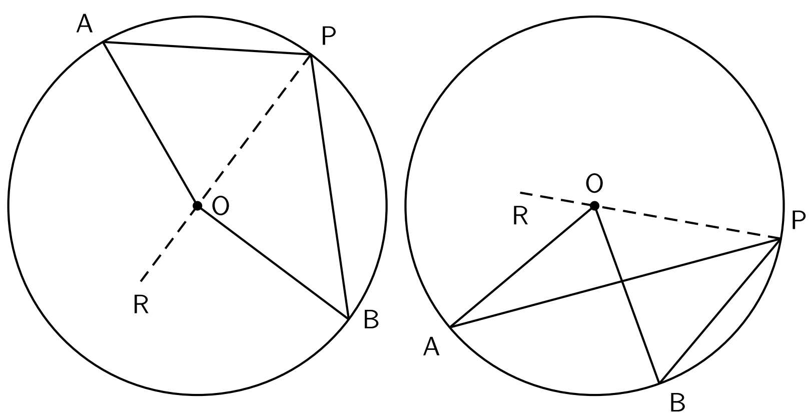
By contradiction. Assume that SBR is not a tangent to the circle and draw XBY such that XBY is a tangent to the circle.

$$\begin{aligned} A\hat{B}X &= A\hat{Q}B \quad \text{(tan-chord theorem)} \\ \text{However,} \quad A\hat{B}S &= A\hat{Q}B \quad \text{(given)} \\ \therefore A\hat{B}X &= A\hat{B}S \qquad (42.2) \\ \text{But since,} \quad A\hat{B}X &= A\hat{B}S + X\hat{B}S \\ \text{(42.2) can only be true if,} \quad X\hat{B}S &= 0 \end{aligned}$$

If $X\hat{B}S$ is zero, then both XBY and SBR coincide and SBR is a tangent to the circle.

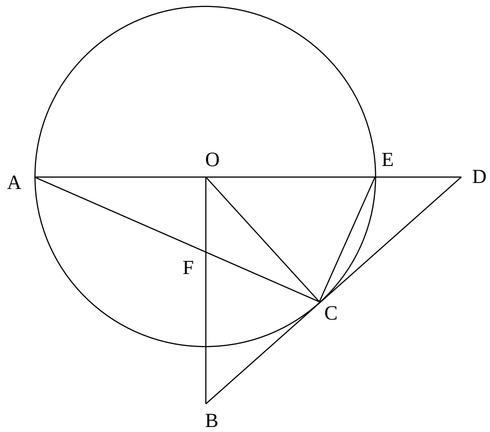
? **Exercise: Applying Theorem 9**

1. *Show that Theorem* 9 also applies to the following two cases:



Worked Example 190: Circle Geometry I

Question:



BD is a tangent to the circle with centre O.
$BO \perp AD$.

Prove that:

1. $CFOE$ is a cyclic quadrilateral
2. $FB = BC$
3. $\triangle COE /// \triangle CBF$
4. $CD^2 = ED.AD$
5. $\frac{OE}{BC} = \frac{CD}{CO}$

Answer

1. **Step 1 : To show a quadrilateral is cyclic, we need a pair of opposite angles to be supplementary, so lets look for that.**

$$\begin{aligned} F\hat{O}E &= 90^\circ \ (BO \perp OD) \\ F\hat{C}E &= 90^\circ \ (\angle \text{ subtended by diameter } AE) \\ &\therefore \ CFOE \text{ is a cyclic quadrilateral (opposite } \angle\text{'s supplementary)} \end{aligned}$$

2. **Step 1 : Since these two sides are part of a triangle, we are proving that triangle to be isosceles. The easiest way is to show the angles opposite to those sides to be equal.**

   Let $O\hat{E}C = x$.

$$\begin{aligned} &\therefore \ F\hat{C}B = x \ (\angle \text{ between tangent } BD \text{ and chord } CE) \\ &\therefore \ B\hat{F}C = x \ (\text{exterior } \angle \text{ to cyclic quadrilateral } CFOE) \\ &\therefore \ BF = BC \ (\text{sides opposite equal } \angle\text{'s in isosceles } \triangle BFC) \end{aligned}$$

3. **Step 1 : To show these two triangles similar, we will need 3 equal angles. We already have 3 of the 6 needed angles from the previous question. We need only find the missing 3 angles.**

$$\begin{aligned} C\hat{B}F &= 180^\circ - 2x \ (\text{sum of } \angle\text{'s in } \triangle BFC) \\ OC &= OE \ (\text{radii of circle } O) \\ &\therefore \ E\hat{C}O = x \ (\text{isosceles } \triangle COE) \\ &\therefore \ C\hat{O}E = 180^\circ - 2x \ (\text{sum of } \angle\text{'s in } \triangle COE) \end{aligned}$$

- $C\hat{O}E = C\hat{B}F$
- $E\hat{C}O = F\hat{C}B$
- $O\hat{E}C = C\hat{F}B$

$$\therefore \ \triangle COE /// \triangle CBF \ (3 \ \angle\text{'s equal})$$

4. **Step 1 : This relation reminds us of a proportionality relation between similar triangles. So investigate which triangles contain these sides and prove them similar. In this case 3 equal angles works well. Start with one triangle.**

   In $\triangle EDC$

$$\begin{aligned} C\hat{E}D &= 180^\circ - x \text{ ($\angle$'s on a straight line AD)} \\ E\hat{C}D &= 90^\circ - x \text{ (complementary $\angle$'s)} \end{aligned}$$

   Step 2 : Now look at the angles in the other triangle.

   In $\triangle ADC$

$$\begin{aligned} A\hat{C}E &= 180^\circ - x \text{ (sum of $\angle$'s $A\hat{C}E$ and $E\hat{C}O$)} \\ C\hat{A}D &= 90^\circ - x \text{ (sum of $\angle$'s in $\triangle CAE$)} \end{aligned}$$

   Step 3 : The third equal angle is an angle both triangles have in common.

   Lastly, $A\hat{D}C = E\hat{D}C$ since they are the same $\angle$.

   Step 4 : Now we know that the triangles are similar and can use the proportionality relation accordingly.

$$\begin{aligned} \therefore\ & \triangle ADC///\triangle CDE \text{ (3 $\angle$'s equal)} \\ \therefore\ & \frac{ED}{CD} = \frac{CD}{AD} \\ \therefore\ & CD^2 = ED.AD \end{aligned}$$

5. **Step 1 : This looks like another proportionality relation with a little twist, since not all sides are contained in 2 triangles. There is a quick observation we can make about the odd side out, OE.**

$$OE = CD \text{ ($\triangle OEC$ is isosceles)}$$

   Step 2 : With this observation we can limit ourselves to proving triangles BOC and ODC similar. Start in one of the triangles.

   In $\triangle BCO$

$$\begin{aligned} O\hat{C}B &= 90^\circ \text{ (radius OC on tangent BD)} \\ C\hat{B}O &= 180^\circ - 2x \text{ (sum of $\angle$'s in $\triangle BFC$)} \end{aligned}$$

   Step 3 : Then we move on to the other one.

   In $\triangle OCD$

$$\begin{aligned} O\hat{C}D &= 90^\circ \text{ (radius OC on tangent BD)} \\ C\hat{O}D &= 180^\circ - 2x \text{ (sum of $\angle$'s in $\triangle OCE$)} \end{aligned}$$

   Step 4 : Again we have a common element.

   Lastly, OC is a common side to both $\triangle$'s.

   Step 5 : Then, once we've shown similarity, we use the proportionality relation , as well as our first observation, appropriately.

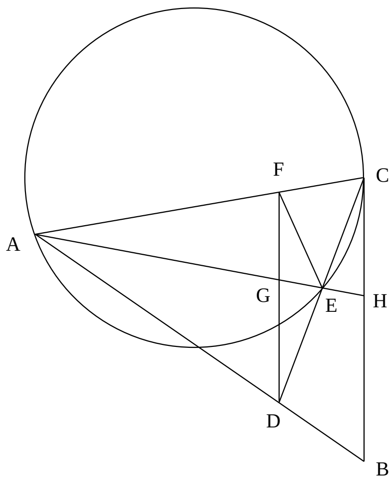
$\therefore \quad \triangle BOC /// \triangle ODC$ (common side and 2 equal $\angle$'s)

$\therefore \quad \frac{CO}{BC} = \frac{CD}{CO}$

$\therefore \quad \frac{OE}{BC} = \frac{CD}{CO}$ ($OE = CD$ isosceles $\triangle OEC$)

Worked Example 191: Circle Geometry II

Question:



FD is drawn parallel to the tangent CB

Prove that:

1. $FADE$ is cyclic
2. $\triangle AFE /// \triangle CBD$
3. $\frac{FC.AG}{GH} = \frac{DC.FE}{BD}$

Answer

1. **Step 1 : In this case, the best way to show $FADE$ is a cyclic quadrilateral is to look for equal angles, subtended by the same chord.**
   Let $\angle BCD = x$

   $\therefore \quad \angle CAH = x$ ($\angle$ between tangent BC and chord CE)
   $\therefore \quad \angle FDC = x$ (alternate $\angle$, $FD \parallel CB$)
   $\therefore \quad FADE$ is a cyclic quadrilateral (chord FE subtends equal $\angle$'s)

2. **Step 1 : To show these 2 triangles similar we will need 3 equal angles. We can use the result from the previous question.**
   Let $\angle FEA = y$

   $\therefore \quad \angle FDA = y$ ($\angle$'s subtended by same chord AF in cyclic quadrilateral $FADE$)
   $\therefore \quad \angle CBD = y$ (corresponding $\angle$'s, $FD \parallel CB$)
   $\therefore \quad \angle FEA = \angle CBD$

   Step 2 : We have already proved 1 pair of angles equal in the previous question.

   $$\angle BCD = \angle FAE \text{ (above)}$$

Step 3 : Proving the last set of angles equal is simply a matter of adding up the angles in the triangles. Then we have proved similarity.

$$\begin{aligned} \angle AFE &= 180° - x - y \text{ (}\angle\text{'s in } \triangle AFE\text{)} \\ \angle CBD &= 180° - x - y \text{ (}\angle\text{'s in } \triangle CBD\text{)} \\ &\therefore \triangle AFE///\triangle CBD \text{ (3 }\angle\text{'s equal)} \end{aligned}$$

3. **Step 1 : This equation looks like it has to do with proportionality relation of similar triangles. We already showed triangles AFE and CBD similar in the previous question. So lets start there.**

$$\begin{aligned} \frac{DC}{BD} &= \frac{FA}{FE} \\ &\therefore \frac{DC.FE}{BD} = FA \end{aligned}$$

Step 2 : Now we need to look for a hint about side FA. Looking at triangle CAH we see that there is a line FG intersecting it parallel to base CH. This gives us another proportionality relation.

$$\begin{aligned} \frac{AG}{GH} &= \frac{FA}{FC} \text{ (}FG \parallel CH \text{ splits up lines } AH \text{ and } AC \text{ proportionally)} \\ &\therefore FA = \frac{FC.AG}{GH} \end{aligned}$$

Step 3 : We have 2 expressions for the side FA.

$$\therefore \frac{FC.AG}{GH} = \frac{DC.FE}{BD}$$

42.3 Co-ordinate Geometry

42.3.1 Equation of a Circle

We know that every point on the circumference of a circle is the same distance away from the centre of the circle. Consider a point (x_1,y_1) on the circumference of a circle of radius r with centre at (x_0,y_0).

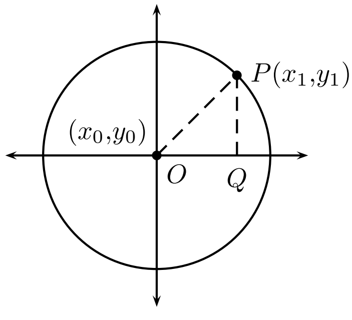


Figure 42.3: Circle h with centre (x_0,y_0) has a tangent, g passing through point P at (x_1,y_1). Line f passes through the centre and point P.

In Figure 42.3, $\triangle OPQ$ is a right-angled triangle. Therefore, from the Theorem of Pythagoras, we know that:

$$OP^2 = PQ^2 + OQ^2$$

But,

$$\begin{aligned} PQ &= y_1 - y_0 \\ OQ &= x_1 - x_0 \\ OP &= r \\ \therefore \quad r^2 &= (y_1 - y_0)^2 + (x_1 - x_0)^2 \end{aligned}$$

But, this same relation holds for any point P on the circumference. In fact, the relation holds for all points P on the circumference. Therefore, we can write:

$$(x - x_0)^2 + (y - y_0)^2 = r^2 \qquad (42.3)$$

for a circle with centre at (x_0,y_0) and radius r.

For example, the equation of a circle with centre $(0,0)$ and radius 4 is:

$$\begin{aligned} (y - y_0)^2 + (x - x_0)^2 &= r^2 \\ (y - 0)^2 + (x - 0)^2 &= 4^2 \\ y^2 + x^2 &= 16 \end{aligned}$$

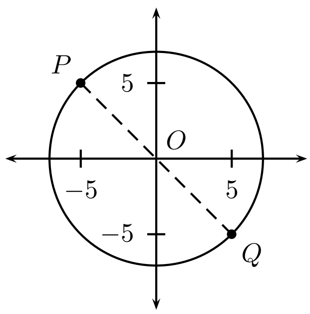
Worked Example 192: Equation of a Circle I

Question: Find the equation of a circle (centre O) with a diameter between two points, P at $(-5,5)$ and Q at $(5,-5)$.

Answer

Step 1 : Draw a picture

Draw a picture of the situation to help you figure out what needs to be done.



Step 2 : Find the centre of the circle

We know that the centre of a circle lies on the midpoint of a diameter. Therefore the co-ordinates of the centre of the circle is found by finding the midpoint of the line between P and Q. Let the co-ordinates of the centre of the circle be (x_0,y_0), let the co-ordinates of P be (x_1,y_1) and let the co-ordinates of Q be (x_2,y_2). Then, the co-ordinates of the midpoint are:

$$\begin{aligned} x_0 &= \frac{x_1 + x_2}{2} \\ &= \frac{-5+5}{2} \\ &= 0 \\ y_0 &= \frac{y_1 + y_2}{2} \\ &= \frac{5+(-5)}{2} \\ &= 0 \end{aligned}$$

The centre point of line PQ and therefore the centre of the circle is at $(0,0)$.

Step 3 : Find the radius of the circle

If P and Q are two points on a diameter, then the radius is half the distance between them.

The distance between the two points is:

$$\begin{aligned} r = \frac{1}{2}PQ &= \frac{1}{2}\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \frac{1}{2}\sqrt{(5-(-5))^2 + (-5-5)^2} \\ &= \frac{1}{2}\sqrt{(10)^2 + (-10)^2} \\ &= \frac{1}{2}\sqrt{100+100} \\ &= \sqrt{\frac{200}{4}} \\ &= \sqrt{50} \end{aligned}$$

Step 4 : Write the equation of the circle

$$x^2 + y^2 = 50$$

Worked Example 193: Equation of a Circle II

Question: Find the center and radius of the circle
$x^2 - 14x + y^2 + 4y = -28$.

Answer

Step 1 : Change to standard form

We need to rewrite the equation in the form $(x - x_0) + (y - y_0) = r^2$

To do this we need to complete the square

i.e. add and subtract ($\frac{1}{2}$ cooefficient of $x)^2$ and ($\frac{1}{2}$ cooefficient of $y)^2$

Step 2 : Adding cooefficients

$x^2 - 14x + y^2 + 4y = -28$

$\therefore \quad x^2 - 14x + (7)^2 - (7)^2 + y^2 + 4y + (2)^2 - (2)^2 = -28$

Step 3 : Complete the squares

$\therefore \quad (x-7)^2 - (7)^2 + (y+2)^2 - (2)^2 = -28$

Step 4 : Take the constants to the other side

$\therefore \quad (x-7)^2 - 49 + (y+2)^2 - 4 = -28$

$\therefore \quad (x-7)^2 + (y+2)^2 = -28 + 49 + 4$

$\therefore \quad (x-7)^2 + (y+2)^2 = 25$

Step 5 : Read the values from the equation

$\therefore$ center is $(7; -2)$ and the radius is 5 units

42.3.2 Equation of a Tangent to a Circle at a Point on the Circle

We are given that a tangent to a circle is drawn through a point P with co-ordinates (x_1,y_1). In this section, we find out how to determine the equation of that tangent.

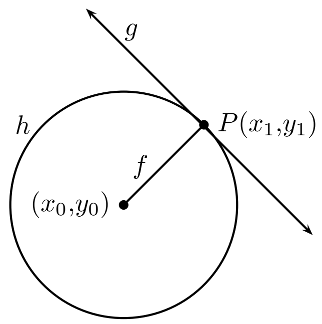


Figure 42.4: Circle h with centre (x_0,y_0) has a tangent, g passing through point P at (x_1,y_1). Line f passes through the centre and point P.

We start by making a list of what we know:

1. We know that the equation of the circle with centre (x_0,y_0) is $(x-x_0)^2+(y-y_0)^2=r^2$.
2. We know that a tangent is perpendicular to the radius, drawn at the point of contact with the circle.

As we have seen in earlier grades, there are two steps to determining the equation of a straight line:

Step 1: Calculate the gradient of the line, m.

Step 2: Calculate the y-intercept of the line, c.

The same method is used to determine the equation of the tangent. First we need to find the gradient of the tangent. We do this by finding the gradient of the line that passes through the centre of the circle and point P (line f in Figure 42.4), because this line is a radius line and the tangent is perpendicular to it.

$$m_f = \frac{y_1 - y_0}{x_1 - x_0} \tag{42.4}$$

The tangent (line g) is perpendicular to this line. Therefore,

$$m_f \times m_g = -1$$

So,

$$m_g = -\frac{1}{m_f}$$

Now, we know that the tangent passes through (x_1, y_1) so the equation is given by:

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - y_1 &= -\frac{1}{m_f}(x - x_1) \\ y - y_1 &= -\frac{1}{\frac{y_1 - y_0}{x_1 - x_0}}(x - x_1) \\ y - y_1 &= -\frac{x_1 - x_0}{y_1 - y_0}(x - x_1) \end{aligned}$$

For example, find the equation of the tangent to the circle at point $(1,1)$. The centre of the circle is at $(0,0)$. The equation of the circle is $x^2 + y^2 = 2$.

Use

$$y - y_1 = -\frac{x_1 - x_0}{y_1 - y_0}(x - x_1)$$

with $(x_0, y_0) = (0,0)$ and $(x_1, y_1) = (1,1)$.

$$\begin{aligned} y - y_1 &= -\frac{x_1 - x_0}{y_1 - y_0}(x - x_1) \\ y - 1 &= -\frac{1 - 0}{1 - 0}(x - 1) \\ y - 1 &= -\frac{1}{1}(x - 1) \\ y &= -(x - 1) + 1 \\ y &= -x + 1 + 1 \\ y &= -x + 2 \end{aligned}$$

?

Exercise: Co-ordinate Geometry

1. Find the equation of the cicle:
   - A with center $(0;5)$ and radius 5
   - B with center $(2;0)$ and radius 4
   - C with center $(5;7)$ and radius 18
   - D with center $(-2;0)$ and radius 6
   - E with center $(-5;-3)$ and radius $\sqrt{3}$
2. A Find the equation of the circle with center $(2;1)$ which passes through $(4;1)$.
   - B Where does it cut the line $y = x + 1$?
   - C Draw a sketch to illustrate your answers.
3. A Find the equation of the circle with center $(-3;-2)$ which passes through $(1;-4)$.
   - B Find the equation of the circle with center $(3;1)$ which passes through $(2;5)$.
   - C Find the point where these two circles cut each other.
4. Find the center and radius of the following circles:
   - A $(x-9)^2 + (y-6)^2 = 36$
   - B $(x-2)^2 + (y-9)^2 = 1$
   - C $(x+5)^2 + (y+7)^2 = 12$
   - D $(x+4)^2 + (y+4)^2 = 23$
   - E $3(x-2)^2 + 3(y+3)^2 = 12$

F $x^2 - 3x + 9 = y^2 + 5y + 25 = 17$

5. Find the $x-$ and $y-$ intercepts of the following graphs and draw a scetch to illustrate your answer:

   A $(x + 7)^2 + (y - 2)^2 = 8$
   B $x^2 + (y - 6)^2 = 100$
   C $(x + 4)^2 + y^2 = 16$
   D $(x - 5)^2 + (y + 1)^2 = 25$

6. Find the center and radius of the following circles:

   A $x^2 + 6x + y^2 - 12y = -20$
   B $x^2 + 4x + y^2 - 8y = 0$
   C $x^2 + y^2 + 8y = 7$
   D $x^2 - 6x + y^2 = 16$
   E $x^2 - 5x + y^2 + 3y = -\frac{3}{4}$
   F $x^2 - 6nx + y^2 + 10ny = 9n^2$

7. Find the equations to the tangent to the circle:

   A $x^2 + y^2 = 17$ at the point $(1; 4)$
   B $x^2 + y^2 = 25$ at the point $(3; 4)$
   C $(x + 1)^2 + (y - 2)^2 = 25$ at the point $(3; 5)$
   D $(x - 2)^2 + (y - 1)^2 = 13$ at the point $(5; 3)$

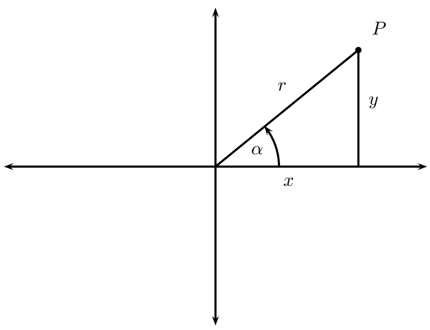
42.4 Transformations

42.4.1 Rotation of a Point about an angle θ

First we will find a formula for the co-ordinates of P after a rotation of θ.

We need to know something about polar co-ordinates and compound angles before we start.

Polar co-ordinates



Notice that : $\sin\alpha = \frac{y}{r} \therefore y = r\sin\alpha$
and $\cos\alpha = \frac{x}{r} \therefore x = r\cos\alpha$
so P can be expressed in two ways:

$P(x; y)$ rectangular co-ordinates

or $P(r\cos\alpha; r\sin\alpha)$ polar co-ordinates.

Compound angles

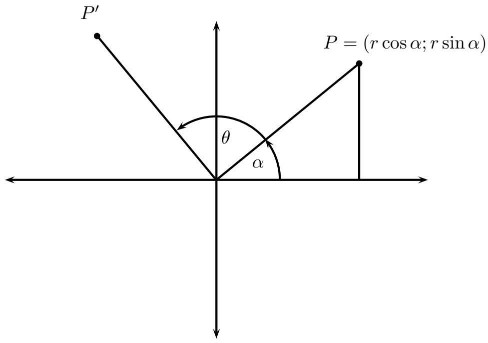
(See derivation of formulae in Ch. 12)

$$\begin{aligned}\sin(\alpha + \beta) &= \sin\alpha\cos\beta + \sin\beta\cos\alpha \\ \cos(\alpha + \beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta\end{aligned}$$

Now consider P' after a rotation of θ

$$P(x;y) = P(r\cos\alpha; r\sin\alpha)$$
$$P'(r\cos(\alpha+\theta); r\sin(\alpha+\theta))$$

Expand the co-ordinates of P'



$$\begin{aligned} x - \text{co-ordinate of } P' &= r\cos(\alpha+\theta) \\ &= r\left[\cos\alpha\cos\theta - \sin\alpha\sin\theta\right] \\ &= r\cos\alpha\cos\theta - r\sin\alpha\sin\theta \\ &= x\cos\theta - y\sin\theta \end{aligned}$$

$$\begin{aligned} y - \text{co-ordinate of } P' &= r\sin(\alpha+\theta) \\ &= r\left[\sin\alpha\cos\theta + \sin\theta\cos\alpha\right] \\ &= r\sin\alpha\cos\theta + r\cos\alpha\sin\theta \\ &= y\cos\theta + x\sin\theta \end{aligned}$$

which gives the formula $P' = [(x\cos\theta - y\sin\theta; y\cos\theta + x\sin\theta)]$.

So to find the co-ordinates of $P(1;\sqrt{3})$ after a rotation of 45°, we arrive at:

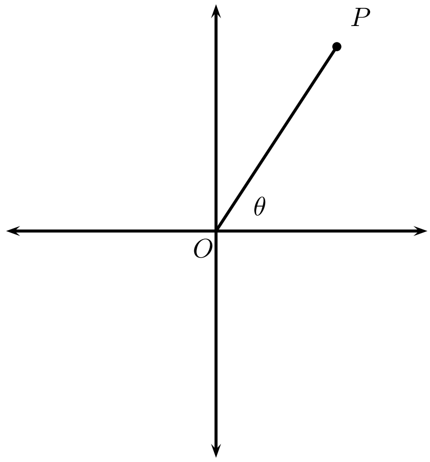
$$\begin{aligned} P' &= [(x\cos\theta - y\sin\theta); (y\cos\theta + x\sin\theta] \\ &= \left[(1\cos 45^\circ - \sqrt{3}\sin 45^\circ); (\sqrt{3}\cos 45^\circ + 1\sin 45^\circ\right] \\ &= \left[\left(\frac{1}{\sqrt{2}} - \frac{\sqrt{3}}{\sqrt{2}}\right); \left(\frac{\sqrt{3}}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right)\right] \\ &= \left(\frac{1-\sqrt{3}}{\sqrt{2}}; \frac{\sqrt{3}+1}{\sqrt{2}}\right) \end{aligned}$$

?

Exercise: Rotations

Any line OP is drawn (not necessarily in the first quadrant), making an angle of θ degrees with the x-axis. Using the co-ordinates of P and the angle α, calculate the co-ordinates of P', if the line OP is rotated about the origin through α degrees.

	P	α
1.	(2, 6)	60°
2.	(4, 2)	30°
3.	(5, -1)	45°
4.	(-3, 2)	120°
5.	(-4, -1)	225°
6.	(2, 5)	-150°



42.4.2 Characteristics of Transformations

Rigid transformations like translations, reflections, rotations and glide reflections preserve shape and size, and that enlargement preserves shape but not size.

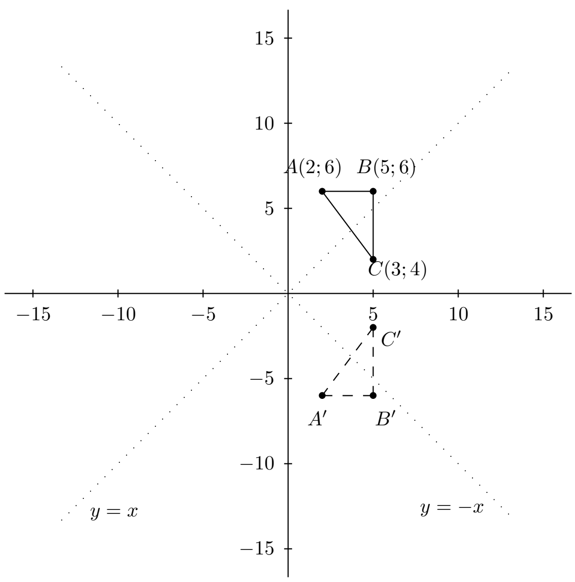
42.4.3 Characteristics of Transformations

Rigid transformations like translations, reflections, rotations and glide reflections preserve shape and size, and that enlargement preserves shape but not size.

Activity :: : Geometric Transformations

Draw a large 15×15 grid and plot $\triangle ABC$ with $A(2;6)$, $B(5;6)$ and $C(5;1)$. Fill in the lines $y = x$ and $y = -x$.

Complete the table below , by drawing the images of $\triangle ABC$ under the given transformations. The first one has been done for you.



Transformation	Description (translation, reflection, rotation, enlargement)	Co-ordinates	Lengths	Angles
$(x;y) \rightarrow (x;-y)$	reflection about the x-axis	$A'(2;-6)$ $B'(5;-6)$ $C'(5;-2)$	$A'B' = 3$ $B'C' = 4$ $A'C' = 5$	$\hat{B}' = 90°$ $\tan\hat{A} = 4/3$ $\therefore \hat{A} = 53°, \hat{C} = 37°$
$(x;y) \rightarrow (x+1;y-2)$				
$(x;y) \rightarrow (-x;y)$				
$(x;y) \rightarrow (-y;x)$				
$(x;y) \rightarrow (-x;-y)$				
$(x;y) \rightarrow (2x;2y)$				
$(x;y) \rightarrow (y;x)$				
$(x;y) \rightarrow (y;x+1)$				

A transformation that leaves lengths and angles unchanged is called a rigid transformation. Which of the above transformations are rigid?

42.5 Exercises

1. ΔABC undergoes several transformations forming $\Delta A'B'C'$. Describe the relationship between the angles and sides of ΔKLM and $\Delta A'B'C'$ (e.g., they are twice as large, the same, etc.)

Transformation	Sides	Angles	Area
Reflect			
Reduce by a scale factor of 3			
Rotate by 90°			
Translate 4 units right			
Enlarge by a scale factor of 2			

2. ΔDEF has $\hat{E} = 30°$, $DE = 4$ cm, $EF = 5$ cm. ΔDEF is enlarged by a scale factor of 6 to form $\Delta D'E'F'$.

   A Solve ΔDEF

   B Hence, solve $\Delta D'E'F'$

3. ΔXYZ has an area of $6\,\text{cm}^2$. Find the area of $\Delta X'Y'Z'$ if the points have been transformed as follows:

   A $(x,y) \rightarrow (x+2;y+3)$

   B $(x,y) \rightarrow (y;x)$

C $(x, y) \rightarrow (4x; y)$

D $(x, y) \rightarrow (3x; y + 2)$

E $(x, y) \rightarrow (-x; -y)$

F $(x, y) \rightarrow (x; -y + 3)$

G $(x, y) \rightarrow (4x; 4y)$

H $(x, y) \rightarrow (-3x; 4y)$

Chapter 43

Trigonometry - Grade 12

43.1 Compound Angle Identities

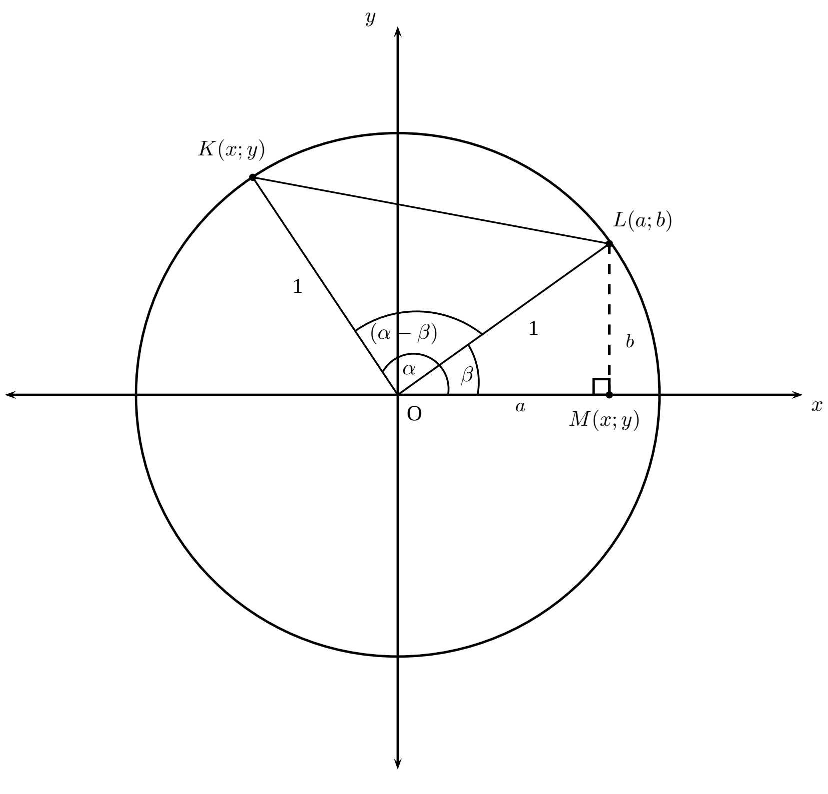
43.1.1 Derivation of $\sin(\alpha + \beta)$

We have, for any angles α and β, that

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \sin\beta\cos\alpha$$

How do we derive this identity? It is tricky, so follow closely.

Suppose we have the unit circle shown below. The two points $L(a,b)$ and $K(x,y)$ are on the circle.



We can get the coordinates of L and K in terms of the angles α and β. For the triangle LOK, we have that

$$\sin\beta = \frac{b}{1} \quad \Longrightarrow \quad b = \sin\beta$$
$$\cos\beta = \frac{a}{1} \quad \Longrightarrow \quad a = \cos\beta$$

Thus the coordinates of L are $(\cos\beta;\sin\beta)$. In the same way as above, we can see that the coordinates of K are $(\cos\alpha;\sin\alpha)$. The identity for $\cos(\alpha-\beta)$ is now determined as follows: Using the distance formula (i.e. $d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$ or $d^2=(x_2-x_1)^2+(y_2-y_1)^2)$, we can find KL^2.

$$\begin{aligned}
TR^2 &= (\cos\alpha-\cos\beta)^2+(\sin\alpha-\sin\beta)^2 \\
&= \cos^2\alpha-2\cos\alpha\cos\beta+\cos^2\beta+\sin^2\alpha-2\sin\alpha\sin\beta+\sin^2\beta \\
&= (\cos^2\alpha+\sin^2\alpha)+(\cos^2\beta+\sin^2\beta)-2\cos\alpha\cos\beta-2\sin\alpha\sin\beta \\
&= 1+1-2(\cos\alpha\cos\beta+\sin\alpha\sin\beta) \\
&= 2-2(\cos\alpha\cos\beta+\sin\alpha\sin\beta)
\end{aligned}$$

Now using the cosine rule for $\triangle KOL$, we get

$$\begin{aligned}
KL^2 &= KO^2+LO^2-2\cdot KO\cdot LO\cdot\cos(\alpha-\beta) \\
&= 1^2+1^2-2(1)(1)\cos(\alpha-\beta) \\
&= 2-2\cdot\cos(\alpha-\beta)
\end{aligned}$$

Equating our two values for TR^2, we have

$$\begin{aligned}
2-2\cdot\cos(\alpha-\beta) &= 2-2(\cos\alpha\cos\beta+\sin\alpha\cdot\sin\beta) \\
\Longrightarrow\quad \cos(\alpha-\beta) &= \cos\alpha\cdot\cos\beta+\sin\alpha\cdot\sin\beta
\end{aligned}$$

Now let $\alpha\to 90^\circ-\alpha$. Then

$$\begin{aligned}
\cos(90^\circ-\alpha-\beta) &= \cos(90^\circ-\alpha)\cos\beta+\sin(90^\circ-\alpha)\sin\beta \\
&= \sin\alpha\cdot\cos\beta+\cos\alpha\cdot\sin\beta
\end{aligned}$$

But $\cos(90^\circ-(\alpha+\beta))=\sin(\alpha+\beta)$. Thus

$$\sin(\alpha+\beta)=\sin\alpha\cdot\cos\beta+\cos\alpha\cdot\sin\beta$$

43.1.2 Derivation of $\sin(\alpha-\beta)$

We can use

$$\sin(\alpha+\beta)=\sin\alpha\cos\beta+\sin\beta\cos\alpha$$

to show that

$$\sin(\alpha-\beta)=\sin\alpha\cos\beta-\sin\beta\cos\alpha$$

We know that

$$\sin(-\theta)=-\sin(\theta)$$

and

$$\cos(-\theta)=\cos\theta$$

Therefore,

$$\begin{aligned}
\sin(\alpha-\beta) &= \sin(\alpha+(-\beta)) \\
&= \sin\alpha\cos(-\beta)+\sin(-\beta)\cos\alpha \\
&= \sin\alpha\cos\beta-\sin\beta\cos\alpha
\end{aligned}$$

43.1.3 Derivation of $\cos(\alpha+\beta)$

We can use

$$\sin(\alpha-\beta)=\sin\alpha\cos\beta-\sin\beta\cos\alpha$$

to show that

$$\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$$

We know that

$$\sin(\theta)=\cos(90-\theta).$$

Therefore,

$$\begin{aligned}\cos(\alpha+\beta) &= \sin(90-(\alpha+\beta)) \\ &= \sin((90-\alpha)-\beta)) \\ &= \sin(90-\alpha)\cos\beta - \sin\beta\cos(90-\alpha) \\ &= \cos\alpha\cos\beta - \sin\beta\sin\alpha\end{aligned}$$

43.1.4 Derivation of $\cos(\alpha - \beta)$

We found this identity in our derivation of the $\sin(\alpha+\beta)$ identity. We can also use the fact that

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \sin\beta\cos\alpha$$

to derive that

$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

As

$$\cos(\theta) = \sin(90-\theta),$$

we have that

$$\begin{aligned}\cos(\alpha-\beta) &= \sin(90-(\alpha-\beta)) \\ &= \sin((90-\alpha)+\beta)) \\ &= \sin(90-\alpha)\cos\beta + \sin\beta\cos(90-\alpha) \\ &= \cos\alpha\cos\beta + \sin\beta\sin\alpha\end{aligned}$$

43.1.5 Derivation of $\sin 2\alpha$

We know that

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \sin\beta\cos\alpha$$

When $\alpha = \beta$, we have that

$$\begin{aligned}\sin(\alpha+\alpha) &= \sin\alpha\cos\alpha + \sin\alpha\cos\alpha \\ &= 2\sin\alpha\cos\alpha \\ &= \sin(2\alpha)\end{aligned}$$

43.1.6 Derivation of $\cos 2\alpha$

We know that

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

When $\alpha = \beta$, we have that

$$\begin{aligned}\cos(\alpha+\alpha) &= \cos\alpha\cos\alpha - \sin\alpha\sin\alpha \\ &= \cos^2\alpha - \sin^2\alpha \\ &= \cos(2\alpha)\end{aligned}$$

However, we can also write

$$\cos 2\alpha = 2cos^2\alpha - 1$$

and

$$\cos 2\alpha = 1 - 2sin^2\alpha$$

by using

$$\sin^2\alpha + \cos^2\alpha = 1.$$

Activity :: $\cos 2\alpha$ **Identity : Use**

$$\sin^2\alpha + \cos^2\alpha = 1$$

to show that:

$$\cos 2\alpha = \begin{cases} 2\cos^2\alpha - 1 \\ 1 - 2\sin^2\alpha \end{cases}$$

43.1.7 Problem-solving Strategy for Identities

The most important thing to remember when asked to prove identities is:

Important: Trigonometric Identities

Never assume that the left hand side is equal to the right hand side. You need to **show** that both sides are equal.

A suggestion for proving identities: It is usually much easier simplifying the more complex side of an identity to get the simpler side than the other way round.

Worked Example 194: Trigonometric Identities 1

Question: Prove that $\sin 75° = \frac{\sqrt{2}(\sqrt{3}+1)}{4}$ without using a calculator.
Answer
Step 1 : Identify a strategy
We only know the exact values of the trig functions for a few special angles ($30°$, $45°$, $60°$, etc.). We can see that $75° = 30° + 45°$. Thus we can use our double-angle identity for $\sin(\alpha+\beta)$ to express $\sin 75°$ in terms of known trig function values.
Step 2 : Execute strategy

$$\begin{aligned} \sin 75° &= \sin(45° + 30°) \\ &= \sin(45°)\cos(30°) + \sin(30°)\cos(45°) \\ &= \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} \cdot \frac{1}{2} \\ &= \frac{\sqrt{3}+1}{2\sqrt{2}} \\ &= \frac{\sqrt{3}+1}{2\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} \\ &= \frac{\sqrt{2}(\sqrt{3}+1)}{4} \end{aligned}$$

Worked Example 195: Trigonometric Identities 2

Question: Deduce a formula for $\tan(\alpha+\beta)$ in terms of $\tan\alpha$ and $\tan\beta$.
Hint: Use the formulae for $\sin(\alpha+\beta)$ *and* $\cos(\alpha+\beta)$
Answer
Step 1 : Identify a strategy
We can reexpress $\tan(\alpha+\beta)$ in terms of cosines and sines, and then use the double-angle formulas for these. We then manipulate the resulting expression in order to get it in terms of $\tan\alpha$ and $\tan\beta$.
Step 2 : Execute strategy

$$\begin{aligned}
\tan(\alpha+\beta) &= \frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)} \\
&= \frac{\sin\alpha\cdot\cos\beta+\sin\beta\cdot\cos\alpha}{\cos\alpha\cdot\cos\beta-\sin\alpha\cdot\sin\beta} \\
&= \frac{\frac{\sin\alpha\cdot\cos\beta}{\cos\alpha\cdot\cos\beta}+\frac{\sin\beta\cdot\cos\alpha}{\cos\alpha\cdot\cos\beta}}{\frac{\cos\alpha\cdot\cos\beta}{\cos\alpha\cdot\cos\beta}-\frac{\sin\alpha\cdot\sin\beta}{\cos\alpha\cdot\cos\beta}} \\
&= \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\cdot\tan\beta}
\end{aligned}$$

Worked Example 196: Trigonometric Identities 3

Question: Prove that

$$\frac{\sin\theta+\sin 2\theta}{1+\cos\theta+\cos 2\theta}=\tan\theta$$

For which values is the identity not valid?
Answer
Step 1 : Identify a strategy
The right-hand side (RHS) of the identity cannot be simplified. Thus we should try simplify the left-hand side (LHS). We can also notice that the trig function on the RHS does not have a 2θ dependance. Thus we will need to use the double-angle formulas to simplify the $\sin 2\theta$ and $\cos 2\theta$ on the LHS. We know that $\tan\theta$ is undefined for some angles θ. Thus the identity is also undefined for these θ, and hence is not valid for these angles. Also, for some θ, we might have division by zero in the LHS, which is not allowed. Thus the identity won't hold for these angles also.
Step 2 : Execute the strategy

$$\begin{aligned}
LHS &= \frac{\sin\theta+2\sin\theta\cos\theta}{1+\cos\theta+(2\cos^2\theta-1)} \\
&= \frac{\sin\theta(1+2\cos\theta)}{\cos\theta(1+2\cos\theta)} \\
&= \frac{\sin\theta}{\cos\theta} \\
&= \tan\theta \\
&= RHS
\end{aligned}$$

We know that $\tan\theta$ is undefined when $\theta=90°+180°n$, where n is an integer. The LHS is undefined when $1+\cos\theta+\cos 2\theta=0$. Thus we need to solve this equation.

$$\begin{aligned}
1+\cos\theta+\cos 2\theta &= 0 \\
\Longrightarrow \quad \cos\theta(1+2\cos\theta) &= 0
\end{aligned}$$

The above has solutions when $\cos\theta = 0$, which occurs when $\theta = 90° + 180°n$, where n is an integer. These are the same values when $\tan\theta$ is undefined. It also has solutions when $1 + 2\cos\theta = 0$. This is true when $\cos\theta = -\frac{1}{2}$, and thus $\theta = \ldots - 240°, -120°, 120°, 240°, \ldots$. To summarise, the identity is not valid when $\theta = \ldots - 270°, -240°, -120°, -90°, 90°, 120°, 240°, 270°, \ldots$

Worked Example 197: Trigonometric Equations

Question: Solve the following equation for y without using a calculator.

$$\frac{1 - \sin y - \cos 2y}{\sin 2y - \cos y} = -1$$

Answer

Step 1 : Identify a strategy

Before we are able to solve the equation, we first need to simplify the left-hand side. We do this using the double-angle formulas.

Step 2 : Execute the strategy

$$\begin{aligned}
& \frac{1 - \sin y - (1 - 2\sin^2 y)}{2\sin y\cos y - \cos y} &=& -1 \\
\implies & \frac{2\sin^2 y - \sin y}{\cos y(2\sin y - 1)} &=& -1 \\
\implies & \frac{\sin y(2\sin y - 1)}{\cos y(2\sin y - 1)} &=& -1 \\
\implies & \tan y &=& -1 \\
\implies & y = 135° + 180°n;\ n \in \mathbb{Z} & &
\end{aligned}$$

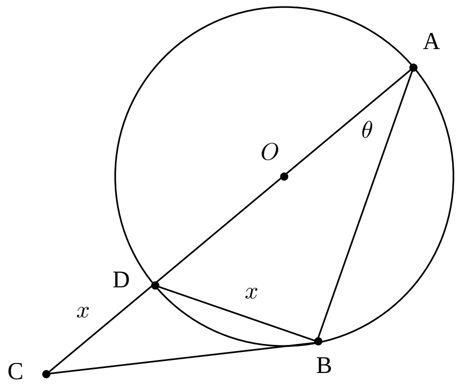
43.2 Applications of Trigonometric Functions

43.2.1 Problems in Two Dimensions

Worked Example 198:

Question: For the figure below, we are given that $BC = BD = x$.

Show that $BC^2 = 2x^2(1 + \sin\theta)$.



Answer

Step 1 : Identify a strategy

We want CB, and we have CD and BD. If we could get the angle $B\hat{D}C$, then we could use the cosine rule to determine DC. This is possible, as $\triangle ABD$ is a right-angled triangle. We know this from circle geometry, that any triangle circumscribed by a circle with one side going through the origin, is right-angled. As we have two angles of $\triangle ABD$, we know $A\hat{D}B$ and hence $B\hat{D}C$. Using the cosine rule, we can getBC^2.

Step 2 : Execute the strategy

$$A\hat{D}B = 180° - \theta - 90° = 90° - \theta$$

Thus

$$\begin{aligned} B\hat{D}C &= 180° - A\hat{D}B \\ &= 180° - (90° - \theta) \\ &= 90° + \theta \end{aligned}$$

Now the cosine rule gives

$$\begin{aligned} BC^2 &= CD^2 + BD^2 - 2 \cdot CD \cdot BD \cdot \cos(B\hat{D}C) \\ &= x^2 + x^2 - 2 \cdot x^2 \cdot \cos(90° + \theta) \\ &= 2x^2 + 2x^2 \left[\sin(90°)\cos(\theta) + \sin(\theta)\cos(90°)\right] \\ &= 2x^2 + 2x^2 \left[1 \cdot \cos(\theta) + \sin(\theta) \cdot 0\right] \\ &= 2x^2(1 - \sin\theta) \end{aligned}$$

?

Exercise:

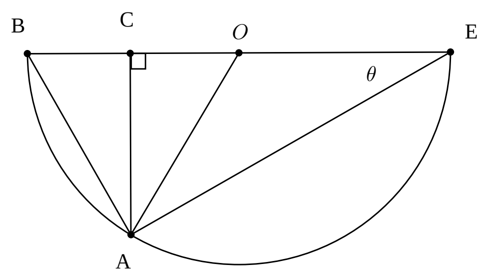
1. For the diagram on the right,
   A Find $A\hat{O}C$ in terms of θ.

   B Find an expression for:

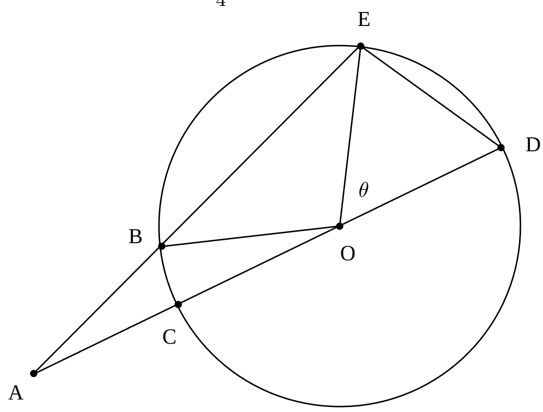
      i. $\cos\theta$
      ii. $\sin\theta$
      iii. $\sin 2\theta$

   C Using the above, show that $\sin 2\theta = 2\sin\theta\cos\theta$.

   D Now do the same for $\cos 2\theta$ and $\tan\theta$.



2. DA is a diameter of circle O with radius r. $CA = r$, $AB = DE$ and $D\hat{O}E = \theta$. Show that $\cos\theta = \frac{1}{4}$.



3. The figure on the right shows a cyclic quadrilateral with $\frac{BC}{CD} = \frac{AD}{AB}$.

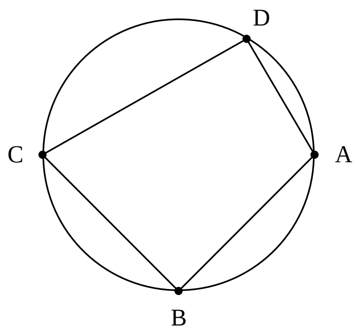
   A Show that the area of the cyclic quadrilateral is $DC \cdot DA \cdot \sin\hat{D}$.

   B Find expressions for $\cos\hat{D}$ and $\cos\hat{B}$ in terms of the quadrilateral sides.

   C Show that $2CA^2 = CD^2 + DA^2 + AB^2 + BC^2$.

   D Suppose that $BC = 10$, $CD = 15$, $AD = 4$ and $AB = 6$. Find CA^2.

   E Find the angle $\hat{D}$ using your expression for $\cos\hat{D}$. Hence find the area of $ABCD$.



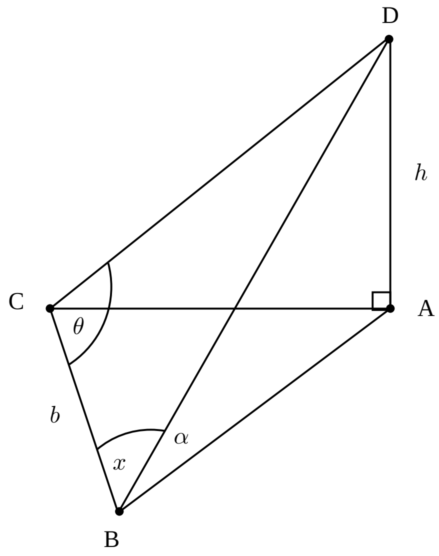
43.2.2 Problems in 3 dimensions

Worked Example 199: Height of tower

Question:

D is the top of a tower of height h. Its base is at C. The triangle ABC lies on the ground (a horizontal plane). If we have that $BC = b$, $D\hat{B}A = \alpha$, $D\hat{B}C = x$ and $D\hat{C}B = \theta$, show that

$$h = \frac{b\sin\alpha\sin x}{\sin(x+\theta)}$$



Answer

Step 1 : Identify a strategy

We have that the triangle ABD is right-angled. Thus we can relate the height h with the angle α and either the length BA or BD (using sines or cosines). But we have two angles and a length for $\triangle BCD$, and thus can work out all the remaining lengths and angles of this triangle. We can thus work out BD.

Step 2 : Execute the strategy

We have that

$$\begin{aligned} \frac{h}{BD} &= \sin\alpha \\ \Longrightarrow \quad h &= BD\sin\alpha \end{aligned}$$

Now we need BD in terms of the given angles and length b. Considering the triangle BCD, we see that we can use the sine rule.

$$\begin{aligned} \frac{\sin\theta}{DB} &= \frac{\sin(D\hat{B}C)}{b} \\ \Longrightarrow \quad DB &= \frac{b\sin\theta}{\sin(D\hat{B}C)} \end{aligned}$$

But $D\hat{B}C = 180^\circ - \alpha - \theta$, and

$$\begin{aligned} \sin(180^\circ - \alpha - \theta) &= -\sin(-\alpha - \theta) \\ &= \sin(\alpha + \theta) \end{aligned}$$

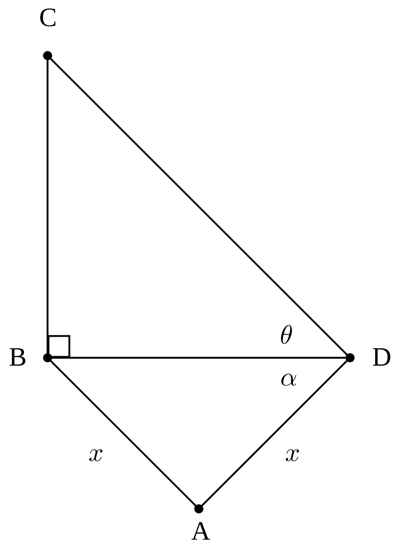
So

$$\begin{aligned} DB &= \frac{b\sin\theta}{\sin(D\hat{B}C)} \\ &= \frac{b\sin\theta}{\sin(\alpha + \theta)} \end{aligned}$$

?

Exercise:

1. The line BC represents a tall tower, with C at its foot. Its angle of elevation from D is θ. We are also given that $BA = AD = x$.



A Find the height of the tower BC in terms of x, $\tan\theta$ and $\cos 2\alpha$.

B Find BC if we are given that $k = 140m$, $\alpha = 21°$ and $\theta = 9°$.

43.3 Other Geometries

43.3.1 Taxicab Geometry

Taxicab geometry, considered by Hermann Minkowski in the 19th century, is a form of geometry in which the usual metric of Euclidean geometry is replaced by a new metric in which the distance between two points is the sum of the (absolute) differences of their coordinates.

43.3.2 Manhattan distance

The metric in taxi-cab geometry, is known as the *Manhattan distance*, between two points in an Euclidean space with fixed Cartesian coordinate system as the sum of the lengths of the projections of the line segment between the points onto the coordinate axes.

For example, in the plane, the Manhattan distance between the point P_1 with coordinates (x_1, y_1) and the point P_2 at (x_2, y_2) is

$$|x_1 - x_2| + |y_1 - y_2| \quad (43.1)$$

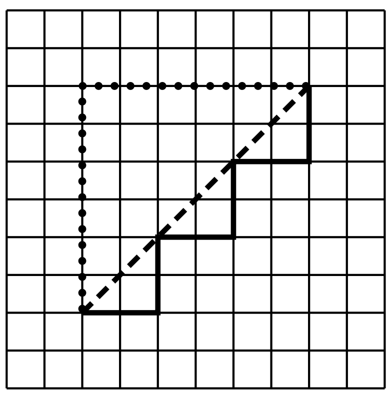


Figure 43.1: Manhattan Distance (dotted and solid) compared to Euclidean Distance (dashed). In each case the Manhattan distance is 12 units, while the Euclidean distance is $\sqrt{36}$

The Manhattan distance depends on the choice on the rotation of the coordinate system, but does not depend on the translation of the coordinate system or its reflection with respect to a coordinate axis.

Manhattan distance is also known as city block distance or taxi-cab distance. It is given these names because it is the shortest distance a car would drive in a city laid out in square blocks.

Taxicab geometry satisfies all of Euclid's axioms except for the side-angle-side axiom, as one can generate two triangles with two sides and the angle between them the same and have them not be congruent. In particular, the parallel postulate holds.

A circle in taxicab geometry consists of those points that are a fixed Manhattan distance from the center. These circles are squares whose sides make a $45\circ$ angle with the coordinate axes.

43.3.3 Spherical Geometry

Spherical geometry is the geometry of the two-dimensional surface of a sphere. It is an example of a non-Euclidean geometry.

In plane geometry the basic concepts are points and line. On the sphere, points are defined in the usual sense. The equivalents of lines are not defined in the usual sense of "straight line" but in the sense of "the shortest paths between points" which is called a geodesic. On the sphere the geodesics are the great circles, so the other geometric concepts are defined like in plane geometry but with lines replaced by great circles. Thus, in spherical geometry angles are defined between great circles, resulting in a spherical trigonometry that differs from ordinary trigonometry in many respects (for example, the sum of the interior angles of a triangle exceeds 180°).

Spherical geometry is the simplest model of elliptic geometry, in which a line has no parallels through a given point. Contrast this with hyperbolic geometry, in which a line has two parallels, and an infinite number of ultra-parallels, through a given point.

Spherical geometry has important practical uses in celestial navigation and astronomy.

Extension: Distance on a Sphere

The great-circle distance is the shortest distance between any two points on the surface of a sphere measured along a path on the surface of the sphere (as opposed to going through the sphere's interior). Because spherical geometry is rather different from ordinary Euclidean geometry, the equations for distance take on a different form. The distance between two points in Euclidean space is the length of a straight line from one point to the other. On the sphere, however, there are no straight lines. In non-Euclidean geometry, straight lines are replaced with geodesics. Geodesics on the sphere are the great circles (circles on the sphere whose centers are coincident with the center of the sphere).

Between any two points on a sphere which are not directly opposite each other, there is a unique great circle. The two points separate the great circle into two arcs. The length of the shorter arc is the great-circle distance between the points. Between two points which are directly opposite each other (called antipodal points) there infinitely many great circles, but all have the same length, equal to half the circumference of the circle, or πr, where r is the radius of the sphere.

Because the Earth is approximately spherical (see spherical Earth), the equations for great-circle distance are important for finding the shortest distance between points on the surface of the Earth, and so have important applications in navigation.

Let ϕ_1,λ_1; ϕ_2,λ_2, be the latitude and longitude of two points, respectively. Let $\Delta\lambda$ be the longitude difference. Then, if r is the great-circle radius of the sphere, the great-circle distance is $r\Delta\sigma$, where $\Delta\sigma$ is the angular difference/distance and can be determined from the spherical law of cosines as:

$$\Delta\sigma = \arccos\left\{\sin\phi_1\sin\phi_2 + \cos\phi_1\cos\phi_2\cos\Delta\lambda\right\}$$

Extension: Spherical Distance on the Earth

The shape of the Earth more closely resembles a flattened spheroid with extreme values for the radius of curvature, or arcradius, of 6335.437 km at the equator (vertically) and 6399.592 km at the poles, and having an average great-circle radius of 6372.795 km.

Using a sphere with a radius of 6372.795 km thus results in an error of up to about 0.5%.

43.3.4 Fractal Geometry

The word "fractal" has two related meanings. In colloquial usage, it denotes a shape that is recursively constructed or self-similar, that is, a shape that appears similar at all scales of magnification and is therefore often referred to as "infinitely complex." In mathematics a fractal is a geometric object that satisfies a specific technical condition, namely having a Hausdorff dimension greater than its topological dimension. The term fractal was coined in 1975 by Benot Mandelbrot, from the Latin fractus, meaning "broken" or "fractured."

Three common techniques for generating fractals are:

- Iterated function systems - These have a fixed geometric replacement rule. Cantor set, Sierpinski carpet, Sierpinski gasket, Peano curve, Koch snowflake, Harter-Heighway dragon curve, T-Square, Menger sponge, are some examples of such fractals.
- Escape-time fractals - Fractals defined by a recurrence relation at each point in a space (such as the complex plane). Examples of this type are the Mandelbrot set, the Burning Ship fractal and the Lyapunov fractal.
- Random fractals, generated by stochastic rather than deterministic processes, for example, fractal landscapes, Lvy flight and the Brownian tree. The latter yields so-called mass- or dendritic fractals, for example, Diffusion Limited Aggregation or Reaction Limited Aggregation clusters.

Fractals in nature

Approximate fractals are easily found in nature. These objects display self-similar structure over an extended, but finite, scale range. Examples include clouds, snow flakes, mountains, river networks, and systems of blood vessels.

Trees and ferns are fractal in nature and can be modeled on a computer using a recursive algorithm. This recursive nature is clear in these examples - a branch from a tree or a frond from a fern is a miniature replica of the whole: not identical, but similar in nature.

The surface of a mountain can be modeled on a computer using a fractal: Start with a triangle in 3D space and connect the central points of each side by line segments, resulting in 4 triangles. The central points are then randomly moved up or down, within a defined range. The procedure is repeated, decreasing at each iteration the range by half. The recursive nature of the algorithm guarantees that the whole is statistically similar to each detail.

Summary of the Trigonomertic Rules and Identities

Pythagorean Identity

$\cos^2\theta + \sin^2\theta = 1$

Cofuntion Identities

$\sin(90° - \theta) = \cos\theta$
$\cos(90° - \theta) = \sin\theta$

Ratio Identities

$\tan\theta = \frac{\sin\theta}{\cos\theta}$

Odd/Even Identities

$\sin(-\theta) = -\sin\theta$
$\cos(-\theta) = \cos\theta$
$\tan(-\theta) = -\tan\theta$

Periodicity Identities

$\sin(\theta \pm 360°) = \sin\theta$
$\cos(\theta \pm 360°) = \cos\theta$
$\tan(\theta \pm 180°) = \tan\theta$

Double angle Identities

$\sin(2\theta) = 2\sin\theta\cos\theta$
$\cos(2\theta) = \cos^2\theta - \sin^2\theta$
$\cos(2\theta) = 2\cos^2\theta - 1$
$\tan(2\theta) = \frac{2\tan\theta}{1-\tan^2\theta}$

Addition/Subtraction Identities

$\sin(\theta + \phi) = \sin\theta\cos\phi + \cos\theta\sin\phi$
$\sin(\theta - \phi) = \sin\theta\cos\phi - \cos\theta\sin\phi$
$\cos(\theta + \phi) = \cos\theta\cos\phi - \sin\theta\sin\phi$
$\cos(\theta - \phi) = \cos\theta\cos\phi + \sin\theta\sin\phi$
$\tan(\theta + \phi) = \frac{\tan\phi + \tan\theta}{1 - \tan\theta\tan\phi}$
$\tan(\theta - \phi) = \frac{\tan\phi - \tan\theta}{1 + \tan\theta\tan\phi}$

Area Rule

$\text{Area} = \frac{1}{2}bc\sin A$
$\text{Area} = \frac{1}{2}ab\sin C$
$Area = \frac{1}{2}ac\sin B$

Cosine rule

$a^2 = b^2 + c^2 - 2bc\cos A$
$b^2 = a^2 + c^2 - 2ac\cos B$
$c^2 = a^2 + b^2 - 2ab\cos C$

Sine Rule

$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$

43.4 End of Chapter Exercises

Do the following without using a calculator.

1. Suppose $\cos\theta = 0.7$. Find $\cos 2\theta$ and $\cos 4\theta$.
2. If $\sin\theta = \frac{4}{7}$, again find $\cos 2\theta$ and $\cos 4\theta$.
3. Work out the following:
   A $\cos 15°$
   B $\cos 75°$
   C $\tan 105°$
   D $\cos 15°$
   E $\cos 3° \cos 42° - \sin 3° \sin 42°$
   F $1 - 2\sin^2(22.5°)$
4. Solve the following equations:
   A $\cos 3\theta \cdot \cos\theta - \sin 3\theta \cdot \sin\theta = -\frac{1}{2}$
   B $3\sin\theta = 2\cos^2\theta$
   C
5. Prove the following identities
   A $\sin^3\theta = \frac{3\sin\theta - \sin 3\theta}{4}$

B $\cos^2\alpha(1-\tan^2\alpha)=\cos 2\alpha$

C $4\sin\theta\cdot\cos\theta\cdot\cos 2\theta=\sin 4\theta$

D $4\cos^3 x-3\cos x=\cos 3x$

E $\tan y=\frac{\sin 2y}{\cos 2y+1}$

6. (Challenge question!) If $a+b+c=180°$, prove that

$$\sin^3 a+\sin^3 b+\sin^3 c=3\cos(a/2)\cos(b/2)\cos(c/2)+\cos(3a/2)\cos(3b/2)\cos(3c/2)$$

Chapter 44

Statistics - Grade 12

44.1 Introduction

In this chapter, you will use the mean, median, mode and standard deviation of a set of data to identify whether the data is normally distributed or whether it is skewed. You will learn more about populations and selecting different kinds of samples in order to avoid bias. You will work with lines of best fit, and learn how to find a regression equation and a correlation coefficient. You will analyse these measures in order to draw conclusions and make predictions.

44.2 A Normal Distribution

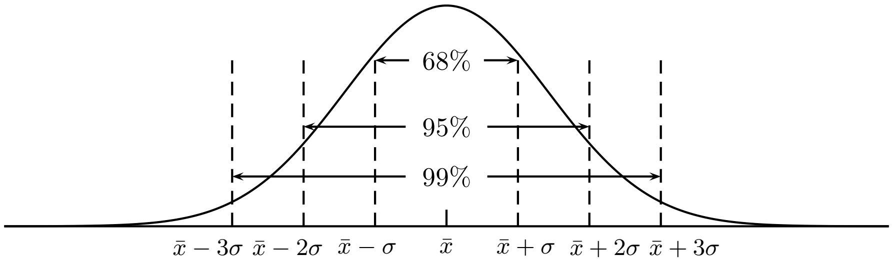
Activity :: Investigation :
You are given a table of data below.

75	67	70	71	71	73	74	75
80	75	77	78	78	78	78	79
91	81	82	82	83	86	86	87

1. Calculate the mean, median, mode and standard deviation of the data.
2. What percentage of the data is within one standard deviation of the mean?
3. Draw a histogram of the data using intervals $60 \leq x < 64$, $64 \leq x < 68$, etc.
4. Join the midpoints of the bars to form a frequency polygon.

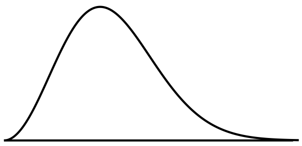
If large numbers of data are collected from a population, the graph will often have a bell shape. If the data was, say, examination results, a few learners usually get very high marks, a few very low marks and most get a mark in the middle range. We say a distribution is *normal* if

- the mean, median and mode are equal.
- it is symmetric around the mean.
- $\pm 68\%$ of the sample lies within one standard deviation of the mean, 95% within two standard deviations and 99% within three standard deviations of the mean.

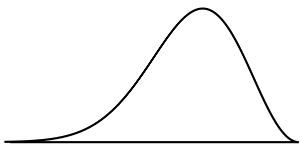


What happens if the test was very easy or very difficult? Then the distribution may not be symmetrical. If extremely high or extremely low scores are added to a distribution, then the mean tends to shift towards these scores and the curve becomes skewed.

If the test was very difficult, the mean score is shifted to the left. In this case, we say the distribution is *positively skewed*, or *skewed right*.


Skewed right

If it was very easy, then many learners would get high scores, and the mean of the distribution would be shifted to the right. We say the distribution is *negatively skewed*, or *skewed left*.


Skewed left

Exercise: Normal Distribution

1. Given the pairs of normal curves below, sketch the graphs on the same set of axes and show any relation between them. An important point to remember is that the area beneath the curve corresponds to 100%.
   A Mean = 8, standard deviation = 4 and Mean = 4, standard deviation = 8
   B Mean = 8, standard deviation = 4 and Mean = 16, standard deviation = 4
   C Mean = 8, standard deviation = 4 and Mean = 8, standard deviation = 8
2. After a class test, the following scores were recorded:

Test Score	Frequency
3	1
4	7
5	14
6	21
7	14
8	6
9	1
Total	64
Mean	6
Standard Deviation	1,2

   A Draw the histogram of the results.
   B Join the midpoints of each bar and draw a frequency polygon.
   C What mark must one obtain in order to be in the top 2% of the class?
   D Approximately 84% of the pupils passed the test. What was the pass mark?
   E Is the distribution normal or skewed?

3. In a road safety study, the speed of 175 cars was monitored along a specific stretch of highway in order to find out whether there existed any link between high speed and the large number of accidents along the route. A frequency table of the results is drawn up below.

Speed (km.h^{-1})	**Number of cars (Frequency)**
50	19
60	28
70	23
80	56
90	20
100	16
110	8
120	5

The mean speed was determined to be around 82 km.h^{-1} while the median speed was worked out to be around 84,5 km.h^{-1}.

A Draw a frequency polygon to visualise the data in the table above.

B Is this distribution symmetrical or skewed left or right? Give a reason fro your answer.

44.3 Extracting a Sample Population

Suppose you are trying to find out what percentage of South Africa's population owns a car. One way of doing this might be to send questionnaires to peoples homes, asking them whether they own a car. However, you quickly run into a problem: you cannot hope to send every person in the country a questionnaire, it would be far to expensive. Also, not everyone would reply. The best you can do is send it to a few people, see what percentage of these own a car, and then use this to estimate what percentage of the entire country own cars. This smaller group of people is called the *sample population*.

The sample population must be carefully chosen, in order to avoid biased results. How do we do this?
First, it must be *representative*. If all of our sample population comes from a very rich area, then almost all will have cars. But we obviously cannot conclude from this that almost everyone in the country has a car! We need to send the questionnaire to rich as well as poor people.
Secondly, the *size* of the sample population must be large enough. It is no good having a sample population consisting of only two people, for example. Both may very well not have cars. But we obviously cannot conclude that no one in the country has a car! The larger the sample population size, the more likely it is that the statistics of our sample population corresponds to the statistics of the entire population.

So how does one ensure that ones sample is representative? There are a variety of methods available, which we will look at now.

Random Sampling. Every person in the country has an equal chance of being selected. It is unbiased and also independant, which means that the selection of one person has no effect on the selection on another. One way of doing this would be to give each person in the country a number, and then ask a computer to give us a list of random numbers. We could then send the questionnaire to the people corresponding to the random numbers.

Systematic Sampling. Again give every person in the country a number, and then, for example, select every hundredth person on the list. So person with number 1 would be selected, person with number 100 would be selected, person with number 200 would be selected, etc.

Stratified Sampling. We consider different subgroups of the population, and take random samples from these. For example, we can divide the population into male and female, different ages, or into different income ranges.

Cluster Sampling. Here the sample is concentrated in one area. For example, we consider all the people living in one urban area.

?

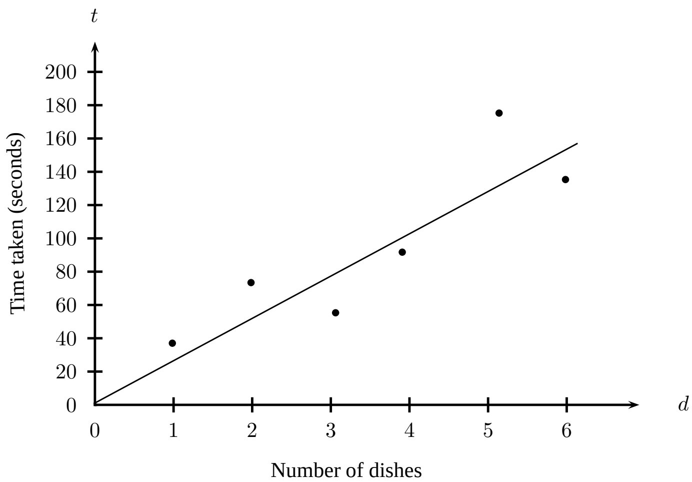
Exercise: Sampling

1. Discuss the advantages, disadvantages and possible bias when using
   A systematic sampling
   B random sampling
   C cluster sampling
2. Suggest a suitable sampling method that could be used to obtain information on:
   A passengers views on availability of a local taxi service.
   B views of learners on school meals.
   C defects in an item made in a factory.
   D medical costs of employees in a large company.
3. 5% of a certain magazines' subscribers is randomly selected. The random number 16 out of 50, is selected. Then subscribers with numbers 16, 66, 116, 166, ... are chosen as a sample. What kind of sampling is this?

44.4 Function Fitting and Regression Analysis

In Grade 11 we recorded two sets of data (bivariate data) on a scatter plot and then we drew a line of best fit as close to as many of the data items as possible. Regression analysis is a method of finding out exactly which function best fits a given set of data. We can find out the equation of the regression line by drawing and estimating, or by using an algebraic method called "the least squared method", or we can use a calculator. The linear regression equation is written $\hat{y} = a + bx$ (we say y-hat) or $y = A + Bx$. Of course these are both variations of a more familiar equation $y = mx + c$.

Suppose you are doing an experiment with washing dishes. You count how many dishes you begin with, and then find out how long it takes to finish washing them. So you plot the data on a graph of time taken versus number of dishes. This is plotted below.



If t is the time taken, and d the number of dishes, then it looks as though t is proportional to d, ie. $t = m \cdot d$, where m is the constant of proportionality. There are two questions that interest us now.

1. How do we find m? One way you have already learnt, is to draw a line of best-fit through the data points, and then measure the gradient of the line. But this is not terribly precise. Is there a better way of doing it?
2. How well does our line of best fit really fit our data? If the points on our plot don't all lie close to the line of best fit, but are scattered everywhere, then the fit is not 'good', and our assumption that $t = m \cdot d$ might be incorrect. Can we find a quantitative measure of how well our line really fits the data?

In this chapter, we answer both of these questions, using the techniques of *regression analysis*.

Worked Example 200: Fitting by hand

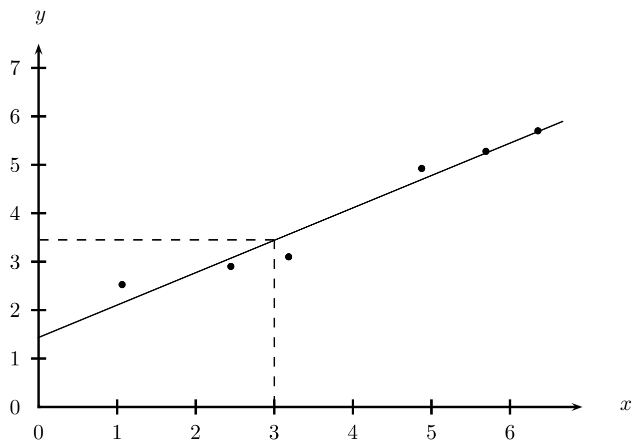
Question: Use the data given to draw a scatter plot and line of best fit. Now write down the equation of the line that best seems to fit the data.

x	1,0	2,4	3,1	4,9	5,6	6,2
y	2,5	2,8	3,0	4,8	5,1	5,3

Answer

Step 1 : Drawing the graph

The first step is to draw the graph. This is shown below.



Step 2 : Calculating the equation of the line

The equation of the line is

$$y = mx + c$$

From the graph we have drawn, we estimate the y-intercept to be 1,5. We estimate that $y = 3{,}5$ when $x = 3$. So we have that points $(3; 3{,}5)$ and $(0; 1{,}6)$ lie on the line. The gradient of the line, m, is given by

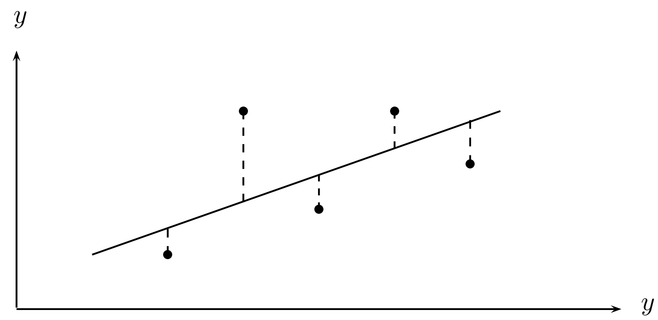
$$\begin{aligned} m &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{3{,}5 - 1{,}5}{3 - 0} \\ &= \frac{2}{3} \end{aligned}$$

So we finally have that the equation of the line of best fit is

$$y = \frac{2}{3}x + 1{,}5$$

44.4.1 The Method of Least Squares

We now come to a more accurate method of finding the line of best-fit. The method is very simple.
Suppose we guess a line of best-fit. Then at at every data point, we find the distance between the data point and the line. If the line fitted the data perfectly, this distance should be zero for all the data points. The worse the fit, the larger the differences. We then square each of these distances, and add them all together.



The best-fit line is then the line that minimises the sum of the squared distances.
Suppose we have a data set of n points $\{(x_1; y_1), (x_2; y_2), \ldots, (x_n,y_n)\}$. We also have a line $f(x) = mx + c$ that we are trying to fit to the data. The distance between the first data point and the line, for example, is

$$\text{distance} = y_1 - f(x) = y_1 - (mx + c)$$

We now square each of these distances and add them together. Lets call this sum $S(m,c)$. Then we have that

$$\begin{aligned} S(m,c) &= (y_1 - f(x_1))^2 + (y_2 - f(x_2))^2 + \ldots + (y_n - f(x_n))^2 \\ &= \sum_{i=1}^{n}(y_i - f(x_i))^2 \end{aligned}$$

Thus our problem is to find the value of m and c such that $S(m,c)$ is minimised. Let us call these minimising values m_0 and c_0. Then the line of best-fit is $f(x) = m_0x + c_0$. We can find m_0 and c_0 using calculus, but it is tricky, and we will just give you the result, which is that

$$\begin{aligned} m_0 &= \frac{n\sum_{i=1}^{n} x_i y_i - \sum_{i=1}^{n} x_i \sum_{i=1}^{n} y_i}{n\sum_{i=1}^{n}(x_i)^2 - \left(\sum_{i=1}^{n} x_i\right)^2} \\ c_0 &= \frac{1}{n}\sum_{i=1}^{n} y_i - \frac{m_0}{n}\sum_{i=0}^{n} x_i = \bar{y} - m_0\bar{x} \end{aligned}$$

Worked Example 201: Method of Least Squares

Question: In the table below, we have the records of the maintenance costs in Rands, compared with the age of the appliance in months. We have data for 5 appliances.

appliance	1	2	3	4	4
age (x)	5	10	15	20	30
cost (y)	90	140	250	300	380

Answer

appliance	x	y	xy	x^2
1	10	15	20	30
2	10	140	1400	100
3	15	250	3750	225
4	20	300	6000	400
5	30	380	11400	900
Total	80	1160	23000	1650

$$\begin{aligned} b &= \frac{n\sum xy - \sum x \sum y}{n\sum x^2 - \left(\sum x\right)^2} = \frac{5 \times 23000 - 80 \times 1160}{5 \times 1650 - 80^2} = 12 \\ a &= \bar{y} - b\bar{x} = \frac{1160}{5} - \frac{12 \times 80}{5} = 40 \\ \therefore &\quad \underline{\hat{y} = 40 + 12x} \end{aligned}$$

44.4.2 Using a calculator

Worked Example 202: Using the Sharp EL-531VH calculator

Question: Find a regression equation for the following data:

Days (x)	1	2	3	4	5
Growth in m (y)	1,00	2,50	2,75	3,00	3,50

Answer

Step 1 : Getting your calculator ready

Using your calculator, change the mode from normal to "Stat xy". This mode enables you to type in bivariate data.

Step 2 : Entering the data

Key in the data as follows:

1	(x,y)	1	DATA	$n = 1$
2	(x,y)	2,5	DATA	$n = 2$
3	(x,y)	2,75	DATA	$n = 3$
4	(x,y)	3,0	DATA	$n = 4$
5	(x,y)	3,5	DATA	$n = 5$

Step 3 : Getting regression results from the calculator

Ask for the values of the regression coefficients a and b.

RCL	a	gives	$a = 0{,}9$
RCL	b	gives	$b = 0{,}55$

$$\therefore \hat{y} = 0{,}9 + 0{,}55x$$

Worked Example 203: Using the CASIO fx-82ES Natural Display calculator

Question: Using a calculator determine the least squares line of best fit for the following data set of marks.

Learner	1	2	3	4	5
Chemistry (%)	52	55	86	71	45
Accounting (%)	48	64	95	79	50

For a Chemistry mark of 65%, what mark does the least squares line predict for Accounting?

Answer

Step 1 : Getting your calculator ready

Switch on the calculator. Press [MODE] and then select STAT by pressing [2]. The following screen will appear:

1	1-VAR	2	A + BX
3	_ + CX^2	4	ln X
5	e^X	6	A . B^X
7	A . X^B	8	1/X

Now press [2] for linear regression. Your screen should look something like this:

	x	y
1		
2		
3		

Step 2 : Entering the data

Press [52] and then [=] to enter the first mark under x. Then enter the other values, in the same way, for the x-variable (the Chemistry marks) in the order in which they are given in the data set. Then move the cursor across and up and enter 48 under y opposite 52 in the x-column. Continue to enter the other y-values (the Accounting marks) in order so that they pair off correctly with the corresponding x-values.

	x	y
1	52	
2	55	
3		

Then press [AC]. The screen clears but the data remains stored.

1:	Type	2:	Data
3:	Edit	4:	Sum
5:	Var	6:	MinMax
7:	Reg		

Now press [SHIFT][1] to get the stats computations screen shown below. Choose Regression by pressing [7].

1:	A	2:	B
3:	r	4:	$\hat{x}$
5:	$\hat{y}$		

Step 3 : Getting regression results from the calculator

a) Press [1] and [=] to get the value of the y-intercept, $a = -5{,}065.. = -5{,}07$(to 2 d.p.)
Finally, to get the slope, use the following key sequence: [SHIFT][1][7][2][=].
The calculator gives $b = 1{,}169.. = 1{,}17$(to 2 d.p.)

The equation of the line of regression is thus:
$\hat{y} = -5{,}07 + 1{,}17x$

b) Press [AC][65][SHIFT][1][7][5][=]
This gives a (predicted) Accounting mark of ˆ$= 70{,}938.. = 71\%$

Exercise:

1. The table below lists the exam results for 5 students in the subjects of Science and Biology.

Learner	1	2	3	4	5
Science %	55	66	74	92	47
Biology %	48	59	68	84	53

   A Use the formulae to find the regression equation coefficients a and b.
   B Draw a scatter plot of the data on graph paper.
   C Now use algebra to find a more accurate equation.

2. Footlengths and heights of 7 students are given in the table below.

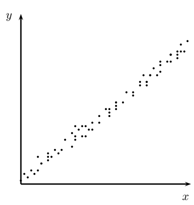
Height (cm)	170	163	131	181	146	134	166
Footlength (cm)	27	23	20	28	22	20	24

   A Draw a scatter plot of the data on graph paper.
   B Indentify and describe any trends shown in the scatter plot.
   C Find the equation of the least squares line by using algebraic methods and draw the line on your graph.
   D Use your equation to predict the height of a student with footlength 21,6 cm.
   E Use your equation to predict the footlength of a student 176 cm tall.

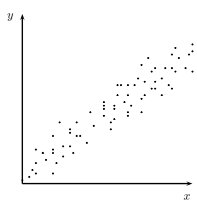
3. Repeat the data in question 2 and find the regression line using a calculator

44.4.3 Correlation coefficients

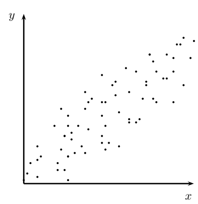
Once we have applied regression analysis to a set of data, we would like to have a number that tells us exactly how well the data fits the function. A correlation coefficient, r, is a tool that tells us to what degree there is a relationship between two sets of data. The correlation coefficient $r \in [-1;1]$ when $r = -1$, there is a perfect negative relationship, when $r = 0$, there is no relationship and $r = 1$ is a perfect positive correlation.



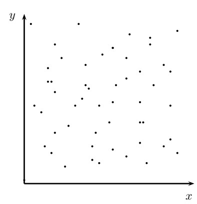
Positive, strong
$r \approx 0{,}9$



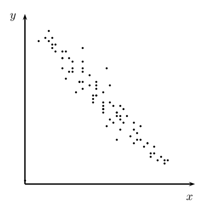
Positive, fairly strong
$r \approx 0{,}7$



Positive, weak
$r \approx 0{,}4$



No association
$r = 0$



Negative, fairly strong
$r \approx -0{,}7$

We often use the correlation coefficient r^2 in order to work with the strength of the correlation only (no whether it is positive or negative).

In this case:

$r^2 = 0$	no correlation
$0 < r^2 <$ 0,25	very weak
0,25 $< r^2 <$ 0,5	weak
0,5 $< r^2 <$ 0,75	moderate
0,75 $< r^2 <$ 0,9	strong
0,9 $< r^2 < 1$	very strong
$r^2 = 1$	perfect correlation

The correlation coefficient r can be calculated using the formula

$$r = \frac{1}{n-1}\sum\left(\frac{x-\bar{x}}{s_x}\right)\left(\frac{y-\bar{y}}{s_y}\right)$$

- where n is the number of data points,
- s_x is the standard deviation of the x-values and
- s_y is the standard deviation of the y-values.

This is known as the Pearson's product moment correlation coefficient. It is a long calculation and much easier to do on the calculator where you simply follow the procedure for the regression equation, and go on to find r.

44.5 Exercises

1. Below is a list of data concerning 12 countries and their respective carbon dioxide (CO_2) emmission levels per person and the gross domestic product (GDP - a measure of products produced and services delivered within a country in a year) per person.

	CO_2 emmissions per capita (x)	GDP per capita (y)
South Africa	8,1	3 938
Thailand	2,5	2 712
Italy	7,3	20 943
Australia	17,0	23 893
China	2,5	816
India	0,9	463
Canada	16,0	22 537
United Kingdom	9,0	21 785
United States	19,9	31 806
Saudi Arabia	11,0	6 853
Iran	3,8	1 493
Indonesia	1,2	986

A Draw a scatter plot of the data set and your estimate of a line of best fit.
B Calculate equation of the line of regression using the method of least squares.
C Draw the regression line equation onto the graph.
D Calculate the correlation coefficient r.
E What conclusion can you reach, regarding the relationship between CO_2 emission and GDP per capita for the countries in the data set?

2. A collection of data on the peculiar investigation into a foot size and height of students was recorded in the table below. Answer the questions to follow.

Length of right foot (cm)	Height (cm)
25,5	163,3
26,1	164,9
23,7	165,5
26,4	173,7
27,5	174,4
24	156
22,6	155,3
27,1	169,3

A Draw a scatter plot of the data set and your estimate of a line of best fit.

B Calculate equation of the line of regression using the method of least squares or your calculator.

C Draw the regression line equation onto the graph.

D Calculate the correlation coefficient r.

E What conclusion can you reach, regarding the relationship between the length of the right foot and height of the students in the data set?

3. A class wrote two tests, and the marks for each were recorded in the table below. Full marks in the first test was 50, and the second test was out of 30.

   A Is there a strong association between the marks for the first and second test? Show why or why not.

   B One of the learners (in row 18) did not write the second test. Given their mark for the first test, calculate an expected mark for the second test.

Learner	Test 1 (Full marks: 50)	Test 2 (Full marks: 30)
1	42	25
2	32	19
3	31	20
4	42	26
5	35	23
6	23	14
7	43	24
8	23	12
9	24	14
10	15	10
11	19	11
12	13	10
13	36	22
14	29	17
15	29	17
16	25	16
17	29	18
18	17	
19	30	19
20	28	17

4. A fast food company produces hamburgers. The number of hamburgers made, and the costs are recorded over a week.

Hamburgers made	Costs
495	R2382
550	R2442
515	R2484
500	R2400
480	R2370
530	R2448
585	R2805

   A Find the linear regression function that best fits the data.

   B If the total cost in a day is R2500, estimate the number of hamburgers produced.

   C What is the cost of 490 hamburgers?

5. The profits of a new shop are recorded over the first 6 months. The owner wants to predict his future sales. The profits so far have been R90 000 , R93 000, R99 500, R102 000, R101 300, R109 000.

   A For the profit data, calculate the linear regression function.

B Give an estimate of the profits for the next two months.

C The owner wants a profit of R130 000. Estimate how many months this will take.

6. A company produces sweets using a machine which runs for a few hours per day. The number of hours running the machine and the number of sweets produced are recorded.

Machine hours	Sweets produced
3,80	275
4,23	287
4,37	291
4,10	281
4,17	286

Find the linear regression equation for the data, and estimate the machine hours needed to make 300 sweets.

Chapter 45

Combinations and Permutations - Grade 12

45.1 Introduction

Mathematics education began with counting. At the beginning, fingers, beans, buttons, and pencils were used to help with counting, but these are only practical for small numbers. What happens when a large number of items must be counted?

This chapter focuses on how to use mathematical techniques to count combinations of items.

45.2 Counting

An important aspect of probability theory is the ability to determine the total number of possible outcomes when multiple events are considered.

For example, what is the total number of possible outcomes when a die is rolled and then a coin is tossed? The roll of a die has six possible outcomes (1, 2, 3, 4, 5 or 6) and the toss of a coin, 2 outcomes (head or tails). Counting the possible outcomes can be tedious.

45.2.1 Making a List

The simplest method of counting the total number of outcomes is by making a list:

1H, 1T, 2H, 2T, 3H, 3T, 4H, 4T, 5H, 5T, 6H, 6T

or drawing up a table.

die	coin
1	H
1	T
2	H
2	T
3	H
3	T
4	H
4	T
5	H
5	T
6	H
6	T

Both these methods result in 12 possible outcomes, but both these methods have a lot of repetition.

45.2.2 Tree Diagrams

One method of eliminating some of the repetition is to use *tree diagrams*. Tree diagrams are a graphical method of listing all possible combinations of events from a random experiment.

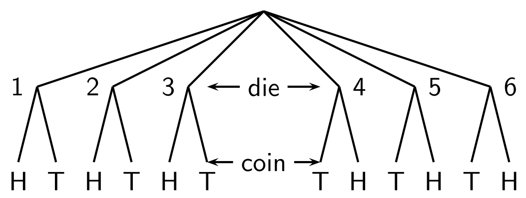


Figure 45.1: Example of a tree diagram. Each possible outcome is a branch of the tree.

45.3 Notation

45.3.1 The Factorial Notation

For an integer n, the notation $n!$ (read n factorial) represents:

$$n \times (n-1) \times (n-2) \times \ldots \times 3 \times 2 \times 1$$

with the special case of $0! = 1$.

The factorial notation will be used often in this chapter.

45.4 The Fundamental Counting Principle

The use of lists, tables and tree diagrams is only feasible for events with a few outcomes. When the number of outcomes grows, it is not practical to list the different possibilities and the fundamental counting principle is used.

The **fundamental counting principle** describes how to determine the total number of outcomes of a series of events.

Suppose that two experiments take place. The first experiment has n_1 possible outcomes, and the second has n_2 possible outcomes. Therefore, the first experiment, followed by the second experiment, will have a total of $n_1 \times n_2$ possible outcomes. This idea can be generalised to m experiments as the total number of outcomes for m experiments is:

$$n_1 \times n_2 \times n_3 \times \ldots \times n_m = \prod_{i=1}^{m} n_i$$

$\prod$ is the multiplication equivalent of $\sum$.

Note: the order in which the experiments are done does not affect the total number of possible outcomes.

Worked Example 204: Lunch Special

Question: A take-away has a 4-piece lunch special which consists of a sandwich, soup, dessert and drink for R25.00. They offer the following choices for :
Sandwich: chicken mayonnaise, cheese and tomato, tuna, and ham and lettuce
Soup: tomato, chicken noodle, vegetable
Dessert: ice-cream, piece of cake
Drink: tea, coffee, coke, Fanta and Sprite. How many possible meals are there?

Answer

Step 1 : Determine how many parts to the meal there are

There are 4 parts: sandwich, soup, dessert and drink.

Step 2 : Identify how many choices there are for each part

Meal Component	Sandwich	Soup	Dessert	Drink
Number of choices	4	3	2	5

Step 3 : Use the fundamental counting principle to determine how many different meals are possible

$$4 \times 3 \times 2 \times 5 = 120$$

So there are 120 possible meals.

45.5 Combinations

The fundamental counting principle describes how to calculate the total number of outcomes when multiple independent events are performed together.

A more complex problem is determining how many combinations there are of selecting a group of objects from a set. Mathematically, a *combination* is defined as an un-ordered collection of unique elements, or more formally, a subset of a set. For example, suppose you have fifty-two playing cards, and select five cards. The five cards would form a combination and would be a subset of the set of 52 cards.

In a set, the order of the elements in the set does not matter. These are represented usually with curly braces, for example $\{2, 4, 6\}$ is a subset of the set $\{1,2,3,4,5,6\}$. Since the order of the elements does not matter, only the specific elements are of interest. Therefore,

$$\{2, 4, 6\} = \{6, 4, 2\}$$

and $\{1, 1, 1\}$ is the same as $\{1\}$ because a set is defined by its elements; they don't usually appear more than once.

Given S, the set of all possible unique elements, a combination is a subset of the elements of S. The order of the elements in a combination is not important (two lists with the same elements in different orders are considered to be the same combination). Also, the elements cannot be repeated in a combination (every element appears uniquely once).

45.5.1 Counting Combinations

Calculating the number of ways that certain patterns can be formed is the beginning of *combinatorics*, the study of combinations. Let S be a set with n objects. Combinations of k objects from this set S are subsets of S having k elements each (where the order of listing the elements does not distinguish two subsets).

Combination without Repetition

When the order does not matter, but each object can be chosen only once, the number of combinations is:

$$\frac{n!}{r!(n-r)!} = \binom{n}{r}$$

where n is the number of objects from which you can choose and r is the number to be chosen.

For example, if you have 10 numbers and wish to choose 5 you would have 10!/(5!(10 - 5)!) $=$ 252 ways to choose.

For example how many possible 5 card hands are there in a deck of cards with 52 cards?

52! / (5!(52-5)!) $= 2\ 598\ 960$ combinations

Combination with Repetition

When the order does not matter and an object can be chosen more than once, then the number of combinations is:

$$\frac{(n+r-1)!}{r!(n-1)!} = \binom{n+r-1}{r} = \binom{n+r-1}{n-1}$$

where n is the number of objects from which you can choose and r is the number to be chosen.

For example, if you have ten types of donuts to choose from and you want three donuts there are (10 + 3 - 1)! / 3!(10 - 1)! = 220 ways to choose.

45.5.2 Combinatorics and Probability

Combinatorics is quite useful in the computation of probabilities of events, as it can be used to determine exactly how many outcomes are possible in a given event.

Worked Example 205: Probability

Question: At a school, learners each play 2 sports. They can choose from netball, basketball, soccer, athletics, swimming, or tennis. What is the probability that a learner plays soccer and either netball, basketball or tennis?

Answer
Step 1 : Identify what events we are counting
We count the events: soccer and netball, soccer and basketball, soccer and tennis. This gives three choices.
Step 2 : Calculate the total number of choices
There are 6 sports to choose from and we choose 2 sports. There are $\binom{6}{2} = 6!/(2!(6-2)!) = 15$ choices.
Step 3 : Calculate the probability
The probability is the number of events we are counting, divided by the total number of choices.
Probability $= \frac{3}{15} = \frac{1}{5} = 0{,}2$

45.6 Permutations

The concept of a combination did not consider the order of the elements of the subset to be important. A permutation is a combination with the order of a selection from a group being important. For example, for the set $\{1,2,3,4,5,6\}$, the combination $\{1,2,3\}$ would be identical to the combination $\{3,2,1\}$, but these two combinations are permutations, because the elements in the set are ordered differently.

More formally, a permutation is an ordered list without repetitions, perhaps missing some elements.

This means that $\{1,2,2,3,4,5,6\}$ and $\{1,2,4,5,5,6\}$ are not permutations of the set $\{1,2,3,4,5,6\}$.

Now suppose you have these objects:

1, 2, 3

Here is a list of all permutations of those:

1 2 3; 1 3 2; 2 1 3; 2 3 1; 3 1 2; 3 2 1;

45.6.1 Counting Permutations

Let S be a set with n objects. Permutations of k objects from this set S refer to sequences of k different elements of S (where two sequences are considered different if they contain the same elements but in a different order, or if they have a different length). Formulas for the number of permutations and combinations are readily available and important throughout combinatorics.

It is easy to count the number of permutations of size r when chosen from a set of size n (with $r \leq n$).

1. Select the first member of all permutations out of n choices because there are n distinct elements in the set.

2. Next, since one of the n elements has already been used, the second member of the permutation has $(n-1)$ elements to choose from the remaining set.

3. The third member of the permutation can be filled in $(n-2)$ ways since 2 have been used already.

4. This pattern continues until there are r members on the permutation. This means that the last member can be filled in $(n-(r-1)) = (n-r+1)$ ways.

5. Summarizing, we find that there is a total of

$$n(n-1)(n-2)...(n-r+1)$$

different permutations of r objects, taken from a pool of n objects. This number is denoted by $P(n,r)$ and can be written in factorial notation as:

$$P(n,r) = \frac{n!}{(n-r)!}.$$

For example, if we have a total of 5 elements, the integers $\{1,2,3,4,5\}$, how many ways are there for a permutation of three elements to be selected from this set? In this case, $n = 10$ and $r = 3$. Then, $P(10,3) = 10!/7! = 720$.

Worked Example 206: Permutations

Question: Show that a collection of n objects has $n!$ permutations.

Answer

Proof: Constructing an ordered sequence of n objects is equivalent to choosing the position occupied by the first object, then choosing the position of the second object, and so on, until we have chosen the position of each of our n objects.

There are n ways to choose a position for the first object. Once its position is fixed, we can choose from (n-1) possible positions for the second object. With the first two placed, there are (n-2) remaining possible positions for the third object; and so on. There are only two positions to choose from for the penultimate object, and the nth object will occupy the last remaining position.

Therefore, according to the multiplicative principle, there are

$$n(n-1)(n-2)...2 \times 1 = n!$$

ways of constructing an ordered sequence of n objects.

Permutation with Repetition

When order matters and an object can be chosen more than once then the number of permutations is:

$$n^r$$

where n is the number of objects from which you can choose and r is the number to be chosen.

For example, if you have the letters A, B, C, and D and you wish to discover the number of ways of arranging them in three letter patterns (trigrams) you find that there are 4^3 or 64 ways. This is because for the first slot you can choose any of the four values, for the second slot you can choose any of the four, and for the final slot you can choose any of the four letters. Multiplying them together gives the total.

Permutation without Repetition

When the order matters and each object can be chosen only once, then the number of permutations is:

$$\frac{n!}{(n-r)!}$$

where n is the number of objects from which you can choose and r is the number to be chosen.

For example, if you have five people and are going to choose three out of these, you will have 5!/(5-3)! = 60 permutations.

Note that if n = r (meaning number of chosen elements is equal to number of elements to choose from) then the formula becomes

$$\frac{n!}{(n-n)!} = \frac{n!}{0!} = n!$$

For example, if you have three people and you want to find out how many ways you may arrange them it would be 3! or $3 \times 2 \times 1 = 6$ ways. The reason for this is because you can choose from three for the initial slot, then you are left with only two to choose from for the second slot, and that leaves only one for the final slot. Multiplying them together gives the total.

45.7 Applications

Extension: The Binomial Theorem

In mathematics, the binomial theorem is an important formula giving the expansion of powers of sums. Its simplest version reads

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

Whenever n is any positive integer, the numbers

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

are the binomial coefficients (the coefficients in front of powers).
For example, here are the cases n = 2, n = 3 and n = 4:

$$(x+y)^2 = x^2 + \mathbf{2}y + y^2$$
$$(x+y)^3 = x^3 + \mathbf{3}x^2y + \mathbf{3}xy^2 + y^3$$
$$(x+y)^4 = x^4 + \mathbf{4}x^3y + \mathbf{6}x^2y^2 + \mathbf{4}xy^3 + y^4$$

The coefficients form a triangle, where each number is the sum of the two numbers above it:

```
        1
      1   1
    1   2   1
  1   3   3   1
1   4   6   4   1
      . . .
```

This formula, and the triangular arrangement of the binomial coefficients, are often attributed to Blaise Pascal who described them in the 17th century. It was, however, known to the Chinese mathematician Yang Hui in the 13th century, the earlier Persian mathematician Omar Khayym in the 11th century, and the even earlier Indian mathematician Pingala in the 3rd century BC.

Worked Example 207: Number Plates

Question: The number plate on a car consists of any 3 letters of the alphabet (excluding the vowels and 'Q'), followed by any 3 digits (0 to 9). For a car chosen at random, what is the probability that the number plate starts with a 'Y' and ends with an odd digit?

Answer

Step 1 : Identify what events are counted

The number plate starts with a 'Y', so there is only 1 choice for the first letter, and ends with an even digit, so there are 5 choices for the last digit (1,3,5,7,9).

Step 2 : Find the number of events

Use the counting principle. For each of the other letters, there are 20 possible choices (26 in the alphabet, minus 5 vowels and 'Q') and 10 possible choices for each of the other digits.

Number of events $= 1 \times 20 \times 20 \times 10 \times 10 \times 5 = 200\ 000$

Step 3 : Find the number of total possible number plates

Use the counting principle. This time, the first letter and last digit can be anything.

Total number of choices $= 20 \times 20 \times 20 \times 10 \times 10 \times 10 = 8\ 000\ 000$

Step 4 : Calculate the probability

The probability is the number of events we are counting, divided by the total number of choices.

Probability $= \frac{200\ 000}{8\ 000\ 000} = \frac{1}{40} = 0{,}025$

Worked Example 208: Factorial

Question: Show that

$$\frac{n!}{(n-1)!} = n$$

Answer

Method 1: Expand the factorial notation.

$$\frac{n!}{(n-1)!} = \frac{n \times (n-1) \times (n-2) \times ... \times 2 \times 1}{(n-1) \times (n-2) \times ... \times 2 \times 1}$$

Cancelling the common factor of $(n-1) \times (n-2) \times ... \times 2 \times 1$ on the top and bottom leaves n.

So $\frac{n!}{(n-1)!} = n$

Method 2: We know that $P(n,r) = \frac{n!}{(n-r)!}$ is the number of permutations of r objects, taken from a pool of n objects. In this case, $r = 1$. To choose 1 object from n objects, there are n choices.

So $\frac{n!}{(n-1)!} = n$

45.8 Exercises

1. Tshepo and Sally go to a restaurant, where the menu is:

Starter	Main Course	Dessert
Chicken wings	Beef burger	Chocolate ice cream
Mushroom soup	Chicken burger	Strawberry ice cream
Greek salad	Chicken curry	Apple crumble
	Lamb curry	Chocolate mousse
	Vegetable lasagne	

   A How many different combinations (of starter, main meal, and dessert) can Tshepo have?

   B Sally doesn't like chicken. How many different combinations can she have?

2. Four coins are thrown, and the outcomes recorded. How many different ways are there of getting three heads? First write out the possibilites, and then use the formula for combinations.

3. The answers in a multiple choice test can be A, B, C, D, or E. In a test of 12 questions, how many different ways are there of answering the test?

4. A girl has 4 dresses, 2 necklaces, and 3 handbags.

   A How many different choices of outfit (dress, necklace and handbag) does she have?

   B She now buys two pairs of shoes. How many choices of outfit (dress, necklace, handbag and shoes) does she now have?

5. In a soccer tournament of 9 teams, every team plays every other team.

   A How many matches are there in the tournament?

   B If there are 5 boys' teams and 4 girls' teams, what is the probability that the first match will be played between 2 girls' teams?

6. The letters of the word 'BLUE' are rearranged randomly. How many new words (a word is any combination of letters) can be made?

7. The letters of the word 'CHEMISTRY' are arranged randomly to form a new word. What is the probability that the word will start and end with a vowel?

8. There are 2 History classes, 5 Accounting classes, and 4 Mathematics classes at school. Luke wants to do all three subjects. How many possible combinations of classes are there?
9. A school netball team has 8 members. How many ways are there to choose a captain, vice-captain, and reserve?
10. A class has 15 boys and 10 girls. A debating team of 4 boys and 6 girls must be chosen. How many ways can this be done?
11. A secret pin number is 3 characters long, and can use any digit (0 to 9) or any letter of the alphabet. Repeated characters are allowed. How many possible combinations are there?

Part V

Exercises

Chapter 46

General Exercises

1. [IEB, Nov. 2004, HG] The notation $\lfloor x \rfloor$ means the largest integer less than or equal to x. For example, $\lfloor 3{,}7 \rfloor = 3$ and $\lfloor 5 \rfloor = 5$

   A i. Show that $x = 35$ is a solution to the equation:

   $$x - \left\lfloor x^{\frac{1}{2}} \right\rfloor^2 = 10$$

   ii. Find any two other values of x that satisfy this equation.

   B Sketch the graph of $y = \lfloor x \rfloor$ for $0 \leq x \leq 4$.

2. [IEB, Nov. 2001, HG] How many digits are there *before* the decimal point in the number $(2{,}673)^{90}$? Explain your answer briefly.

Chapter 47

Exercises - Not covered in Syllabus

1. [IEB, Nov. 2001, HG]

   A Sketch two curves which will enable you to solve $|x - 2| \leq |x| - 1$ graphically.

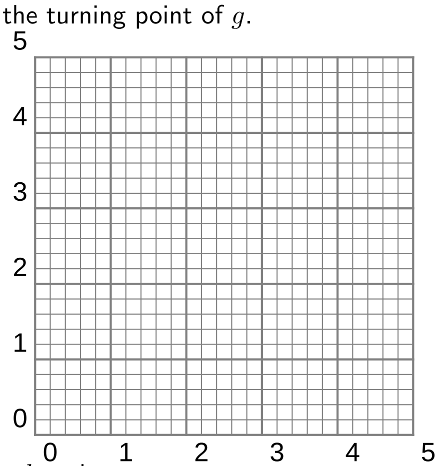
   B *Calculate* the point(s) of intersection of the two curves, and *hence*, or otherwise, write down the solution set to the inequality above.

2. [IEB, Nov. 2002, HG] Evaluate without using a calculator: $|2x - 1| - |-5x|$ given that $x = -1$

3. [IEB, Nov. 2003, HG] In the figure below, two graphs are shown:

$$f(x) = ax^2 + bx + c \qquad \text{and} \qquad g(x) = |x + c|$$

   $x = -1$ is the equation of the axis of symmetry of the parabola. f contains the points (2; 0) and (4; -8), and P is the turning point of g.



   A Find the values of a, b and c.

   B Find the length of MN if MN is perpendicular to the y-axis and MN produced passes through the y-intercept of $f(x)$ and $g(x)$.

   C Determine the equation of the graph that results when $g(x)$ is reflected about the line $x = 1$.

4. [IEB, Nov. 2003, HG]

   A Sketch the graph of $f(x) = x^3 - 9x^2 + 24x - 20$, showing all intercepts with the axes and turning points.

   B Find the equation of the tangent to $f(x)$ at $x = 4$.

   C Sketch the graph of $y = |f(x)|$ on a new set of axes, giving coordinates of the turning points and intercepts with the axes.

5. [IEB, Nov. 2004, HG] Solve: $18 \leq |x - 3|$

6. [IEB, Nov. 2005, HG] Solve for x: $25^{|1-2x|} = 5^4$

7. [IEB, Nov. 2005, HG] If $f(x) = |4 - x|$, find the value of $f'(3)$.

Appendix A

GNU Free Documentation License

PREAMBLE

The purpose of this License is to make a manual, textbook, or other functional and useful document "free" in the sense of freedom: to assure everyone the effective freedom to copy and redistribute it, with or without modifying it, either commercially or non-commercially. Secondarily, this License preserves for the author and publisher a way to get credit for their work, while not being considered responsible for modifications made by others.

This License is a kind of "copyleft", which means that derivative works of the document must themselves be free in the same sense. It complements the GNU General Public License, which is a copyleft license designed for free software.

We have designed this License in order to use it for manuals for free software, because free software needs free documentation: a free program should come with manuals providing the same freedoms that the software does. But this License is not limited to software manuals; it can be used for any textual work, regardless of subject matter or whether it is published as a printed book. We recommend this License principally for works whose purpose is instruction or reference.

APPLICABILITY AND DEFINITIONS

This License applies to any manual or other work, in any medium, that contains a notice placed by the copyright holder saying it can be distributed under the terms of this License. Such a notice grants a world-wide, royalty-free license, unlimited in duration, to use that work under the conditions stated herein. The "Document", below, refers to any such manual or work. Any member of the public is a licensee, and is addressed as "you". You accept the license if you copy, modify or distribute the work in a way requiring permission under copyright law.

A "Modified Version" of the Document means any work containing the Document or a portion of it, either copied verbatim, or with modifications and/or translated into another language.

A "Secondary Section" is a named appendix or a front-matter section of the Document that deals exclusively with the relationship of the publishers or authors of the Document to the Document's overall subject (or to related matters) and contains nothing that could fall directly within that overall subject. (Thus, if the Document is in part a textbook of mathematics, a Secondary Section may not explain any mathematics.) The relationship could be a matter of historical connection with the subject or with related matters, or of legal, commercial, philosophical, ethical or political position regarding them.

The "Invariant Sections" are certain Secondary Sections whose titles are designated, as being those of Invariant Sections, in the notice that says that the Document is released under this License. If a section does not fit the above definition of Secondary then it is not allowed to be designated as Invariant. The Document may contain zero Invariant Sections. If the Document does not identify any Invariant Sections then there are none.

The "Cover Texts" are certain short passages of text that are listed, as Front-Cover Texts or Back-Cover Texts, in the notice that says that the Document is released under this License. A Front-Cover Text may be at most 5 words, and a Back-Cover Text may be at most 25 words.

A "Transparent" copy of the Document means a machine-readable copy, represented in a format whose specification is available to the general public, that is suitable for revising the document straightforwardly with generic text editors or (for images composed of pixels) generic paint programs or (for drawings) some widely available drawing editor, and that is suitable for input to text formatters or for automatic translation to a variety of formats suitable for input to text formatters. A copy made in an otherwise Transparent file format whose markup, or absence of markup, has been arranged to thwart or discourage subsequent modification by readers is not Transparent. An image format is not Transparent if used for any substantial amount of text. A copy that is not "Transparent" is called "Opaque".

Examples of suitable formats for Transparent copies include plain ASCII without markup, Texinfo input format, LaTeX input format, SGML or XML using a publicly available DTD and standard-conforming simple HTML, PostScript or PDF designed for human modification. Examples of transparent image formats include PNG, XCF and JPG. Opaque formats include proprietary formats that can be read and edited only by proprietary word processors, SGML or XML for which the DTD and/or processing tools are not generally available, and the machine-generated HTML, PostScript or PDF produced by some word processors for output purposes only.

The "Title Page" means, for a printed book, the title page itself, plus such following pages as are needed to hold, legibly, the material this License requires to appear in the title page. For works in formats which do not have any title page as such, "Title Page" means the text near the most prominent appearance of the work's title, preceding the beginning of the body of the text.

A section "Entitled XYZ" means a named subunit of the Document whose title either is precisely XYZ or contains XYZ in parentheses following text that translates XYZ in another language. (Here XYZ stands for a specific section name mentioned below, such as "Acknowledgements", "Dedications", "Endorsements", or "History".) To "Preserve the Title" of such a section when you modify the Document means that it remains a section "Entitled XYZ" according to this definition.

The Document may include Warranty Disclaimers next to the notice which states that this License applies to the Document. These Warranty Disclaimers are considered to be included by reference in this License, but only as regards disclaiming warranties: any other implication that these Warranty Disclaimers may have is void and has no effect on the meaning of this License.

VERBATIM COPYING

You may copy and distribute the Document in any medium, either commercially or non-commercially, provided that this License, the copyright notices, and the license notice saying this License applies to the Document are reproduced in all copies, and that you add no other conditions whatsoever to those of this License. You may not use technical measures to obstruct or control the reading or further copying of the copies you make or distribute. However, you may accept compensation in exchange for copies. If you distribute a large enough number of copies you must also follow the conditions in section A.

You may also lend copies, under the same conditions stated above, and you may publicly display copies.

COPYING IN QUANTITY

If you publish printed copies (or copies in media that commonly have printed covers) of the Document, numbering more than 100, and the Document's license notice requires Cover Texts, you must enclose the copies in covers that carry, clearly and legibly, all these Cover Texts: Front-Cover Texts on the front cover, and Back-Cover Texts on the back cover. Both covers must also clearly and legibly identify you as the publisher of these copies. The front cover must present the full title with all words of the title equally prominent and visible. You may add other material on the covers in addition. Copying with changes limited to the covers, as long as they preserve the title of the Document and satisfy these conditions, can be treated as verbatim copying in other respects.

If the required texts for either cover are too voluminous to fit legibly, you should put the first ones listed (as many as fit reasonably) on the actual cover, and continue the rest onto adjacent pages.

If you publish or distribute Opaque copies of the Document numbering more than 100, you must either include a machine-readable Transparent copy along with each Opaque copy, or state in or with each Opaque copy a computer-network location from which the general network-using public has access to download using public-standard network protocols a complete Transparent copy of the Document, free of added material. If you use the latter option, you must take reasonably prudent steps, when you begin distribution of Opaque copies in quantity, to ensure that this Transparent copy will remain thus accessible at the stated location until at least one year after the last time you distribute an Opaque copy (directly or through your agents or retailers) of that edition to the public.

It is requested, but not required, that you contact the authors of the Document well before redistributing any large number of copies, to give them a chance to provide you with an updated version of the Document.

MODIFICATIONS

You may copy and distribute a Modified Version of the Document under the conditions of sections A and A above, provided that you release the Modified Version under precisely this License, with the Modified Version filling the role of the Document, thus licensing distribution and modification of the Modified Version to whoever possesses a copy of it. In addition, you must do these things in the Modified Version:

1. Use in the Title Page (and on the covers, if any) a title distinct from that of the Document, and from those of previous versions (which should, if there were any, be listed in the History section of the Document). You may use the same title as a previous version if the original publisher of that version gives permission.
2. List on the Title Page, as authors, one or more persons or entities responsible for authorship of the modifications in the Modified Version, together with at least five of the principal authors of the Document (all of its principal authors, if it has fewer than five), unless they release you from this requirement.
3. State on the Title page the name of the publisher of the Modified Version, as the publisher.
4. Preserve all the copyright notices of the Document.
5. Add an appropriate copyright notice for your modifications adjacent to the other copyright notices.
6. Include, immediately after the copyright notices, a license notice giving the public permission to use the Modified Version under the terms of this License, in the form shown in the Addendum below.
7. Preserve in that license notice the full lists of Invariant Sections and required Cover Texts given in the Document's license notice.
8. Include an unaltered copy of this License.
9. Preserve the section Entitled "History", Preserve its Title, and add to it an item stating at least the title, year, new authors, and publisher of the Modified Version as given on the Title Page. If there is no section Entitled "History" in the Document, create one stating the title, year, authors, and publisher of the Document as given on its Title Page, then add an item describing the Modified Version as stated in the previous sentence.

10. Preserve the network location, if any, given in the Document for public access to a Transparent copy of the Document, and likewise the network locations given in the Document for previous versions it was based on. These may be placed in the "History" section. You may omit a network location for a work that was published at least four years before the Document itself, or if the original publisher of the version it refers to gives permission.

11. For any section Entitled "Acknowledgements" or "Dedications", Preserve the Title of the section, and preserve in the section all the substance and tone of each of the contributor acknowledgements and/or dedications given therein.

12. Preserve all the Invariant Sections of the Document, unaltered in their text and in their titles. Section numbers or the equivalent are not considered part of the section titles.

13. Delete any section Entitled "Endorsements". Such a section may not be included in the Modified Version.

14. Do not re-title any existing section to be Entitled "Endorsements" or to conflict in title with any Invariant Section.

15. Preserve any Warranty Disclaimers.

If the Modified Version includes new front-matter sections or appendices that qualify as Secondary Sections and contain no material copied from the Document, you may at your option designate some or all of these sections as invariant. To do this, add their titles to the list of Invariant Sections in the Modified Version's license notice. These titles must be distinct from any other section titles.

You may add a section Entitled "Endorsements", provided it contains nothing but endorsements of your Modified Version by various parties–for example, statements of peer review or that the text has been approved by an organisation as the authoritative definition of a standard.

You may add a passage of up to five words as a Front-Cover Text, and a passage of up to 25 words as a Back-Cover Text, to the end of the list of Cover Texts in the Modified Version. Only one passage of Front-Cover Text and one of Back-Cover Text may be added by (or through arrangements made by) any one entity. If the Document already includes a cover text for the same cover, previously added by you or by arrangement made by the same entity you are acting on behalf of, you may not add another; but you may replace the old one, on explicit permission from the previous publisher that added the old one.

The author(s) and publisher(s) of the Document do not by this License give permission to use their names for publicity for or to assert or imply endorsement of any Modified Version.

COMBINING DOCUMENTS

You may combine the Document with other documents released under this License, under the terms defined in section A above for modified versions, provided that you include in the combination all of the Invariant Sections of all of the original documents, unmodified, and list them all as Invariant Sections of your combined work in its license notice, and that you preserve all their Warranty Disclaimers.

The combined work need only contain one copy of this License, and multiple identical Invariant Sections may be replaced with a single copy. If there are multiple Invariant Sections with the same name but different contents, make the title of each such section unique by adding at the end of it, in parentheses, the name of the original author or publisher of that section if known, or else a unique number. Make the same adjustment to the section titles in the list of Invariant Sections in the license notice of the combined work.

In the combination, you must combine any sections Entitled "History" in the various original documents, forming one section Entitled "History"; likewise combine any sections Entitled "Acknowledgements", and any sections Entitled "Dedications". You must delete all sections Entitled "Endorsements".

COLLECTIONS OF DOCUMENTS

You may make a collection consisting of the Document and other documents released under this License, and replace the individual copies of this License in the various documents with a single copy that is included in the collection, provided that you follow the rules of this License for verbatim copying of each of the documents in all other respects.

You may extract a single document from such a collection, and distribute it individually under this License, provided you insert a copy of this License into the extracted document, and follow this License in all other respects regarding verbatim copying of that document.

AGGREGATION WITH INDEPENDENT WORKS

A compilation of the Document or its derivatives with other separate and independent documents or works, in or on a volume of a storage or distribution medium, is called an "aggregate" if the copyright resulting from the compilation is not used to limit the legal rights of the compilation's users beyond what the individual works permit. When the Document is included an aggregate, this License does not apply to the other works in the aggregate which are not themselves derivative works of the Document.

If the Cover Text requirement of section A is applicable to these copies of the Document, then if the Document is less than one half of the entire aggregate, the Document's Cover Texts may be placed on covers that bracket the Document within the aggregate, or the electronic equivalent of covers if the Document is in electronic form. Otherwise they must appear on printed covers that bracket the whole aggregate.

TRANSLATION

Translation is considered a kind of modification, so you may distribute translations of the Document under the terms of section A. Replacing Invariant Sections with translations requires special permission from their copyright holders, but you may include translations of some or all Invariant Sections in addition to the original versions of these Invariant Sections. You may include a translation of this License, and all the license notices in the Document, and any Warranty Disclaimers, provided that you also include the original English version of this License and the original versions of those notices and disclaimers. In case of a disagreement between the translation and the original version of this License or a notice or disclaimer, the original version will prevail.

If a section in the Document is Entitled "Acknowledgements", "Dedications", or "History", the requirement (section A) to Preserve its Title (section A) will typically require changing the actual title.

TERMINATION

You may not copy, modify, sub-license, or distribute the Document except as expressly provided for under this License. Any other attempt to copy, modify, sub-license or distribute the Document is void, and will automatically terminate your rights under this License. However, parties who have received copies, or rights, from you under this License will not have their licenses terminated so long as such parties remain in full compliance.

FUTURE REVISIONS OF THIS LICENSE

The Free Software Foundation may publish new, revised versions of the GNU Free Documentation License from time to time. Such new versions will be similar in spirit to the present version, but may differ in detail to address new problems or concerns. See http://www.gnu.org/copyleft/.

Each version of the License is given a distinguishing version number. If the Document specifies that a particular numbered version of this License "or any later version" applies to it, you have the option of following the terms and conditions either of that specified version or of any later version that has been published (not as a draft) by the Free Software Foundation. If the Document does not specify a version number of this License, you may choose any version ever published (not as a draft) by the Free Software Foundation.

ADDENDUM: How to use this License for your documents

To use this License in a document you have written, include a copy of the License in the document and put the following copyright and license notices just after the title page:

> Copyright © YEAR YOUR NAME. Permission is granted to copy, distribute and/or modify this document under the terms of the GNU Free Documentation License, Version 1.2 or any later version published by the Free Software Foundation; with no Invariant Sections, no Front-Cover Texts, and no Back-Cover Texts. A copy of the license is included in the section entitled "GNU Free Documentation License".

If you have Invariant Sections, Front-Cover Texts and Back-Cover Texts, replace the "with...Texts." line with this:

with the Invariant Sections being LIST THEIR TITLES, with the Front-Cover Texts being LIST, and with the Back-Cover Texts being LIST.

If you have Invariant Sections without Cover Texts, or some other combination of the three, merge those two alternatives to suit the situation.

If your document contains nontrivial examples of program code, we recommend releasing these examples in parallel under your choice of free software license, such as the GNU General Public License, to permit their use in free software.